Topological constraints of hyperkähler manifolds

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Introduction

A *hyperkähler manifold* is a simply connected compact Kähler manifold X such that
$$H^{2,0}(X) = H^0(X, \Omega_X^2) = \mathbf{C}\sigma$$
is generated by a nowhere vanishing holomorphic 2-form σ (a *holomorphic symplectic form*).

A few immediate consequences

- X must be of even (complex) dimension $2n$ since it is symplectic.
- The symplectic form σ induces an isomorphism $\sigma\colon \mathcal{T}_X \xrightarrow{\sim} \Omega_X$. In particular, all the odd Chern classes and Chern characters vanish:
$$\forall\, k \in \mathbf{Z} \quad c_{2k+1}(\mathcal{T}_X) = 0, \quad \mathrm{ch}_{2k+1}(\mathcal{T}_X) = 0.$$
- The canonical bundle $\omega_X = \Omega_X^{2n}$ is trivial: a trivialization is given by $\sigma^n := \sigma \wedge \cdots \wedge \sigma$.

Some examples

- In dimension 2: K3 surface (*e.g.*, a quartic surface). They all have the same topological type, and $b_2 = 22$.
- $\mathrm{K3}^{[n]}$: Hilbert scheme of "n points" on a K3 surface (and deformation), with $b_2 = 23$.
- Kum_n: generalized Kummer variety (and deformation), with $b_2 = 7$.
- Two examples found by O'Grady: one of dimension 6 with $b_2 = 8$, and one of dimension 10 with $b_2 = 24$.

This in fact gives a complete list of *all* known examples.

It is interesting to study *a priori* constraints that a hyperkähler manifold should satisfy.

A key conjecture

We call a partition *even* if it only contains even integers, and *odd* otherwise. For a hyperkähler manifold X, any Chern number/Chern character number given by an odd partition automatically vanishes.

Conjecture (Ellingsrud–Göttsche–Lehn, Nieper-Wißkirchen, ...)

Let X be a compact hyperkähler manifold of dimension $2n$. Then

- $\int_X c_\lambda > 0$ *for all even partitions λ of $2n$.*
- *Similarly,* $(-1)^n \int_X \mathrm{ch}_\lambda > 0$ *for all even partitions λ of $2n$.*

For example, when $n = 3$, this means that the integrals of c_6, c_4c_2, and c_2^3 are all positive, while the integrals of $\mathrm{ch}_6, \mathrm{ch}_4 \mathrm{ch}_2$, and ch_2^3 are all negative.

Cobordism classes

Review: complex cobordism ring

We write Ω^* for the *complex cobordism ring*.

For a complex manifold X of dimension n, its cobordism class is denoted by $[X]$. The ring structure on Ω^* satisfies the following

$$[X] + [Y] = [X \sqcup Y], \quad [X] \times [Y] = [X \times Y].$$

Theorem (Milnor, Novikov, Thom)

(1) *The cobordism class of a complex manifold X of dimension m is uniquely determined by its Chern numbers $\left\{\int_X c_\lambda\right\}_{\lambda\vdash m}$, or equivalently, by the Chern character numbers $\left\{\int_X \mathrm{ch}_\lambda\right\}_{\lambda\vdash m}$.*

(2) *Consider a sequence $(X_k)_{k\in\mathbf{Z}_{>0}}$ of manifolds such that*

$$\dim(X_k) = k \quad \textit{and} \quad \int_{X_k} \mathrm{ch}_k \neq 0.$$

Then the complex cobordism ring with rational coefficients $\Omega^*_{\mathbf{Q}} := \Omega^* \otimes \mathbf{Q}$ is isomorphic to a polynomial ring with infinitely many variables via

$$\begin{aligned} \mathbf{Q}[x_1, x_2, \ldots] &\xrightarrow{\sim} \Omega^*_{\mathbf{Q}}. \\ x_k &\longmapsto [X_k] \end{aligned}$$

Note that since $\int_{\mathbf{P}^n} \mathrm{ch}_n = \frac{n+1}{n!}$, such a sequence indeed exists.

Idea of proof for the isomorphism

We illustrate the idea of the proof of (2) using (1).

The goal is to show that, in each dimension m, the products $X_\mu := \prod_i X_{\mu_i}$ for all partitions $\mu \vdash m$ form an additive basis.

In other words, for any given manifold Y of dimension m, we need to find rational coefficients $(a_\mu)_{\mu\vdash m}$ such that $\sum_\mu a_\mu [X_\mu] = [Y]$.

Thanks to (1), this is equivalent to solving a system of linear equations

$$\forall\, \lambda \vdash m \quad \sum_\mu a_\mu \cdot \left(\int_{X_\mu} \mathrm{ch}_\lambda\right) = \int_Y \mathrm{ch}_\lambda\,.$$

For simplicity, we look at the case of $m = 2$: we obtain the linear equations

$$\begin{cases} a_2 \int_{X_2} \mathrm{ch}_2 + a_{1,1} \int_{X_{1,1}} \mathrm{ch}_2 &= \int_Y \mathrm{ch}_2 \\ a_2 \int_{X_2} \mathrm{ch}_{1,1} + a_{1,1} \int_{X_{1,1}} \mathrm{ch}_{1,1} &= \int_Y \mathrm{ch}_{1,1} \end{cases}$$

where $X_{1,1}$ stands for $X_1 \times X_1$ and $\mathrm{ch}_{1,1}$ for $\mathrm{ch}_1 \times \mathrm{ch}_1$.

Due to the additivity of the Chern character, we have

$$\begin{aligned} \mathrm{ch}_2(X_1 \times X_1) &= \mathrm{ch}_2(\mathcal{T}_{X_1} \boxplus \mathcal{T}_{X_1}) \\ &= \mathrm{ch}_2(X_1) \boxplus \mathrm{ch}_2(X_1) = 0. \end{aligned}$$

On the other hand, $\int_{X_2} \mathrm{ch}_2$ and $\int_{X_{1,1}} \mathrm{ch}_{1,1}$ are both non-zero by the assumption that $\int_{X_k} \mathrm{ch}_k \neq 0$. So the coefficient matrix is a lower triangular matrix with non-zero entries on the diagonal, and is therefore invertible. We may then invert it to solve a_2 and $a_{1,1}$.

In the general case, we have the following lemma which generalizes the vanishing argument that we saw when $m = 2$.

Lemma

Let $\lambda, \mu \vdash m$ be two partitions. If λ is not a refinement of μ, then for all manifolds $X_1, \ldots, X_{\ell(\mu)}$ with $\dim(X_i) = \mu_i$, we have

$$\int_{X_\mu} \mathrm{ch}_\lambda = 0,$$

where we write X_μ for the product manifold $\prod_i X_i$ and ch_λ for the product of Chern characters $\prod_j \mathrm{ch}_{\lambda_j}(X_\mu)$.

We can then sort the partitions in the reverse lexicographic order: the coefficient matrix would then be a lower triangular matrix with non-zero diagonal entries so it is invertible. This concludes the proof.

For us, the interest is to study the subring of $\Omega^*_{\mathbf{Q}}$ generated by elements with vanishing odd Chern numbers/Chern character numbers:

$$\Omega^*_{\mathbf{Q},\mathrm{even}} := \left\langle [X] \;\middle|\; \textstyle\int_X c_\lambda = 0 \text{ for all odd } \lambda \vdash \dim X \right\rangle.$$

It clearly contains the cobordism classes of all hyperkähler manifolds.

By repeating the same argument, we deduce the following result.

Proposition

Consider a sequence $(X_k)_{k\in\mathbf{Z}_{>0}}$ of manifolds with vanishing odd Chern numbers such that

$$\dim(X_k) = 2k \quad \textit{and} \quad \int_{X_k} \mathrm{ch}_{2k} \neq 0.$$

Then the even complex cobordism ring $\Omega^*_{\mathbf{Q},\mathrm{even}}$ is isomorphic to a polynomial ring $\mathbf{Q}[x_1, x_2, \ldots]$ by sending x_k to $[X_k]$.

The two known infinite families both satisfy the required property: in fact, we obtained explicit formulae for the integral of the top degree Chern character.

Proposition (Oberdieck–S.–Voisin)

For $n \geq 1$, we have

$$\int_{\mathrm{K3}^{[n]}} \mathrm{ch}_{2n} = (-1)^n \frac{(2n+2)!}{(2n-1)\cdot n!^4},$$

and

$$\int_{\mathrm{Kum}_n} \mathrm{ch}_{2n} = (-1)^n \frac{(2n+2)!}{n!^4}.$$

Consequently, both infinite families can be used as generators for the even complex cobordism ring $\Omega^*_{\mathbf{Q},\mathrm{even}}$.

Remarks

The proof of these formulae uses the explicit descriptions of the cohomology ring for these two examples in terms of *Nakajima operators*, and the computation is essentially an analysis of the combinatorial properties of these objects.

We see that both formulae confirm the conjecture on the sign of the top degree Chern character. For other products of Chern classes/characters, we have also verified them in small dimensions using a computer, although we do not have a closed formula in general.

Neither family can be used to express all hyperkähler manifolds using only *positive* linear combinations.

b_2 and c_2

One of the most important objects in the study of hyperkähler manifolds is the second cohomology group $H^2(X, \mathbf{Z})$, which carries a natural quadratic form.

Theorem (Beauville–Bogomolov–Fujiki form)

There exists a unique primitive integral quadratic form q_X on $H^2(X, \mathbf{Z})$ of signature $(3, b_2 - 3)$ and a constant $C_X \in \mathbf{Q}$ (Fujiki constant) satisfying

$$\forall\, \beta \in H^2(X, \mathbf{Z}), \quad \int_X \beta^{2n} = C_X \cdot q_X(\beta)^n.$$

More generally, let $\alpha \in H^{4k}(X, \mathbf{Q})$ be a class that remains of type $(2k, 2k)$ on all small deformations of X (e.g., any characteristic class), then there exists a constant $C(\alpha) \in \mathbf{Q}$ (generalized Fujiki constant of α) such that

$$\forall\, \beta \in H^2(X, \mathbf{Z}), \quad \int_X \alpha \cdot \beta^{2n-2k} = C(\alpha) \cdot q_X(\beta)^{n-k}.$$

- The usual Fujiki constant C_X is the same as $C(1_X)$.
- In top degree, the generalized Fujiki constants give us characteristic numbers (*i.e.*, integral of product of Chern classes).
- We have $C(c_2) > 0$, which is explained by results from differential geometry.
  Namely, for a Kähler manifold X of dimension m with trivial canonical bundle, one can choose a Ricci flat metric and obtain the following pointwise relation

  $$8\pi^2 \, c_2 \, \omega^{m-2} = \frac{\|R\|^2}{m(m-1)} \, \omega^m,$$

  where ω is the Kähler form and R is the curvature tensor.
  By taking $\frac{\omega^m}{m!}$ as the volume form and integrating over X, we get

  $$\int_X c_2 \cdot \omega^{m-2} = \frac{(m-2)! \, \|R\|^2}{8\pi^2}.$$

  Hence for a hyperkähler manifold X, we have $C(c_2) > 0$ using the Fujiki relations. Equivalently, we have $C(\mathrm{ch}_2) = C(-c_2) < 0$.

This motivates us to extend the positivity conjecture to generalized Fujiki constants as well.

Conjecture

Let X be a compact hyperkähler manifold of dimension $2n$. Then

- $C(c_\lambda) > 0$ *for all even partitions* λ *of* $2k \leq 2n$.
- *Similarly,* $(-1)^k C(\mathrm{ch}_\lambda) > 0$ *for all even partitions* λ *of* $2k \leq 2n$.

Moreover, we expect that these positivity results should follow from a similar local argument. In other words, there are algebraic identities that provide the pointwise positivity, and the global positivity is obtained by integrating over X.

Boundedness

Question. In each dimension $2n$, is the second Betti number $b_2(X)$ bounded for all hyperkähler manifolds X of dimension $2n$?

We have the following affirmative result in dimension 4.

Theorem (Guan)

When $n = 2$, we have $b_2 = 23$ or $b_2 \leq 8$ for all hyperkähler fourfolds X. The bound is sharp and is attained by $\mathrm{K3}^{[2]}$.

Corollary

For a hyperkähler fourfold X, we have $C(\mathrm{ch}_4) > 0$.

Our result is the following upper bound on b_2, subject to the positivity of $C(\mathrm{ch}_4)$.

Theorem (Beckmann–S.)

For a hyperkähler manifold X of dimension $2n$, if $C(\mathrm{ch}_4) > 0$, or equivalently, $C(c_2^2) > 2C(c_4)$, then we have the following inequality

$$b_2(X) \leq \frac{10}{\dfrac{C(c_2^2)}{C(c_4)} - 2} - (2n - 9).$$

The inequality takes a rather strange form. We now introduce another notion to rewrite it in a more natural form.

Riemann–Roch polynomial

Consider the Hirzebruch–Riemann–Roch formula: for a line bundle $L \in \mathrm{Pic}(X)$, we have

$$\begin{aligned}\chi(X, L) &= \int_X \mathrm{ch}(L)\,\mathrm{td}_X \\ &= \int_X \mathrm{td}_{2n} + \mathrm{td}_{2n-2}\,\frac{L^2}{2} + \mathrm{td}_{2n-4}\,\frac{L^4}{24} + \cdots \\ &= C(\mathrm{td}_{2n}) + C(\mathrm{td}_{2n-2})\frac{q_X(L)}{2} + C(\mathrm{td}_{2n-4})\frac{q_X(L)^2}{24} + \cdots\end{aligned}$$

We get the following *Riemann–Roch polynomial* of X

$$\mathrm{RR}_X(q) := \sum_{k=0}^{n} \frac{C(\mathrm{td}_{2n-2k})}{(2k)!} q^k,$$

which satisfies the property

$$\forall\, L \in \mathrm{Pic}(X) \quad \chi(X, L) = \mathrm{RR}_X(q_X(L)).$$

Among the known examples, there are only two types of Riemann–Roch polynomials.

- (Ellingsrud–Göttsche–Lehn, Ríos Ortiz) For $\mathrm{K3}^{[n]}$ and OG_{10}, we have

$$\mathrm{RR}_X(q) = \binom{q/2+n+1}{n};$$

- (Nieper-Wißkirchen, Ríos Ortiz) For Kum_n $(n \geq 2)$ and OG_6, we have

$$\mathrm{RR}_X(q) = (n+1)\binom{q/2+n}{n}.$$

In terms of the Riemann–Roch polynomial, the bound on b_2 takes an alternative form.

Theorem (Beckmann–S.)

Let X be a hyperkähler manifold of dimension $2n$ for $n \geq 2$. If the Riemann–Roch polynomial RR_X factorizes as a product of linear factors (and not all identical), then $C(\mathrm{ch}_4) > 0$, and we have the inequality

$$b_2(X) \leq \frac{n-1}{\frac{n(\sum \lambda_i^2)}{(\sum \lambda_i)^2} - 1} - (2n-2),$$

where λ_i are the roots of RR_X.

Here we see that the bound measures the dispersion of the roots: it gets smaller as the roots get more dispersed.

Using this description, we can examine the bound for the two known types of Riemann–Roch polynomials.

- For $\mathrm{RR}_{\mathrm{K3}^{[n]}}$, the bound is

$$b_2 \leq n + 17 + \frac{12}{n+1}.$$

  Among the known examples, it is attained by $\mathrm{K3}^{[2]}$, $\mathrm{K3}^{[3]}$, and OG_{10}.

- For $\mathrm{RR}_{\mathrm{Kum}_n}$, the bound is

$$b_2 \leq n + 5.$$

  Among the known examples, it is attained by Kum_2 and OG_6.

We remark that the bound also holds for hyperkähler orbifolds in dimension 4. This further suggests that the generalized Fujiki constants for characteristic classes and consequently the Riemann–Roch polynomial RR_X are largely governed by properties that are of local nature.

It is therefore useful if one could say more about the shape of the Riemann–Roch polynomial for an arbitrary hyperkähler manifold.

Theorem (Jiang)

Let X be a hyperkähler manifold of dimension $2n$. The coefficients of RR_X are all positive. In other words, $C(\mathrm{td}_{2k}) > 0$ for $0 \leq k \leq n$.

Conjecture

Let X be a hyperkähler variety of dimension $2n$ (possibly singular).

1. *The Riemann–Roch polynomial RR_X factorizes as a product of linear factors, and the roots form an arithmetic progression;*
2. *When X is smooth, the difference between two roots is equal to 2.*

Generalized Hitchin–Sawon formula

Theorem (Nieper-Wißkirchen)

Consider the following polynomial

$$\mathrm{RR}_{X,1/2}(q) := \sum_{k=0}^{n} \frac{C(\mathrm{td}^{1/2}_{2n-2k})}{(2k)!} q^k.$$

It factorizes as a power

$$\mathrm{RR}_{X,1/2}(q) = C(\mathrm{td}^{1/2}_{2n}) \left(1 + \tfrac{1}{2r_X} q\right)^n.$$

for some positive constant r_X.

The proof uses the *Rozansky–Witten theory*.

Idea of proof for the bound

Consider the second Chern class $c_2 \in H^4(X, \mathbf{Z})$.

Inside $H^4(X, \mathbf{Z})$ we have the image of

$$\smile : H^2(X, \mathbf{Z}) \times H^2(X, \mathbf{Z}) \longrightarrow H^4(X, \mathbf{Z}).$$

The cup product is in fact injective, so we have $\mathrm{SH}^4(X) := \mathrm{Sym}^2 H^2(X, \mathbf{Z})$ sitting inside $H^4(X, \mathbf{Z})$.

A natural question is to ask whether c_2 lies in $\mathrm{SH}^4(X)$ or not.

Idea of proof for the bound

We can project c_2 to $\mathrm{SH}^4(X)$

$$c_2 = \overline{c_2} + z,$$

and study the difference z, which is a primitive $(2,2)$-class.
By the Hodge–Riemann bilinear relations, we get

$$\int_X z^2 \omega^{2n-4} \geq 0,$$

where equality holds if and only if $z = 0$.

Idea of proof for the bound

If we now look at c_2^2, we have

$$c_2^2 = \overline{c_2}^2 + 2\overline{c_2}z + z^2,$$

and by considering generalized Fujiki constant, we get

$$C(c_2^2) = C(\overline{c_2}^2) + C(z^2) \geq C(\overline{c_2}^2).$$

This gives the main inequality. By computing the values of the generalized Fujiki constants, we get the desired statement involving $C(\mathrm{ch}_4)$ and b_2.

In other words, the bound on b_2 is essentially given by a triangle inequality involving c_2. This also gives us the following corollaries on the second Chern class.

Corollary

Let X be a hyperkähler manifold of dimension $2n$ with $n \geq 2$. Then $c_2 \in \mathrm{SH}^4(X)$ if and only if $C(\mathrm{ch}_4) > 0$ and b_2 attains the upper bound.

Corollary

Among known smooth hyperkähler manifolds of dimension $2n$ with $n \geq 2$, we have $c_2 \in \mathrm{SH}^4(X)$ if and only if X is one of the following

$$\mathrm{K3}^{[2]}, \quad \mathrm{K3}^{[3]}, \quad \mathrm{Kum}_2, \quad \mathrm{OG}_6, \quad \mathrm{OG}_{10}.$$

Thank you!