TOPOLOGICAL CONSTRAINTS OF HYPERKÄHLER MANIFOLDS

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Abstract. Hyperkähler manifolds are important among compact Kähler manifolds with a trivial canonical bundle, but very few examples are known. We study the topological constraints for these manifolds. I will present two results: a description of their cobordism classes in terms of known examples (joint with Georg Oberdieck and Claire Voisin), and a conditional bound on the second Betti number (joint with Thorsten Beckmann). I will also talk about some conjectural properties on the positiveness of certain numerical invariants called generalized Fujiki constants, and explain the consequences.

1. Introduction

Definition 1.1. A simply connected compact Kähler manifold X is called a *hyperkähler manifold* if the vector space $H^{2,0}(X) \coloneqq H^0(X, \Omega_X^2)$ is generated by a symplectic holomorphic 2-form σ (such manifolds are also known as *irreducible symplectic varieties*).

The existence of a symplectic form gives immediately the following.

Proposition 1.2. *Let X be a hyperkähler manifold.*

- *The dimension of X is even. We will write* $\dim(X) = 2n$ *throughout these notes.*
- *The symplectic form σ induces an isomorphism $\sigma\colon \mathcal{T}_X \xrightarrow{\sim} \Omega_X$. In particular, all odd Chern classes and Chern characters vanish:*

$$\forall\, k \in \mathbf{Z} \quad c_{2k+1}(\mathcal{T}_X) = 0, \quad \mathrm{ch}_{2k+1}(\mathcal{T}_X) = 0.$$

Compact hyperkähler manifolds are extremely important in the study of manifolds with trivial canonical bundle. Notably, by the Beauville–Bogomolov decomposition theorem, they are one of the three building blocks.

Theorem 1.3 (Beauville–Bogomolov)**.** *Let X be a compact Kähler manifold with trivial canonical bundle. Then there exists a finite étale cover*

$$T \times \prod_i Y_i \times \prod_j K_j \longrightarrow X,$$

where T is a complex torus, Y_i are strict Calabi–Yau manifolds,[1] *and K_j are hyperkähler manifolds.*

Compact hyperkähler manifolds are also mysterious in that only very few examples are known, in sharp contrast to strict Calabi–Yau manifolds. We list all the known examples below.

Example 1.4.

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[1]A strict Calabi–Yau manifold is a simply connected Kähler manifold Y with trivial canonical bundle such that $H^{k,0}(Y) = 0$ for all $k \notin \{0, \dim(Y)\}$.

- In dimension 2, these are precisely K3 surfaces. They all have the same topological type, and the second Betti number b_2 is equal to 22.
- $\mathrm{K3}^{[n]}$ for $n \geq 2$: for a K3 surface S, one can consider the Hilbert scheme of points $S^{[n]}$, which is a hyperkähler manifold of dimension $2n$. More generally, their deformations are also hyperkähler. They have $b_2 = 23$.
- Kum_n for $n \geq 2$: similarly, for an Abelian surface A, one can consider the Hilbert scheme of points; but to produce a simply connected manifold, one use the sum map $\Sigma\colon A^{[n+1]} \to A$ and take the preimage of a point $\mathrm{Kum}(A) \coloneqq \Sigma^{-1}(0)$. When $n = 1$ this gives the Kummer surface of A, which is why the higher dimensional analogues and their deformations are called generalized Kummer varieties. They have $b_2 = 7$.
- Two sporadic examples discovered by O'Grady, using desingularizations of moduli spaces of sheaves: one example OG_6 of dimension 6 and $b_2 = 8$, and another OG_{10} of dimension 10 and $b_2 = 24$.

The talk will mainly revolve around the following conjectural properties of a hyperkähler manifold.

Conjecture 1.5. *Let X be a compact hyperkähler manifold of dimension $2n$. Then*

- $\int_X c_\lambda > 0$ *for all even partitions λ of $2n$, that is, partitions that only contain even integers. For example, when $n = 3$, this means that the integrals of c_6, c_4c_2, and c_2^3 are positive.*
- *Similarly,* $(-1)^n \int_X \mathrm{ch}_\lambda > 0$ *for all even partitions λ of $2n$.*

We will provide some evidence for this conjecture, as well as some consequences and applications of it. We will also introduce a generalized version of the conjecture later.

2. Cobordism classes

Consider Ω^* the complex cobordism ring, for which we omit the definition.[2] The cobordism class of a complex manifold is denoted by $[X]$. The ring structure on Ω^* is given by

$$[X] + [Y] = [X \sqcup Y], \quad [X] \cdot [Y] = [X \times Y].$$

We have the following nice description.

Theorem 2.1 (Milnor, Novikov, Thom)**.**

(1) The cobordism class of a complex manifold X of dimension m is uniquely determined by its Chern numbers $\left\{\int_X c_\lambda\right\}_{\lambda \vdash m}$, or equivalently, by the Chern character numbers $\left\{\int_X \mathrm{ch}_\lambda\right\}_{\lambda \vdash m}$.

(2) Consider a sequence $(X_k)_{k \in \mathbf{Z}_{>0}}$ of manifolds such that $\dim(X_k) = k$ and $\int_{X_k} \mathrm{ch}_k \neq 0$. Then the complex cobordism ring with rational coefficients $\Omega^*_{\mathbf{Q}} \coloneqq \Omega^* \otimes \mathbf{Q}$ is isomorphic to the polynomial ring $\mathbf{Q}[x_1, x_2, \dots]$ by sending $[X_k]$ to x_k. Note that since $\int_{\mathbf{P}^n} \mathrm{ch}_n = \frac{n+1}{n!}$, such a sequence indeed exists.

[2] To define complex cobordism, one needs to leave the category of complex manifolds and consider *stably almost complex manifolds* instead, which are pairs (X, α) consisting of X, a differential manifold, and α, an almost complex structure on $\mathcal{T}_X \oplus \underline{\mathbf{R}}^k$, the direct sum of the tangent bundle with some trivial real vector bundle of rank k. This is necessary because the boundary of a manifold with even real dimension is of odd real dimension, so one needs the extra component to define an almost complex structure.

We briefly explain the proof of (2) using (1) (see the book *Characteristic Classes* of Milnor–Stasheff, Theorem 16.7, where this result is attributed to Thom). Denote by $\ell(\lambda)$ the length of a partition λ. A partition λ is said to be a refinement of another partition μ, if we can regroup some subsets of λ to get μ. For example, $(1,1,1)$ is a refinement of $(2,1)$. We state the following lemma.

Lemma 2.2. *Let $\lambda, \mu \vdash m$ be two partitions. If λ is not a refinement of μ, then for all manifolds $X_1, \ldots, X_{\ell(\mu)}$ with $\dim(X_i) = \mu_i$, we have*

$$\int_{X_\mu} \mathrm{ch}_\lambda = 0,$$

where we write X_μ for the product manifold $\prod_i X_i$ and ch_λ for the product of Chern characters $\prod_j \mathrm{ch}_{\lambda_j}(X_\mu)$.

Proof. We first prove the simple case where $\lambda = (m)$ and $\mu = (m_1, m_2)$ with $m > m_1 \geq m_2 > 0$ and $m_1 + m_2 = m$ to illustrate the idea. Consider two manifolds X_1 and X_2 with $\dim(X_i) = m_i$. Due to the additivity of the Chern character, we have

$$\begin{aligned} \mathrm{ch}_\lambda(X_\mu) = \mathrm{ch}_m(X_1 \times X_2) &= \mathrm{ch}_m(\mathcal{T}_{X_1} \boxplus \mathcal{T}_{X_2}) \\ &= \mathrm{ch}_m(X_1) \boxplus \mathrm{ch}_m(X_2). \end{aligned}$$

Here, for a product $X_1 \times X_2$, the symbol $\boxplus$ for bundles or classes on the two components means that we pull back them via the projection maps and take the sum over the product space. Since each X_i has dimension $m_i < m$, the two m-th Chern characters both vanish so $\mathrm{ch}_\lambda(X_\mu) = 0$.

In the general case, if we denote $\ell(\mu)$ by l, each term that appears in the product

$$\begin{aligned} \mathrm{ch}_\lambda(X_\mu) &= \mathrm{ch}_{\lambda_1}(X_\mu) \cdots \mathrm{ch}_{\lambda_{\ell(\lambda)}}(X_\mu) \\ &= \big(\mathrm{ch}_{\lambda_1}(X_1) \boxplus \cdots \boxplus \mathrm{ch}_{\lambda_1}(X_l)\big) \cdots \big(\mathrm{ch}_{\lambda_{\ell(\lambda)}}(X_1) \boxplus \cdots \boxplus \mathrm{ch}_{\lambda_{\ell(\lambda)}}(X_l)\big). \end{aligned}$$

has the form

$$\mathrm{ch}_{\lambda^1}(X_1) \boxtimes \mathrm{ch}_{\lambda^2}(X_2) \boxtimes \cdots \boxtimes \mathrm{ch}_{\lambda^l}(X_l),$$

where $(\lambda^i)_{1 \leq i \leq l}$ are disjoint subsets of λ viewed as partitions. Similarly, the symbol $\boxtimes$ here means pulling back via the projection maps then taking the product. For each i, if $|\lambda^i| > \dim(X_i)$ then the Chern character $\mathrm{ch}_{\lambda^i}(X_i)$ vanishes. Since $|\lambda| = |\mu| = m$, for this term to have a non-zero integral, we must have $|\lambda^i| = \dim(X_i)$ for all $1 \leq i \leq l$, which means that λ would be a refinement of μ. Since this is not the case by assumption, we may conclude that every term in the product ch_λ has a vanishing integral. □

For a given integer m, we can sort the partitions of m in reverse lexicographic order: for example when $m = 3$, we have $(3), (2,1), (1,1,1)$. For two partitions λ, μ, if λ is to the left of μ then λ is not a refinement of μ, so Lemma 2.2 applies.

Proof of Theorem 2.1(2). The goal is to show that, in each dimension m, the products $X_\mu := \prod_i X_{\mu_i}$ for all partitions $\mu \vdash m$ form an additive basis. In other words, for a given manifold Y of dimension m, we need to find rational coefficients $(a_\mu)_{\mu \vdash m}$ such that $\sum_\mu a_\mu [X_\mu] = [Y]$. Thanks to (1), this is equivalent to the linear equations

$$\forall\, \lambda \vdash m \quad \sum_\mu a_\mu \cdot \big(\textstyle\int_{X_\mu} \mathrm{ch}_\lambda\big) = \textstyle\int_Y \mathrm{ch}_\lambda,$$

or in terms of row vectors,

$$(a_\mu)_{\mu\vdash m} \cdot \left(\int_{X_\mu} \mathrm{ch}_\lambda\right)_{\mu\vdash m, \lambda\vdash m} = \left(\int_Y \mathrm{ch}_\lambda\right)_{\lambda\vdash m}.$$

If we sort the partitions in reverse lexicographic order, the coefficient matrix is an upper triangular matrix by the above argument. Moreover, the assumption $\int_{X_k} \mathrm{ch}_k \neq 0$ guarantees that $\int_{X_\lambda} \mathrm{ch}_\lambda \neq 0$, hence the entries on the diagonal are non-zero and the matrix is invertible, which proves the statement. □

For us, the interest here is to study the subring of $\Omega^*_{\mathbf{Q}}$ generated by elements with vanishing odd Chern numbers/Chern character numbers: we define

$$\Omega^*_{\mathbf{Q},\mathrm{even}} := \left\langle [X] \;\middle|\; \int_X c_\lambda = 0 \text{ for all } \lambda \vdash \dim X \text{ containing an odd integer} \right\rangle,$$

which, by Proposition 1.2, contains the cobordism classes of all hyperkähler manifolds.

By repeating the same argument as above, we deduce the following result.

Proposition 2.3. *Consider a sequence $(X_k)_{k\in\mathbf{Z}_{>0}}$ of manifolds with vanishing odd Chern numbers such that $\dim(X_k) = 2k$ and $\int_{X_k} \mathrm{ch}_{2k} \neq 0$. Then the even complex cobordism ring $\Omega^*_{\mathbf{Q},\mathrm{even}}$ is isomorphic to the polynomial ring $\mathbf{Q}[x_1, x_2, \ldots]$ by sending $[X_k]$ to x_k.*

The two known infinite families both satisfy the required property: in fact, we obtained explicit formulae for the integral of the top degree Chern character.

Proposition 2.4 (Oberdieck–S.–Voisin)**.** *For $n \geq 1$, we have*

$$\int_{\mathrm{K3}^{[n]}} \mathrm{ch}_{2n} = (-1)^n \frac{(2n+2)!}{(2n-1)\cdot n!^4},$$

and

$$\int_{\mathrm{Kum}_n} \mathrm{ch}_{2n} = (-1)^n \frac{(2n+2)!}{n!^4}.$$

Consequently, both infinite families can be used as generators for the even complex cobordism ring $\Omega^*_{\mathbf{Q},\mathrm{even}}$.

We remark that neither family can be used to express all hyperkähler manifolds using only *positive* linear combinations.

The proof of these formulae uses the explicit descriptions of the cohomology ring for these two examples in terms of Nakajima operators, and the computation is essentially an analysis of the combinatorial properties of these objects.

We see that both formulae confirm Conjecture 1.5 for the top degree Chern character. For other products of Chern classes/characters, we have also verified them in small dimensions using a computer, although we do not have a closed formula in general (Oberdieck has a conjectural one for $\mathrm{K3}^{[n]}$).

3. b_2 AND c_2

The analysis of cobordism classes in the previous section is very coarse. In this section, we will make better use of properties that are specific to hyperkähler manifolds.

One of the most important objects in the study of hyperkähler manifolds is the second cohomology group $H^2(X, \mathbf{Z})$.

Theorem 3.1 (Beauville–Bogomolov–Fujiki)**.** *Let X be a compact hyperkähler manifold of dimension $2n$. There exist a primitive quadratic form q on $H^2(X,\mathbf{Z})$ and a constant C_X such that*

$$\forall\,\beta \in H^2(X,\mathbf{Z}) \qquad \int_X \beta^{2n} = C_X \cdot q(\beta)^n.$$

The form q is called the BBF form. It is of signature $(3, b_2-3)$. The constant C_X is called the Fujiki constant of X. It is also common to normalize C_X by letting $C_X = (2n-1)!!\cdot c_X$, and we will refer to c_X as the small Fujiki constant.

A lot of information of a hyperkähler manifold is encoded in its second cohomology group. For example, the global Torelli theorem states that one can (almost) recover a hyperkähler manifold just from the polarized Hodge structure on H^2.

The following question naturally arises if we expect some boundedness results for hyperkähler manifolds.

Question 3.2. *In each dimension $2n$, is the second Betti number $b_2(X)$ bounded for all hyperkähler manifolds X of dimension $2n$?*

We have the following affirmative result.

Theorem 3.3 (Guan)**.** *When $n=2$, we have $b_2(X) \leq 23$ for all hyperkähler fourfolds X. The bound is sharp and is attained by* $\mathrm{K3}^{[2]}$.

We will present a conditional bound for b_2. We still need to introduce some extra notions.

Theorem 3.4 (Fujiki)**.** *Let X be a hyperkähler manifold of dimension $2n$, and let α be a characteristic class of degree $(2k,2k)$ (for example, a product of even Chern classes).*[3] *Then there exists a constant $C(\alpha)$ such that*

$$\forall\,\beta \in H^2(X,\mathbf{Z}) \qquad \int_X \alpha\cdot\beta^{2n-2k} = C(\alpha)\cdot q(\beta)^{n-k}.$$

We call it the generalized Fujiki constant *of α.*

- This generalizes the usual Fujiki constant C_X: we have $C(1_X) = C_X$.
- It also generalizes characteristic numbers: for products of Chern classes in top degree, we have $C(c_\lambda) = \int_X c_\lambda$, and similarly for Chern characters.
- We have $C(c_2) > 0$. This positivity is explained by results from differential geometry. Namely, for a Kähler manifold X of dimension m with trivial canonical bundle, one can choose a Ricci flat metric and obtain the following pointwise relation

$$8\pi^2\, c_2\,\omega^{m-2} = \frac{\|R\|^2}{m(m-1)}\,\omega^m,$$

  where ω is the Kähler form and R is the curvature tensor. By taking $\frac{\omega^m}{m!}$ as the volume form and integrating over X, we get

$$\int_X c_2\cdot\omega^{m-2} = \frac{(m-2)!\,\|R\|^2}{8\pi^2}.$$

  Hence for a hyperkähler manifold X, we have $C(c_2) > 0$ using the Fujiki relations. Equivalently, we have $C(\mathrm{ch}_2) = C(-c_2) < 0$.

[3] The result holds more generally for any class α that remains of type $(2k,2k)$ on all small deformations of X. This was proved by Huybrechts.

This motivates us to extend Conjecture 1.5 to generalized Fujiki constants as well. Similar conjectures have been made by Sawon and Cao–Oberdieck–Toda.

Conjecture 3.5. *Let X be a compact hyperkähler manifold of dimension $2n$. Then*

- $C(c_\lambda) > 0$ *for all even partitions λ of $2k \leq 2n$.*
- *Similarly,* $(-1)^k C(\mathrm{ch}_\lambda) > 0$ *for all even partitions λ of $2k \leq 2n$.*

Moreover, we expect that these positivity results should follow from a similar local argument. In other words, there are algebraic identities that provide the pointwise positivity, and the global positivity is obtained by integrating over X.

Again, the conjecture has been verified in small dimensions for the known examples. The first open case is $C(\mathrm{ch}_4)$. When $n = 2$, it is *a posteriori* positive by the bound of Guan.

We now present the bound on b_2, subject to the positivity of $C(\mathrm{ch}_4)$. The same inequality has been independently obtained by Sawon.

Theorem 3.6 (Beckmann–S.)**.** *For a hyperkähler manifold X of dimension $2n$, if $C(\mathrm{ch}_4) > 0$, or equivalently, $C(c_2^2) > 2C(c_4)$, then we have the following inequality*

$$b_2(X) \leq \frac{10}{\dfrac{C(c_2^2)}{C(c_4)} - 2} - (2n - 9). \tag{1}$$

The inequality takes a quite strange form. We will introduce another notion to rewrite it in a more natural form. Consider the Hirzebruch–Riemann–Roch formula. For a line bundle $L \in \mathrm{Pic}(X)$, we have

$$\begin{aligned} \chi(X, L) = \int_X \mathrm{ch}(L)\,\mathrm{td}_X &= \int_X e^L\,\mathrm{td}_X \\ &= \int_X \mathrm{td}_{2n} + \mathrm{td}_{2n-2}\,\frac{L^2}{2} + \mathrm{td}_{2n-4}\,\frac{L^4}{24} + \cdots \\ &= C(\mathrm{td}_{2n}) + C(\mathrm{td}_{2n-2})\frac{q(L)}{2} + C(\mathrm{td}_{2n-4})\frac{q(L)^2}{24} + \cdots \end{aligned}$$

The last equality is obtained using the Fujiki relations. So we get the following polynomial which plays an important role

$$\mathrm{RR}_X(q) := \sum_{k=0}^{n} \frac{C(\mathrm{td}_{2n-2k})}{(2k)!} q^k.$$

We will refer to this polynomial as the (Huybrechts–)Riemann–Roch polynomial of X. Among the known examples, there are only two types of Riemann–Roch polynomials.

- (Ellingsrud–Göttsche–Lehn, Ríos Ortiz) For $\mathrm{K3}^{[n]}$ and OG_{10}, we have $\mathrm{RR}_X(q) = \binom{q/2+n+1}{n}$;
- (Nieper–Wißkirchen, Ríos Ortiz) For Kum_n ($n \geq 2$) and OG_6, we have $\mathrm{RR}_X(q) = (n+1)\binom{q/2+n}{n}$.

In terms of the Riemann–Roch polynomial, the bound on b_2 has an alternative form.

Theorem 3.6′. *Let X be a hyperkähler manifold of dimension $2n$ for $n \geq 2$. If the Riemann–Roch polynomial RR_X factorizes as a product of linear factors (and not all identical),*[4] *then $C(\mathrm{ch}_4) > 0$, and the inequality (1) becomes*

$$b_2(X) \leq \frac{n-1}{\dfrac{n(\sum \lambda_i^2)}{(\sum \lambda_i)^2} - 1} - (2n-2),$$

where λ_i are the roots of RR_X.

Here we see that the bound measures the dispersion of the roots: it gets smaller as the roots get more dispersed.

Using this description, we can examine the bound for the two known types of Riemann–Roch polynomials.

- For $\mathrm{RR}_{\mathrm{K3}^{[n]}}$, the bound is $b_2 \leq n + 17 + \frac{12}{n+1}$;
- For $\mathrm{RR}_{\mathrm{Kum}_n}$, the bound is $b_2 \leq n + 5$.

An interesting remark on the inequality is that it holds also for 4-dimensional singular irreducible symplectic varieties, since all the required ingredients are available for such varieties. There are many more examples in the singular case, and a lot of them actually attain the bound for b_2. This again suggests that the generalized Fujiki constants for characteristic classes and consequently the Riemann–Roch polynomial RR_X are largely governed by properties that are of local nature, and motivates the following conjecture by Jiang.

Conjecture 3.7. *Let X be a hyperkähler variety of dimension $2n$ (possibly singular).*

(1) The Riemann–Roch polynomial RR_X factorizes as a product of linear factors, and the roots form an arithmetic progression;
(2) When X is smooth, the difference between two roots is equal to 2.

The first point is equivalent to some extra relations on the coefficients of RR_X, which we recall are given by generalized Fujiki constants $\frac{C(\mathrm{td}_{2n-2k})}{(2k)!}$. We expect such relations to follow from a local argument, so they should also hold for singular examples. One evidence for this is the result of Hitchin–Sawon and Nieper–Wißkirchen: they showed that if one takes the square root of the Todd class instead, the polynomial

$$\mathrm{RR}_{X,1/2}(q) \coloneqq \sum_{k=0}^{n} \frac{C(\mathrm{td}_{2n-2k}^{1/2})}{(2k)!} q^k$$

always factorizes as an n-th power, and the argument is purely local using Rozansky–Witten theory.

On the other hand, the second point relies on the smoothness of X and already fails for known singular examples. We think this is related to the global geometry of X, and in particular to Lagrangian fibrations (assuming their existence).

We also remark that, as an immediate consequence of the above result of Nieper-Wißkirchen, all the values $C(\mathrm{td}_{2k}^{1/2})$ are positive for $0 \leq k \leq n$. Moreover, Jiang has recently proved the

[4]In fact one only needs the following weaker assumption: write $\mathrm{RR}_X(q) = A_0 q^n + A_1 q^{n-1} + A_2 q^{n-2} + \cdots$, then $C(\mathrm{ch}_4) > 0$ if and only if $2nA_0A_2 < (n-1)A_1^2$. In this case, the bound becomes $b_2(X) \leq \big(1 - \frac{2nA_0A_2}{(n-1)A_1^2}\big)^{-1} - (2n-2)$.

positivity for all the coefficients of the Riemann–Roch polynomial, in other words, $C(\mathrm{td}_{2k}) > 0$ for $0 \le k \le n$. For a fixed $\alpha > 0$, using the description

$$\mathrm{td}_X^\alpha = \exp\left(-2\alpha \sum_{k=1}^{\infty} b_{2k}\,(2k)!\,\mathrm{ch}_{2k}\right)$$

and taking into account the signs of the modified Bernoulli numbers b_{2k}, one sees that each term td_{2k}^α is a linear combination of products of Chern characters where all coefficients are of sign $(-1)^k$. This provides further evidence for the conjecture on the positivity of the generalized Fujiki constants $(-1)^k C(\mathrm{ch}_\lambda)$.

Finally, we explain the proof of the conditional bound.

Idea of proof for the bound. Consider the second Chern class $c_2 \in H^4(X, \mathbf{Z})$. Inside $H^4(X, \mathbf{Z})$ we have the image of

$$\smile\colon H^2(X, \mathbf{Z}) \times H^2(X, \mathbf{Z}) \longrightarrow H^4(X, \mathbf{Z}).$$

The cup product is in fact injective, so we have $\mathrm{SH}^2(X) \coloneqq \mathrm{Sym}^2 H^2(X, \mathbf{Z})$ sitting inside $H^4(X, \mathbf{Z})$. This is part of a more general result by Verbitsky, who studied the subalgebra of $H^*(X, \mathbf{Q})$ generated by $H^2(X, \mathbf{Q})$, which is now known as the *Verbitsky component*.

A natural question is whether c_2 lies in $\mathrm{SH}^2(X)$ or not. Hence we can project c_2 to $\mathrm{SH}^2(X)$

$$c_2 = \overline{c_2} + z,$$

and study the difference z, which is a primitive $(2, 2)$-class. By the Hodge–Riemann bilinear relations, we get

$$\int_X z^2 \omega^{2n-4} \ge 0,$$

where equality holds if and only if $z = 0$.

If we now look at c_2^2, we have

$$c_2^2 = \overline{c_2}^2 + 2\overline{c_2} z + z^2,$$

and by considering generalized Fujiki constant, we get

$$C(c_2^2) = C(\overline{c_2}^2) + C(z^2) \ge C(\overline{c_2}^2).$$

This gives the main inequality. By computing the values of the generalized Fujiki constants, we get the desired statement involving $C(\mathrm{ch}_4)$ and b_2. ☐

In other words, the bound (1) is essentially given by a triangle inequality involving c_2. This also gives us the following corollaries on the second Chern class.

Corollary 3.8. *Let X be a hyperkähler manifold of dimension $2n$ with $n \ge 2$. Then $c_2 \in \mathrm{SH}^2(X)$ if and only if $C(\mathrm{ch}_4) > 0$ and equality holds in* (1).

Corollary 3.9. *Among known smooth hyperkähler manifolds, we have $c_2 \in \mathrm{SH}^4(X)$ if and only if X is one of the following*

$$\mathrm{K3}\ \textit{(trivial)},\quad \mathrm{K3}^{[2]},\quad \mathrm{K3}^{[3]},\quad \mathrm{Kum}_2,\quad \mathrm{OG}_6,\quad \mathrm{OG}_{10}.$$