Completeness of compact Lorentzian manifolds with special holonomy

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Overview

Introduction: Known Completeness Results

Remark

*Compactness does **not** imply completeness in the Lorentzian case:*

- $\mathbb{R}^2$ *with* $g = -2dudv + (\cos^4(v) - 1)du^2$ *is incomplete (e.g.* $t \mapsto (\frac{1}{t} - t, \arctan(t))$ *is inextensible)*
- *constitutes an incomplete Lorentz-metric on* $\mathbb{T}^2$

Proposition

*A **compact** Lorentzian manifold* (M, g) *is complete, if*

- *it is homogeneous* [Mar73],
- *it is flat* [Car89] *or of constant curvature* [Kli96],
- $\dim M = 3$ *and it is locally homogeneous* [DZ10],
- *has a time-like conformal vector field* [RS95].

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Introduction: Special Holonomy I

Definition

The holonomy group $\mathrm{Hol}_x(M,g)$ of an $(n+2)$-dimensional semi-Riemannian manifold (M,g) is defined as

$$\mathrm{Hol}_x(M,g) := \{\mathcal{P}_\gamma \mid \gamma \text{ loop in } x\} \subset \mathrm{O}(T_xM, g_x),$$

where $x \in M$.

Its connected component we call the reduced holonomy group $\mathrm{Hol}^0_x(M,g)$.

Introduction: Special Holonomy II

Definition

We say that a Lorentzian manifold (M^{n+2}, g) has special holonomy if:

(i) $\mathrm{Hol}^0(M,g) \neq \mathrm{SO}^0(1,n+1)$ and

(ii) g degenerates on every $\mathrm{Hol}^0(M,g)$-invariant subspace.

Remark

Difference to the Riemannian case:

- *exists no proper irreducible* $H^0 \subset \mathrm{SO}^0(1,n+1)$ [DSO01]. *Hence, ex.* $\mathrm{Hol}^0(M,g)$*-invariant null line* $\mathbb{L} \subset TM$.
- $\mathrm{Hol}(M,g) \subset \mathrm{Nor}_{O(1,n+1)}(\mathrm{Hol}^0(M,g)) \subset \mathrm{Stab}_{O(1,n+1)}(\mathbb{L})$ [BLL12]. *Hence, ex.* $\mathrm{Hol}(M,g)$*-invariant null line* $\mathbb{L} \subset TM$.

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Introduction: Special Holonomy III

Corollary

$\mathbb{L}$ *constitutes:*

- *a filtration* $\mathbb{L} \subset \mathbb{L}^{\perp} \subset TM$,
- *an integrable codimension one foliation* $\mathcal{F}$ *on* M *with* $T\mathcal{F} = \mathbb{L}^{\perp}$,
- *a screen bundle* Σ *by* $\Sigma := \mathbb{L}^{\perp}/\mathbb{L}$ *with* $\nabla^{\Sigma}_X[Y] := [\nabla^g_X Y]$,
- *a screen distribution* $\mathbb{S}$ *by a split of* $0 \to \mathbb{L} \to \mathbb{L}^{\perp} \to \Sigma \to 0$,
  ($\mathbb{S}$ is not unique but depends on the split.).

Remark

If $\mathbb{L}$ *is spanned by a global* $V \in \mathfrak{X}(M)$

$\Rightarrow$ *screen vector field* $Z \in \mathfrak{X}(M)$ *with* $g(Z,Z) = 0, g(Z,V) = 1$

$\leftrightarrow$ $\mathbb{S} := V^{\perp} \cap Z^{\perp}$

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Introduction: pp-waves I

("plane fronted with parallel rays")

Definition

A Lorentzian manifold (M, g) is called a pp-wave iff it admits a parallel null vector field V and $R^g(\mathbb{L}^\perp, \mathbb{L}^\perp) = 0$, where $\mathbb{L} = \mathbb{R}V$.

Remark

Recall $\mathrm{Hol}(M, g) \subset \mathrm{Stab}_{\mathrm{O}(1,n+1)}(\mathbb{L}) \cong (\mathbb{R}^* \times \mathrm{O}(n)) \ltimes \mathbb{R}^n$.

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Then $\mathrm{pr}_{\mathrm{SO}(n)} \mathrm{Hol}(M, g) = \mathrm{Hol}(\Sigma, \nabla^\Sigma)$.

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$\Leftrightarrow$ $\nabla^g V = 0$ *and* $R(X, Y)\Gamma(\mathbb{L}^\perp) \longrightarrow \Gamma(\mathbb{L})$ *for all* $X, Y \in \mathfrak{X}(M)$*,*

$\Leftrightarrow$ $\nabla^g V = 0$ *and exist local* $s_i \in \Gamma(\mathbb{L}^\perp)$ *with* $g(s_i, s_j) = \delta_{ij}$ *and* $\nabla^g s_i = \alpha_i \otimes V$*,* α_i *local one-forms with* $d\alpha_i|_{\mathbb{L}^\perp \wedge \mathbb{L}^\perp} = 0$*,*

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$\Leftrightarrow$ $\mathrm{Hol}^0(M, g) \subset \mathbb{R}^n$*,*

$\Leftrightarrow$ $\mathrm{Hol}(M, g) \subset \Gamma \ltimes \mathbb{R}^n$ *for discrete* $\Gamma \subset \mathrm{O}(n)$*.*

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Introduction: pp-waves II

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Main Results

Theorem (Leistner/S. [LS13])

Let (M, g) be an $(n+2)$-dimensional compact pp-wave. Then:

(i) *The universal cover is globally isometric to $\mathbb{R}^{n+2}$ equipped with $\Theta = 2dudv + 2H du^2 + \delta_{ij} dx^i dx^j$.*

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Problem (Ehlers/Kundt [EK62])

"Prove the plane waves to be the only g-complete pp-waves, no matter which topology one chooses."

((M, g) is a plane-wave iff it is a pp-wave and $\nabla^g_X R^g = 0 \; \forall X \in \Gamma(\mathbb{L}^\perp)$)

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Outline of the Proofs: Theorem (i)

Step 1:

An $(n+2)$-dimensional compact pp-wave (M, g) is covered by $\mathbb{R}^{n+2}$.

Proof.

- fix a screen distribution $\mathbb{S} := V^\perp \cap Z^\perp$
- Riemannian metric: $h|_{\mathbb{S}\times\mathbb{S}} := g$, $h(Z,\cdot) := g(V,\cdot)$ and $h(V,\cdot) := g(Z,\cdot)$
- let $\widetilde{M}$ denote the universal covering
- $\sigma := g(V,\cdot)$ is closed $\Longrightarrow \widetilde{M} \simeq \mathbb{R} \times \widetilde{L}$, $\widetilde{L}$ a leaf of $\widetilde{\mathbb{L}}^\perp$
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Step 2:

The universal covering metric of a compact pp-wave is isometric to $\Theta = 2dudv + 2H du^2 + \delta_{ij} dx^i dx^j$

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- fix a screen distribution $\mathbb{S} := V^\perp \cap Z^\perp$
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Outline of the Proofs: Theorem (ii)

To show:

Any compact pp-wave is complete.

Proof.

- fix a screen distribution $\mathbb{S} := V^{\perp} \cap Z^{\perp}$
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- $(\mathbb{R}^{n+2}, \Theta)$ is complete if all $\frac{\partial^2 H}{\partial x_i \partial x_j}$ are bounded [CFS03, LS13]
- define $Q := R^g(\cdot, Z, Z, \cdot) \in \Gamma(\bigodot^2 T^*M)$
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- $g(Q, Q) = \sum_{i,j=1}^n R^g(s_i, Z, Z, s_j)^2$ is bounded on M
- hence so is $(\Phi^* g)(\Phi^* Q, \Phi^* Q) = \sum_{i,j=1}^n (\partial_i \partial_j H)^2$ on $\mathbb{R}^{n+2}$

□

Outline of the Proofs: Theorem (ii)

To show:

Any compact pp-wave is complete.

Proof.

- fix a screen distribution $\mathbb{S} := V^\perp \cap Z^\perp$
- let $\Phi : (\mathbb{R}^{n+2}, \Theta) \to (M, g)$ as before
- $(\mathbb{R}^{n+2}, \Theta)$ is complete if all $\frac{\partial^2 H}{\partial x_i \partial x_j}$ are bounded [CFS03, LS13]
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Outline of the Proofs: Corollary

To show:

Every compact Ricci-flat pp-wave is a plane wave.

Proof.

- recall: $F_{ij} := \partial_i(\partial_j(H))$ are bounded
- $\mathrm{Ric}^\Theta = \Delta(H)du^2$ for $\Delta(H) = -\sum_{i,j=1}^n \partial_i^2 \Longrightarrow \Delta H = 0$
- $\Delta H = 0 \Longrightarrow \Delta F_{ij} = 0 \Longrightarrow F_{ij}$ are x_i-independent
- $H(u,x) = \sum_{i,j=1}^n a_{ij}(u)x_ix_j + \sum_{i=1}^n b_i(u)x_i + c(u)$

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Thank you for your attention!

Appendix: Open Questions

Problem

Generalize results to the case where $\nabla^g V = \omega \otimes V$.

Example

Completeness cannot be achieved in general. For instance, equip $\mathbb{R}^{n+2}$ with the metric

$$g_{(u,v,x_1,\ldots,x_n)} := 2dudv - 2H(v,x_1,\ldots,x_n)du^2 + \delta_{ij}dx^i dx^j$$

for

$$H(v,x_1,\ldots,x_n) := \sin(v) - \sum_{i=1}^{n} a_i(\cos(x_i) - 1),$$

$a_i \in \mathbb{R}$. This metric decends to the torus $\mathbb{T}^{n+2} = \mathbb{R}^{n+2}/2\pi\mathbb{Z}^{n+2}$ and so does the inextensible geodesic $\gamma(t) := (\ln(t), 0, \ldots, 0)$.

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