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On an Approximate Method for Random Differential Equations

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Abstract

The approximate method of /13/ for random differential equations is further developed into two directions. First a result on a.s.-convergence of random solutions is proved for the case that the domain of the right-hand side is "stochastic" using an approximation concept for random operator equations. Secondly a concept for the discrete approximation of Banach space-valued random variables is sketched and used for a new derivation of a discrete approximation of continuous Gaussian processes by conditional expectations.

1. Introduction

In recent years there has been considerable interest in the theory, approximation and applications of random equations, especially random ordinary differential equations. The theory of random differential equations is well-developed (see the monographs /1/, /7/, /15/). Recently the use of random fixed point theorems has unified and generalized the approaches to the existence and uniqueness of random solutions (see /4/, /8/, /10/). Diverse problems in the real world, e.g. in mechanics and electrical circuit theory, lead to more suitable models if random equations are used (see also /15/). Thus, approximate methods for the solution of random equations are of considerable interest. Much progress has been made in this direction up to now. We refer to the surveys /5/, /11/ and to /2/.

The purpose of this paper is the further development of the approximate method of /13/ and /12, chapt. 7/. Let us consider the following random differential equation

$$x(t) = f(z(\omega,t),x(t),t), \quad t \in [t_o,t_1], \qquad (1.1)$$
$$x(t_o) = x_o(\omega),$$

where x_o is a random variable and z a stochastic process defined on some underlying probability space with a given joint distribution. The problem is the approximate computation of some characteristics of the distribution of a random solution.

The approach of /13/ and /12/ for solving this problem is exactly of the type which is called "direct numerical method" in /5,p.174/. This approach involves two essential steps:

(i) Replace the random input by a "discrete" random variable using suitable conditional expectations;

(ii) Solve the resulting finite family of deterministic ordinary differential equations by usual numerical integration methods and by usual numerical software, respectively.

In the next two chapters we give results on the two main problems of this approach:

(a) Convergence of approximate random solution to a random solution of (1.1) if x_o and f (that means z) are approximated: We focus our attention on the case of a.s.-convergence and note that approximations by conditional expectations converge almost surely (because of martingale convergence results). Our approach is based on the general approximation concept of /12/ and /9/ for random operator equations. Generalizing /12, chapt. 7/ we treat the case that the domain of f is "stochastic", i.e., it depends on "ω".

(b) Discrete approximations of random variables by conditional expectations: We give a short introduction to a concept for the discrete approximation of Banach space-valued random variables using conditional expectations and sketch some basic properties. As an application, a new description of a discrete approximation for continuous Gaussian processes (see /13, chapt. 6/) is obtained.

2. Convergence of approximate solutions of random ordinary differential equations

We consider initial value problems for random ordinary differential equations of the following kind:

$$\dot{x}(t) = f(\omega,x(t),t), \quad t \in [t_o,t_1], \qquad (2.1)$$
$$x(t_o) = x_o(\omega).$$

Throughout this chapter, let $(\Omega,\mathfrak{A},P)$ be a complete probability space and let us assume that

$x_o : \Omega \to R^n$ is a random variable on $(\Omega,\mathfrak{A},P)$; (2.2)

$f : \mathrm{Gr}\, D \times [t_o,t_1] \to R^n$ is a mapping such that (2.3)

(i) for all $\omega\in\Omega$ $f(\omega,.,.): D(\omega) \times [t_o,t_1] \to R^n$ is continuous;

(ii) for all $\omega\in\Omega$ and $t\in[t_o,t_1]$ $f(\omega,.,t): D(\omega) \to R^n$ is Lipschitzian on $D(\omega)$ with the Lipschitz-constant $L(\omega)$;

(iii) f is a random function on stochastic domain $\tilde{D}$, where $\tilde{D}(\omega) := D(\omega) \times [t_o,t_1]$, $\omega\in\Omega$;

where D is the multifunction defined on Ω into $\mathfrak{P}(R^n)$ (the set of all nonempty subsets of R^n)

$$D(\omega) := \{x\in R^n \mid |x-x_o(\omega)| \leq r(\omega)\}, \; \omega\in\Omega,$$

where $|.|$ denotes some fixed norm on R^n and the "radius" $r : \Omega \to (0,\infty)$ is a random variable on $(\Omega,\mathfrak{A},P)$.

Applying the general concept of /12/, /9/ for the approximate solution of random operator equations we now define the appropriate setting for (2.1). Let X be the Banach space $C([t_o,t_1];R^n)$ endowed with the usual norm

$$\|x\| := \max_{t\in[t_o,t_1]} |x(t)|$$

and we define:

$$C:\Omega \to \mathcal{P}(X) \quad C(\omega) := \{x\in X \mid \|x-x_o(\omega)\| \leqq r(\omega)\},\ \omega\in\Omega,$$

$$T: \mathrm{Gr}\, C \to X\ [T(\omega,x)](t) := x(t)-x_o(\omega)-\int_{t_o}^{t} f(\omega,x(s),s)ds,$$

$$t\in[t_o,t_1],\quad x\in C(\omega),\ \omega\in\Omega.$$

We note that by (2.3) the multifunction C and the mapping T are well-defined and we consider the "random operator equation"

$$T(\omega,x) = 0. \tag{2.4}$$

<u>Proposition 2.1</u>. Let (2.2) and (2.3) be satisfied.

a) $T: \mathrm{Gr}\, C \to X$ is a continuous random operator on stochastic domain C.

b) Assume that (2.1) has a wide-sense solution. Then there exists a random solution of (2.4) which is almost surely unique.

c) For all $\omega\in\Omega$ and all $x,y\in C(\omega)$ it holds:

$$\|x - y\| \leqq 2e^{2L(\omega)(t_1-t_o)}\,\|T(\omega,x) - T(\omega,y)\|.$$

<u>Proof</u>. a) First we note that $\tilde{D}:\Omega\to\mathcal{P}(R^{n+1})$ and $C:\Omega\to\mathcal{P}(X)$ are weakly measurable, separable, closed-valued multifunctions (/8,p.70/). The components f_i, i=1,...,n, of f are continuous real random functions on stochastic domain D (by (2.3) (i), (iii)).

Therefore, /3, Theorem 2/ applies to our situation and there exist continuous real random functions $\tilde{f}_i: \Omega\times R^n\times[t_o,t_1]\to R^n$ such that

$$\tilde{f}_i(\omega,x,t) = f_i(\omega,x,t),\quad x\in D(\omega),\ t\in[t_o,t_1],$$
$$\omega\in\Omega,\ i=1,\dots,n.$$

If we define $\tilde{T}:\Omega\times X\to X$ analogous to T, but with $\tilde{f}$ instead of f, then it is clear that $\tilde{T}(\omega,x) = T(\omega,x)$, for all $(\omega,x)\in\mathrm{Gr}\, C$.

With the same arguments as in /8, p.75/ it results that $\tilde{T}:\Omega\times X\to X$

is a random operator. By definition of $\tilde{T}$ this yields that $T: \mathrm{Gr}\, C\to X$ is a random operator on stochastic domain C. Furthermore (2.3) implies that $T(\omega,.): C(\omega)\to X$ is Lipschitzian for all $\omega\in\Omega$.

b) This follows from a), usual random fixed point theorems (see /8, Corollary 7/) and from condition (2.3), which implies that a wide-sense solution of (2.1) is a.s.-unique.

c) Let $\omega\in\Omega$ and $x,y\in C(\omega)$ be arbitrary, but fixed. Then we define

$$\|x\|_\omega := \max_{t\in[t_o,t_1]} \{e^{-2L(\omega)(t-t_o)}|x(t)|\} \leqq \|x\|,$$

$$[S(\omega,x)](t) := x_o(\omega) + \int_{t_o}^{t} f(\omega,x(s),s)ds,\quad t\in[t_o,t_1].$$

Because of (2.3) we have

$$\|S(\omega,x) - S(\omega,y)\|_\omega \leqq \tfrac{1}{2}\|x - y\|_\omega,$$

and it follows that

$$\|T(\omega,x)-T(\omega,y)\|_\omega \geqq \|x-y\|_\omega - \|S(\omega,x)-S(\omega,y)\|_\omega \geqq \tfrac{1}{2}\|x-y\|_\omega$$
$$\geqq \tfrac{1}{2}e^{-2L(\omega)(t_1-t_o)}\|x-y\|.$$

This completes the proof. q.e.d.

In addition to (2.1) we now consider its following "approximations"

$$\left.\begin{aligned} \dot{x}(t) &= f_m(\omega,x(t),t),\quad t\in[t_o,t_1],\\ x(t_o) &= x_{om}(\omega). \end{aligned}\right\}\ m\in N \tag{2.5}$$

We assume for (2.5) that for all $m\in N$

$x_{om}: \Omega\to R^n$ is a random variable on $(\Omega,\mathfrak{A},P)$; (2.6)

f_m fulfills condition (2.3) with D_m, L_m, x_{om} and r_m instead of D, L, x_o and r. (2.7)

Analogous to the above we define for all $m\in N$

$$C_m:\Omega\to\mathcal{P}(X)\quad C_m(\omega) := \{x\in X\mid \|x-x_{om}(\omega)\| \leqq r_m(\omega)\},\ \omega\in\Omega,$$

$$T_m: \mathrm{Gr}\, C_m\to X\ [T_m(\omega,x)](t) := x(t)-x_{om}(\omega)-\int_{t_o}^{t} f_m(\omega,x(s),s)ds,$$

$$t\in[t_o,t_1],\quad x\in C_m(\omega),\ \omega\in\Omega,$$

and consider the "approximate random operator equations"

$$T_m(\omega,x) = 0 \qquad (m \in N). \tag{2.8}$$

We note that analogous to Propos. 2.1 a) T_m: Gr $C_m \to X$, $m \in N$, are continuous random operators on stochastic domains C_m. Especially T and T_m, $m \in N$, are $\mathfrak{A} \times \mathfrak{B}(X)$-measurable (see /12, Def.1 and Lemma 2/) and our situation is the same as in /12, chapt.3/.

Theorem 2.2. Let the assumptions (2.2), (2.3), (2.6), (2.7) be fulfilled. In addition, we assume that for all $\omega \in \Omega \setminus A$, $P(A)=0$:

(i) $\lim_{m\to\infty} |x_{om}(\omega) - x_o(\omega)| = 0$ and $\lim_{m\to\infty} |r_m(\omega) - r(\omega)| = 0$;

(ii) for all $x \in C(\omega)$, $x_m \in C_m(\omega)$, $m \in N$, such that $\lim_{m\to\infty} \|x_m - x\| = 0$, it holds:
$$\lim_{m\to\infty} \max_{t \in [t_o,t_1]} |f_m(\omega,x_m(t),t) - f(\omega,x(t),t)| = 0;$$

(iii) $L_m(\omega) \leqq L(\omega)$, for all $m \in N$.

Let wide-sense solutions of (2.1) and (2.5), for all $m \in N$, exist. Then there exist random solutions x^* of (2.1) and x_m^* of (2.5), for all $m \in N$, and it holds
$$\lim_{m\to\infty} \max_{t \in [t_o,t_1]} |x^*(\omega,t) - x_m^*(\omega,t)| = 0 \quad \text{a.s.}$$

Proof. From Propos. 2.1 b) it is clear that random solutions of (2.1) and (2.5) ($m \in N$) exist. We only have to prove their convergence.

To this end we apply /12, Theorem 2/ and have to check the assumptions of that Theorem. Because of Propos. 2.1 c) using (2.7) it holds for all $m \in N$, $\omega \in \Omega$ and $x,y \in C_m(\omega)$:
$$\|x - y\| \leqq 2e^{2L_m(\omega)(t_1-t_o)} \|T_m(\omega,x) - T_m(\omega,y)\|.$$

The above assumption (iii) yields that the "inverse stability" assumption b) of /12, Theorem 2/ is fulfilled.

Using (i) and (ii) it follows immediately that for all $\omega \in \Omega \setminus A$ and all $x \in C(\omega)$, $x_m \in C_m(\omega)$, $m \in N$, such that $\lim_{m\to\infty} \|x_m - x\| = 0$, it holds
$$\lim_{m\to\infty} \|T_m(\omega,x_m) - T(\omega,x)\| = 0.$$

It remains to show that for every measurable selector x of C there exists a sequence x_m' of measurable selectors of C_m, $m \in N$, which converges almost surely to x. We conclude from /9, Theorem 4.1/ that the following condition is sufficient for this measurable selector convergence:

There is an $A_o \in \mathfrak{A}$ with $P(A_o)=0$ such that for all $\omega \in \Omega \setminus A_o$ and all $x \in C(\omega)$,
$$\lim_{m\to\infty} d(x,C_m(\omega)) = \lim_{m\to\infty} \inf_{y \in C_m(\omega)} \|x - y\| = 0.$$

Because of
$$d(x,C_m(\omega)) = \max\{0, \|x - x_{om}(\omega)\| - r_m(\omega)\}$$
(/9, Example 4.4/)
$$\xrightarrow[m\to\infty]{} \max\{0, \|x - x_o(\omega)\| - r(\omega)\} \quad \text{a.s. (using (i))}$$
the latter condition is obviously fulfilled. This completes the proof.

q.e.d.

3. On an approximation of continuous Gaussian processes by conditional expectations

In /13/ and /12, chapt. 6/ an approximation method for stochastic processes and Banach space-valued random variables, using conditional expectations was developed. In the following we present an approach which seems to simplify the convergence proof and to unify the description of this approximation method. Our aim in this context is to give a new derivation of the numerical method for continuous Gaussian processes presented in /13, chapt. 6/.

Let Z be a real separable Banach space, Z^* its dual, and let $\langle .,. \rangle$ denote the dual pairing between Z^* and Z. Let $\mathfrak{B}(Z)$ be the σ-algebra of Borel sets of Z, $F=\{z_i^*\}_{i\in N}$ be a countable and total subset of Z^* and let $\mathfrak{C}_F(Z) \subset \mathfrak{B}(Z)$ be the set of cylinder sets (with respect to F)
$$\mathfrak{C}_F(Z) := \{(\langle z_{i_1}^*,.\rangle,\dots,\langle z_{i_m}^*,.\rangle)^{-1}(B) \mid i_j \in N,\ j=1,\dots,m,\ B \in \mathfrak{B}(R^m),\ m \in N\}.$$

Furthermore, let $(\Omega,\mathfrak{A},P)$ be a probability space, $z:\Omega\to Z$ be a Z-valued random variable on $(\Omega,\mathfrak{A},P)$ and let for all $m\in N$ $\{B_{ml}\}_{l=1,\dots,s_m}$ be a finite partition of R^m into Borel subsets of R^m.

We define

$$A_{ml} := (\langle z_1^*,z(.)\rangle,\dots,\langle z_m^*,z(.)\rangle)^{-1}(B_{ml}) \tag{3.1}$$
$$= z^{-1}((\langle z_1^*,.\rangle,\dots,\langle z_m^*,.\rangle)^{-1}(B_{ml}))$$
$$(l=1,\dots,s_m)$$

$$\mathfrak{A}_m := \sigma(\{A_{ml}\}_{l=1,\dots,s_m})\subset\mathfrak{A},\quad m\in N,$$

and consider the conditional expectations

$$E(z|\mathfrak{A}_m) \quad\text{if}\quad E[\|z(\omega)\|]<\infty\ ; \tag{3.2}$$

$$E(\langle z^*,z\rangle|\mathfrak{A}_m) \quad\text{if}\quad E[|\langle z^*,z(\omega)\rangle|]<\infty \quad\text{for some } z^*\in Z^*. \tag{3.3}$$

Remark 3.1. a) We note the following well-known conclusions:

If $E[\|z(\omega)\|]<\infty$, then it holds for all $z^*\in Z^*$:

$$\langle z^*,E(z|\mathfrak{A}_m)\rangle = E(\langle z^*,z\rangle|\mathfrak{A}_m) \quad \text{a.s.} \tag{3.4}$$

If $P(A_{ml})>0$ for $l\in\{1,\dots,s_m\}$, then it holds for $\omega\in A_{ml}$:

$$E(\langle z^*,z\rangle|\mathfrak{A}_m)(\omega) = \frac{1}{P(A_{ml})}\int_{A_{ml}}\langle z^*,z(\omega)\rangle dP \tag{3.5}$$

(in the case (3.3) and analogously for (3.2)).

b) If the sequence $\{\{B_{ml}\}_l\}_{m\in N}$ is chosen in such a way that $\mathfrak{A}_m$, $m\in N$, is increasing and that

$$\{z^{-1}(B)\mid B\in\mathfrak{B}(Z)\}\subseteq\sigma(\{\{z^{-1}((\langle z_1^*,.\rangle,\dots,\langle z_m^*,.\rangle)^{-1}(B_{ml}))\}_l\}_{m\in N}),$$

it follows from the usual martingale convergence that

(i) $\lim\limits_{m\to\infty} E(\langle z^*,z\rangle|\mathfrak{A}_m) = \langle z^*,z\rangle$ a.s., in the case (3.3);

(ii) $\lim\limits_{m\to\infty}\|E(z|\mathfrak{A}_m)-z\| = 0$ a.s., in the case (3.2).

(see e.g. /6, p.77/).

Note that also convergence in p-th mean $(1\le p<\infty)$ results if

$$E[|\langle z^*,z(\omega)\rangle|^p]<\infty \quad\text{and}\quad E[\|z(\omega)\|^p]<\infty\ ,$$

respectively.

c) Let $z:\Omega\to Z$ be a Gaussian random variable, i.e., for all $z^*\in Z^*$ $\langle z^*,z(.)\rangle:\Omega\to R^1$ is Gaussian.

It follows that $E[|\langle z^*,z(\omega)\rangle|^p]<\infty$, for all $1\le p<\infty$, and that $E(\langle z^*,z\rangle|\mathfrak{A}_m)$, $m\in N$, is well-defined and the above martingale convergence results apply.

Proposition 3.2. Let $z:\Omega\to Z$ be Gaussian and let $z_j:=\langle z_j^*,z\rangle:\Omega\to R^1$, $j=1,\dots,m$, be independent $N(0,1)$-distributed random variables. Then it holds

$$E(\langle z^*,z\rangle|\mathfrak{A}_m) = \sum_{j=1}^{m} E[\langle z^*,z\rangle z_j]E(z_j|\mathfrak{A}_m),\quad z^*\in Z^*. \tag{3.6}$$

The proof is an immediate consequence e.g. of /14, p.324/.

We now turn to the special case of a Gaussian random variable $z:\Omega\to C([0,1])$, i.e. a continuous Gaussian process. We suppose for simplicity that $E[z(t)]=0$, for all $t\in[0,1]$, and that z does not degenerate in $[0,1]$. Applying the general concept of this chapter we choose a countable, dense subset

$\{t_j\}_{j\in N}$ of $[0,1]$ such that for all $k\in N$ the covariance matrices $(E[z(t_i)z(t_j)])_{i,j=1,\dots,k}$ are regular.

We define the subset $F\subset(C([0,1]))^*$:

$$F := \{z_j^*\}_{j\in N} \qquad \langle z_j^*,x\rangle := \frac{1}{a_{jj}}\Big(x(t_j) - \sum_{i=1}^{j-1} a_{ij}\langle z_i^*,x\rangle\Big), \tag{3.7}$$
$$x\in C([0,1]),\quad j\in N,$$

where

$$\sum_{i=1}^{j} a_{ij}a_{ik} = E[z(t_j)z(t_k)],\quad j=1,\dots,k,\ k\in N,$$

which corresponds to the Cholesky-decomposition of the above covariance matrices.

Proposition 3.3. $F\subset(C([0,1]))^*$ is total and $z_j:=\langle z_j^*,z(.)\rangle$, $j\in N$, are independent $N(0,1)$-distributed random variables.

If $\mathfrak{A}_m$, $m\in N$, are defined according to (3.1), it holds:

$$E(z(t)|\mathfrak{A}_m) = \sum_{j=1}^{m} E[z(t)z_j]E(z_j|\mathfrak{A}_m),\quad t\in[0,1],\quad m\in N, \tag{3.8}$$

where

$$\sum_{i=1}^{j} a_{ij} E[z(t)z_i] = E[z(t)z(t_j)], \; j=1,\dots,m, \; m\in N. \qquad (3.9)$$

Proof. Let $\langle z_j^*, x\rangle = 0$, for all $j\in N$. Because of (3.7) $x(t_j)=0$ results, for all $j\in N$. It follows that $x(t)\equiv 0$, since $\{t_j\}_{j\in N}$ is dense in $[0,1]$, i.e., F is total. From the usual transformation of multivariate Gaussian random vectors (see e.g. /14, p.322/) if follows that z_j, $j\in N$, are independent $N(0,1)$-distributed random variables. The remainder of the proof is a consequence of Propos. (3.2) and (3.7). q.e.d.

Remark 3.4. a) Applying Remark 3.1 b) and c) to the above discrete approximation of a continuous Gaussian process, convergence results for (3.8) can be obtained.

b) Using the extremal property of conditional expectations with respect to the quadratic mean, the following "estimate" is an immediate consequence:

$$E[|z(t)-E(z(t)|\mathfrak{A}_m)|^2] = E[z^2(t)] - \sum_{j=1}^{m}(E[z(t)z_j])^2 E[(E(z_j|\mathfrak{A}_m))^2]$$

$(t\in[0,1], \; m\in N)$.

c) The finite number of sample paths of $E(z(.)|\mathfrak{A}_m)$ can be computed efficiently from (3.8), (3.9) if the finite number of realizations of the discrete random variables $E(z_j|\mathfrak{A}_m)$, $j=1,\dots,m$, are known. The realizations of $E(z_j|\mathfrak{A}_m)$ can be easily computed from the $N(0,1)$-distribution (see /13, p.531/) if the B_{ml} (see 3.1) are chosen to be m-dimensional intervals.

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