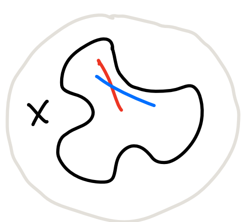
Geometry of Debarre-Voisin varieties

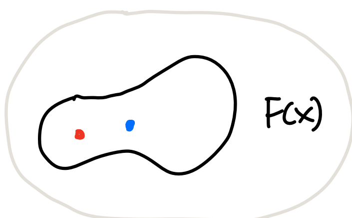
(Joint work with Vladimiro Benedetti)

I] Motivation example

- Let $X \subset \mathbb{P}^5$ be a cubic 4-fold defined by $f \in \mathrm{Sym}^3 V_6^\vee$

X is Fano
$\Rightarrow$ rationally connected
"covered by $\mathbb{P}^1$"





- $F := F(X) = \{ [V_2] \in Gr(2, V_6) \mid \mathbb{P}(V_2) \subset X \} \subset Gr(2, V_6)$

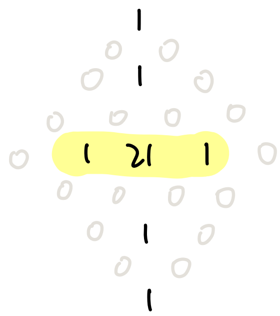
  is the Fano variety of lines of X

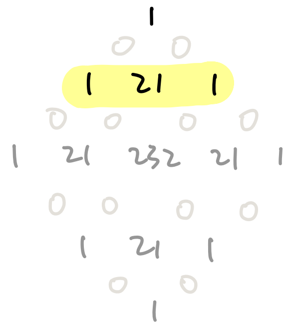
<u>Thm</u> (Beauville-Donagi)

(1) F is hyperkähler of $K3^{[2]}$-type

- $H^1(F, \mathcal{O}_F) = 0$
- $H^0(F, \Omega_F^2) = \mathbb{C}\omega$
- deformation equivalent to $S^{[2]}$ for S K3

Hodge numbers of X and F





- $(H^4(X,\mathbb{Z}), \cdot)$ is a unimodular lattice
- $\mathbb{Z}h^2 \oplus H^4(X,\mathbb{Z})_{van} \hookrightarrow H^4(X,\mathbb{Z})$

- $H^2(F,\mathbb{Z})$ is equipped with q_F, the Beauville–Bogomolov–Fujiki form
- $(H^2(F,\mathbb{Z}), q_X)$ is of disc. 2
- $\mathbb{Z}H \oplus H^2(F,\mathbb{Z})_{prim} \hookrightarrow H^2(F,\mathbb{Z})$

<u>Q</u>: Can we relate the two HS (Hodge structures)? Over $\mathbb{Q}, \mathbb{Z}$?

"point-line correspondence"

$$I = \{(x,\ell) \mid x \in \ell \subset X\}$$

$$\begin{tikzcd} & I \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & F \end{tikzcd}$$

$\mathbb{P}^1$-bundle

$$(x,\ell) \mapsto x, \quad (x,\ell) \mapsto \ell$$

<u>Thm</u> (Beauville–Donagi)

② $q_* p^* : H^4(X,\mathbb{Z})_{van} \xrightarrow{\sim} H^2(F,\mathbb{Z})_{prim}(-1)$

is a Hodge isometry

<u>proof</u>: using the description of the cohomology ring of a $\mathbb{P}^1$-bundle

This can be used to deduce the IHC of X.

<u>thm</u> (Mongardi–Ottem)

For F a projective hyperkähler variety of $K3^{[n]}$-type

IHC (integral Hodge conjecture) holds for $H^{4n-2}(F) \simeq H_2(F)$

(i.e., 1-dim. Hodge classes are algebraic)

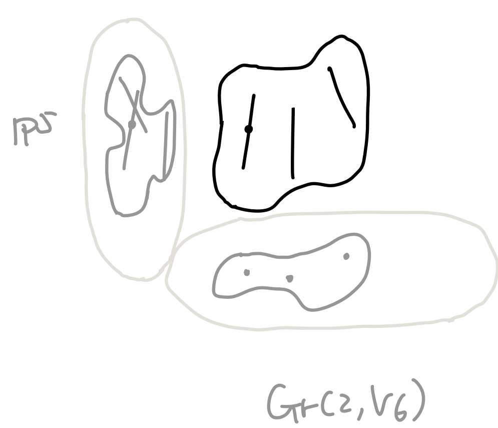
<u>Cor</u> IHC holds for $H^4(X,\mathbb{Z})$ (hence for $H^*(X,\mathbb{Z})$)

<u>Proof</u> $p_* q^* : H^6(F,\mathbb{Z}) \longrightarrow H^4(X,\mathbb{Z})$

$$\begin{tikzcd} & I \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & F \end{tikzcd}$$

BD's thm $\Rightarrow$

$$\mathbb{Z} h^2 + p_* q^* H^6(F,\mathbb{Z}) = H^4(X,\mathbb{Z})$$



II | Debarre-Voisin varieties

Starting from an element $\sigma \in \Lambda^3 V_{10}^\vee$

- $[V_1] \in \mathbb{P}(V_{10})$, $\sigma(V_1, -, -)$ is a skew-symmetric 2-form on V_{10}

  generally it has rank 8

  $X_1 = \{[V_1] \mid \operatorname{rank} \sigma(V_1, -, -) \leq 6\} \subset \mathbb{P}(V_{10})$ Peskine variety expected dim. 6

- $X_3 = \{[V_3] \mid \sigma|_{V_3} = 0\} \subset Gr(3, V_{10})$ a hyperplane section $\Rightarrow$ dim 20

- $X_6 = \{[V_6] \mid \sigma|_{V_6} = 0\} \subset Gr(6, V_{10})$ expected dim. 4

<u>Thm</u> (Debarre-Voisin)

X_6 is hyperkähler of $K3^{[2]}$-type $\Rightarrow$ Hodge diamond of X_6 is the same as F

1
0 0
1 21 1
... ...

Hodge numbers of X_3 and X_1

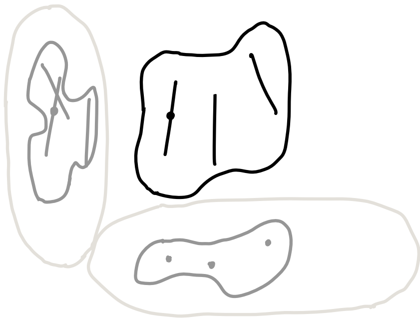
1
1
2
3
4
5
7
8
9
10
H^{20} 1 30 1
10
⋮

1
0 0
0 1 0
0 0 0 0
H^4 0 1 22 1 0
0 0 0 0 0 0
H^6 0 0 1 22 1 0 0
0 0 0 0 0 0
H^8 0 1 22 1 0
0 0 0 0
0 1 0
0 0
1

- (Lefschetz) 1, 1, 2, 3, ⋯ 7, 8, 9, 10 are generated by pullbacks of Schubert classes
- $(H^{20}(X_3, \mathbb{Z}), \cdot)$ unimodular
- $(H^6(X_1, \mathbb{Z}), \cdot)$ unimodular
- $\exists\ h^3, \pi \in H^6(X_1, \mathbb{Z})$

- $\langle 10 \text{ Schubert classes}\rangle \oplus H^{20}(X_3,\mathbb{Z})_{van} \hookrightarrow H^{20}(X_3,\mathbb{Z})$ $\qquad$ $\langle h^3, \Pi\rangle \oplus H^6(X_1,\mathbb{Z})_{van} \hookrightarrow H^6(X_1,\mathbb{Z})$

Incidence correspondences



$$\begin{tikzcd} & I_{3,6} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X_3 & & X_6 \end{tikzcd} \qquad \begin{tikzcd} & I_{1,6} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X_1 & & X_6 \end{tikzcd}$$

<u>Thm</u> (Benedetti - S)

From these correspondences, we get

$$q_* p^* : H^{20}(X_3,\mathbb{Z})_{van} \xrightarrow{\sim} H^2(X_6,\mathbb{Z})_{prim}(-1)$$

and

$$q_* p^* L_h : H^6(X_1,\mathbb{Z})_{van} \xrightarrow{\sim} H^2(X_6,\mathbb{Z})_{prim}(-1) \qquad L_h : \text{Lefschetz}$$

$$\left(\text{also} \quad H^4(X_1,\mathbb{Z})_{van} \xrightarrow[\sim]{L_h} H^6(X_1,\mathbb{Z})_{van} \xrightarrow[\sim]{L_h} H^8(X_1,\mathbb{Z})_{van}\right)$$

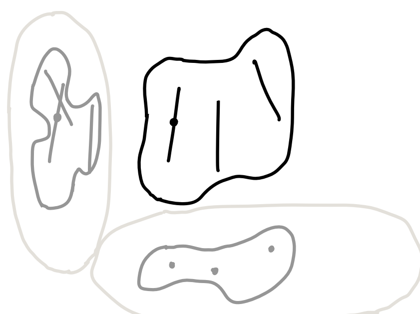
<u>Cor</u> IHC holds for $H^*(X_1,\mathbb{Z})$ and $H^*(X_3,\mathbb{Z})$ (Hence the ncIHC holds for $\mathcal{K}_{X_3}$)

<u>Proof</u>

$$\langle 10 \text{ Schubert classes}\rangle + p_* q^* H^6(X_6,\mathbb{Z}) = H^{20}(X_3,\mathbb{Z})$$

$$\mathbb{Z}h^3 \oplus \mathbb{Z}\Pi + L_h\, p_* q^* H^6(X_6,\mathbb{Z}) = H^6(X_1,\mathbb{Z})$$

- For $H^{22}(X_3,\mathbb{Z})$, the pullback of the Schubert classes only generate an index 3 subgroup
  The extra algebraic class is provided by the class of a $Gr(3,V_6)$
- similar arguments for X_1



Rmk Bernadara-Fatighenti-Manivel also related the HS of X_1 and X_3

via the study of another variety $X_2 \subset Gr(2, V_{10})$

$$\begin{tikzcd}[column sep=small, row sep=small] & X_2 \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\ X_1 \arrow[dr, dash, dashed] & & X_3 \arrow[dl, dash, dashed] \\ & X_6 & \end{tikzcd}$$

Idea of proof of the theorem

① (topological invariant)

for X_6 very general, $H^2(X_6, \mathbb{Q})_{prim}$ is a simple HS
i.e. no nontrivial sub-HS

$\Rightarrow$ over $\mathbb{Q}$ $\quad q_* p^*$ either 0 or isom (of $\mathbb{Q}$ vector space)

② To show the quadratic forms coincide, it suffices to determine the scalar $\Rightarrow$ we specialize to special members

III Divisors in the moduli space of DV varieties

There are 3 incarnations of the moduli space

GIT quotient M $:= \mathbb{P}(\Lambda^3 V_{10}^\vee) /\!/ SL(V_{10})$	coarse moduli space M_{HK}	period domain $\mathcal{P}$

$$M \underset{\sim \atop [\text{O'Grady}]}{\overset{[DV]}{\dashrightarrow}} M_{HK} \xhookrightarrow[\text{Torelli thm}]{\text{period map}} \mathcal{P}$$

$(H^2(X_6, \mathbb{Z}), \cdot) \xrightarrow{\sim} \Lambda$

$\Omega = \{ x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid x \cdot \bar{x} > 0,\ x^2 = 0 \}$

$\Omega / \widetilde{O}^+(\Lambda) =: \mathcal{P}$

- Inside the period domain $\mathcal{P}$ there are natural divisors given as follows

  - $v \in \Lambda$ with $v^2 < 0$ $\Rightarrow$ H_v hyperplane in $\mathbb{P}(\Lambda \otimes \mathbb{C})$

$\overset{\text{induce}}{\Rightarrow}$ divisor D_v in $\mathcal{P}$ (Heegner divisor)

- In $K3^{[2]}$ case, the Heegner divisors can be characterized by their discriminants $D_v = D_{2d}$

- In M, the divisors come from $SL(V_{10})$-invariant hypersurfaces on $\mathbb{P}(\Lambda^3 V_{10}^\vee)$ $\Rightarrow$ this allows us to describe their geometry

<u>Result</u> We consider the following 3 divisors in M coming from $SL(V_{10})$-invariant hypersurfaces

- $D^{3,3,10} := \{[\sigma] \mid \exists V_3 \text{ s.t. } \sigma(V_3, V_3, V_{10}) = 0\}$
- $D^{1,6,10} := \{[\sigma] \mid \exists V_1 < V_6 \text{ s.t. } \sigma(V_1, V_6, V_{10}) = 0\}$
- $D^{4,7,7} := \{[\sigma] \mid \exists V_4 < V_7 \text{ s.t. } \sigma(V_4, V_7, V_7) = 0\}$

Under the period map, they are mapped respectively to D_{22}, D_{24}, and D_{28}

And we can describe the geometry of X_1, X_3, X_6 in these cases.

Precisely
- $D^{4,7,7} \Rightarrow \mathbb{P}((V_7/V_4)^\vee) = \{[V_6] \mid V_4 < V_6 < V_7\} \subset X_6$ is a Lagrangian plane

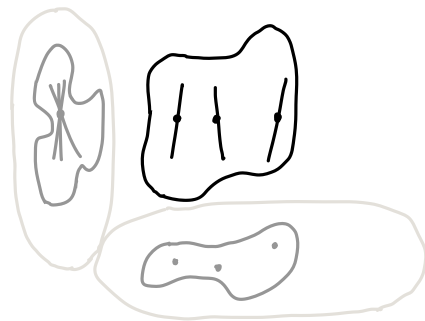
(the inverse holds too)

this case allows us to conclude the proof of the Hodge isometries.

- $D^{3,3,10} \Rightarrow [V_3]$ is a singular point of X_3

  X_6 singular along a K3 surface of degree 22

  (the inverse holds too)



- $D^{1,6,10} \Rightarrow [V_1]$ is a singular point of X_6
  - X_6 contains a uniruled divisor D
  - D admits a conic fibration to

    a K3 surface S of degree 6

  $\rightsquigarrow$ S is equipped with a Brauer twist β

THANK YOU !