Université Paris Cité
École doctorale de sciences mathématiques de Paris centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

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Geometry of hyperkähler manifolds
Géométries des variétés hyperkählériennes

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Soutenance le 5 juillet 2022 devant le jury composé de :

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Résumé

Cette thèse concerne la géométrie des variétés hyperkählériennes. Elle est composée de deux parties et cinq chapitres. Dans la première partie, on étudie quelques propriétés générales de ces variétés. Dans la deuxième partie, on se concentre sur une famille particulière de variétés hyperkählériennes projectives, dite de *Debarre–Voisin*, et étudie leur géométrie explicite.

Au chapitre 1, on rappelle quelques résultats de base bien connus sur les variétés hyperkählériennes.

Au chapitre 2, on étudie quelques aspects numériques des variétés hyperkählériennes. Premièrement, on montre une borne supérieure conditionnelle sur le deuxième nombre de Betti, en termes des *constantes de Fujiki généralisées*, ou de manière équivalente, en termes du *polynôme de Riemann–Roch*. Ensuite, on étudie la classe cohomologique d'un sous-espace lagrangien. On montre une formule pour la projection de la classe cohomologique vers la composante de Verbitsky. On propose aussi une formule conjecturale pour la classe entière dans le cas de $\mathrm{K3}^{[n]}$, qui a été vérifiée pour n jusqu'à 6.

Au chapitre 3, on étudie les espaces de modules et les applications des périodes pour les variétés hyperkählériennes projectives. En général, l'espace de modules pour les variétés hyperkählériennes polarisées avec un type de polarisation fixé n'est pas nécessairement connexe. Pour les $\mathrm{K3}^{[n]}$ et les Kum_n, on obtient une formule précise pour le nombre de composantes connexes, ainsi que le nombre des types de polarisation ayant un carré et une divisibilité donnés. Puis on étudie l'image de l'application des périodes polarisée, et on montre que lorsque l'espace de modules n'est pas connexe, les images de l'application des périodes peuvent être différentes si l'on se restreint sur des composantes différentes.

Au chapitre 4, on étudie les propriétés générales des variétés de Debarre–Voisin. Une telle variété est définie à partir d'un *trivecteur*, et on peut aussi lui associer deux autres variétés qui sont Fano de type K3. On obtient d'abord les critères de lissité pour ces trois variétés, et on donne un aperçu de la géométrie de l'espace de modules et de l'application des périodes. Ensuite, on relie les structures de Hodge entières sur les trois variétés, et on montre que les deux qui sont Fano satisfont la conjecture de Hodge entière. Ces résultats sont obtenus par une analyse détaillée de la géométrie de ces variétés le long de trois diviseurs spéciaux dans l'espace de modules.

Au chapitre 5, on étudie une variété de Debarre–Voisin spéciale qui admet un groupe d'automorphismes très grand, en appliquant les résultats généraux obtenus au chapitre 4.

Mots-clés. variétés hyperkählériennes, variétés symplectiques holomorphes, variétés de Debarre–Voisin, variétés Fano de type K3

Abstract

This thesis concerns the geometry of hyperkähler manifolds. It is divided into two parts and five chapters. In the first part, we study some general properties for such manifolds. In the second part, we focus on one particular family of projective hyperkähler manifolds called *Debarre–Voisin fourfolds* and study their explicit geometry.

In Chapter 1, we recall some well-known basic results on hyperkähler manifolds.

In Chapter 2, we study some numerical aspects of hyperkähler manifolds. First, we produce a conditional upper bound on the second Betti number, in terms of the *generalized Fujiki constants*, or equivalently, in terms of the *Riemann–Roch polynomial.* Then we study the cohomology class of a Lagrangian plane. We deduce a formula for the projection of the cohomology class to the Verbitsky component. We also propose a conjectural formula for the full class in the $\mathrm{K3}^{[n]}$-type case, which is verified for n up to 6.

In Chapter 3, we study the moduli spaces and period maps for projective hyperkähler manifolds. In general, the moduli space for polarized hyperkähler manifolds with a fixed polarization type is not necessarily connected. For $\mathrm{K3}^{[n]}$-type and Kum_n-type, we deduce a precise formula for the number of connected components, as well as the number of polarization types with fixed square and divisibility. Then we study the image of the polarized period map, focusing on the known examples. We show that when the moduli space is not connected, the images of the period map restricted to different connected components can be different.

In Chapter 4, we study the general properties of Debarre–Voisin fourfolds. Such manifolds are defined from a *trivector*, and there are two Fano varieties of K3-type that can also be associated with it. We first deduce the smoothness criteria for these three varieties, and provide a picture of the moduli space and the period map. Then we relate the integral Hodge structures of the three varieties, and show that the two Fano varieties satisfy the integral Hodge conjecture. This is obtained as a detailed analysis of the geometry of these varieties along three special divisors in the moduli space.

In Chapter 5, we study a special Debarre–Voisin fourfold with a large automorphism group, using the general results obtained in Chapter 4.

Keywords. hyperkähler manifolds, holomorphic symplectic varieties, Debarre–Voisin varieties, Fano varieties of K3-type

Contents

Introduction (version française)

Cette thèse est consacrée à l'étude des géométries des variétés hyperkählériennes.

Définition. Une variété *hyperkählérienne* est une variété compacte kählérienne simplement connexe dont l'espace des 2-formes holomorphes est engendré par une 2-forme holomorphe partout non dégénérée, autrement dit, une forme holomorphe symplectique. Une telle variété est donc aussi appelée *holomorphe symplectique irréductible.*

Les variétés hyperkählériennes sont extrêmement importantes dans l'étude des variétés du fibré canonique trivial. Par le théorème de décomposition de Beauville–Bogomolov [**Bea83**, Théorème 2], ils sont parmi l'un des trois types les plus fondamentaux.

Théorème (Beauville–Bogomolov). *Soit X une variété compacte kählérienne du fibré canonique trivial. Alors il existe un revêtement étale*

$$T \times \prod_i Y_i \times \prod_j K_j \longrightarrow X,$$

où T est un tore complexe, Y_i sont des variétés de Calabi–Yau strictes[1] *et K_j sont des variétés hyperkählériennes.*

En plus, des résultats structuraux assez forts ont été démontrés pour ces variétés. Nous nous contentons d'en nommer quelques uns ici. Un traitement plus détaillé des résultats connus sera donné au chapitre 1.

En premier lieu, l'anneau de cohomologie $H^*(X, \mathbf{Q})$ d'une variété hyperkählérienne X possède une structure très riche : le deuxième groupe de cohomologie $H^2(X, \mathbf{Z})$ porte une forme quadratique q_X qui s'appelle la forme de Beauville–Bogomolov–Fujiki, et la cohomologie entière $H^*(X, \mathbf{Q})$ admet l'action d'une algèbre de Lie dite de l'algèbre de Looijenga–Lunts–Verbitsky. Par conséquent, on obtient une décomposition naturelle de $H^*(X, \mathbf{Q})$ en sous-représentations irréductibles. Une composante particulièrement importante est la sous-algèbre engendrée par $H^2(X, \mathbf{Q})$, qui s'appelle la composante de Verbitsky. On peut déjà établir beaucoup de résultats intéressants juste en regardant les propriétés numériques de l'anneaux $H^*(X, \mathbf{Q})$, sans connaitre concrètement la géométrie de la variété X.

[1]Une variété de Calabi–Yau stricte Y est une variété compacte kählérienne simplement connexe de fibre canonique trivial, telle que $H^{k,0}(Y) = 0$ pour tout $k \notin \{0, \dim(Y)\}$.

Deuxièmement, beaucoup d'information d'une variété hyperkählérienne X est incorporée dans son deuxième groupe de cohomologie $\big(H^2(X,\mathbf{Z}),q_X\big)$ muni de la forme de Beauville–Bogomolov–Fujiki, qui devient une structure de Hodge entière polarisée. Par exemple, une forme du théorème de Torelli global par Verbitsky indique que l'on peut (quasiment) identifier la variété hyperkählérienne X juste en regardant cette structure de Hodge. Plus concrètement, il existe une application des périodes partant de l'espace de modules des variétés hyperkählériennes (d'un type de déformation fixé) vers l'espace paramétrant les structures de Hodge correspondantes. Le théorème de Torelli local affirme que c'est un isomorphisme local, et le théorème de Torelli global montre qu'elle est surjective et génériquement injective. L'étude de cette application nous permet de mieux comprendre la géométrie de ces espaces de modules.

Les variétés hyperkählériennes sont aussi très mystérieux dans le sens que très peu d'exemples sont connus, contrairement au cas des variétés de Calabi–Yau strictes. Voici une liste complète de tous les exemples connus à ce jour.

- En dimension 2, les variétés hyperkählériennes ne sont autres que les surfaces K3. Elles ont toutes le même type de topologie, et le deuxième nombre de Betti est égal à 22.
- $\mathrm{K3}^{[n]}$ pour $n\geq 2$: pour une surface K3 S, on peut considérer le schéma de Hilbert des points $S^{[n]}$, qui est une variété hyperkählérienne de dimension $2n$. Plus généralement, une déformation d'une telle variété est elle aussi hyperkählérienne. Le b_2 est égal à 23.
- Kum_n pour $n\geq 2$: de même, pour une surface abélienne A, on peut construire son schéma de Hilbert des points ; par contre, pour produire une variété simplement connexe, il faut considérer l'application de somme $\Sigma\colon A^{[n+1]}\to A$ et prendre la préimage d'un point $\mathrm{Kum}(A)\coloneqq\Sigma^{-1}(0)$. Lorsque $n=1$ cela donne la surface de Kummer de A ; c'est la raison pour laquelle les analogues en dimensions supérieures et leurs déformations sont appelées les variétés de Kummer généralisées. Le b_2 est égal à 7.
- Deux exemples sporadiques découverts par O'Grady en partant des désingularisations des espaces de modules des faisceaux : un exemple OG_6 en dimension 6 avec $b_2=8$, et un autre OG_{10} en dimension 10 avec $b_2=24$.

Dans la première partie de la thèse, on tente d'étudier quelques propriétés générales des variétés hyperkählériennes, notamment celles liées avec les aspects numériques et avec les applications des périodes. On tient aussi à examiner ces propriétés dans le cas des exemples connus.

D'un point de vue algébro-géométrique, il est naturel d'étudier des variétés hyperkählériennes qui sont projectives ou *polarisées*, c'est-à-dire une variété X munie d'une classe ample H qui est primitive. Nous nous intéressons aux familles localement completes : pour une telle famille, on possède la description du modèle projectif d'un membre général de la famille. Par exemple, les modèles projectifs pour les surfaces K3 sont connus en degrés bas : les résultats en degrés $2, 4, 6, 8$ sont classiques, et pour les degrés qui suivent, ils ont été étudiés par Mukai dans une série de travaux (voir le survol [**Deb18**, Section 2.3] pour une liste complète des modèles projectifs connus).

La question est beaucoup plus difficile en dimensions supérieures : il n'y a que quelques familles localement complètes connues (voir [**Deb18**, Section 3.6] pour une liste des exemples). La famille la plus célèbre est celle des variétés des droites pour les cubiques de dimension 4 : elles sont hyperkählériennes de type $\mathrm{K3}^{[2]}$. Des nombreuses travaux ont été réalisés sur ces variétés et sur leurs relations avec la cubique correspondante : il y a beaucoup de phénomènes intéressantes au niveau de leurs structures de Hodge, anneaux de Chow, catégories dérivées, ainsi que la question de rationalité. Les double sextiques d'EPW donnent une autre famille qui est bien étudiée, toujours de type $\mathrm{K3}^{[2]}$. Il y a plusieurs similarités entre les deux familles : par exemple, pour une double sextique d'EPW générale, il y a aussi une variété de Fano de dimension 4 associée qui s'appelle une variété de Gushel–Mukai.

Debarre et Voisin ont construit une autre famille localement complète de variétés hyperkählériennes de type $\mathrm{K3}^{[2]}$ dans [**DV10**]. Cette famille n'est pas aussi bien étudiée que les deux autres, partiellement dû au fait que la variété de Fano associée est de dimension 20 au lieu de 4. On peut néanmoins constater un intérêt croissant à l'égard de ces variétés [**DHOV20, BFM21**].

Dans la deuxième partie de la thèse, on se concentre sur ces variétés hyperkählériennes construites par Debarre–Voisin et on étudie leur géométrie explicite. On retrouvera beaucoup d'éléments clés pour les deux familles mentionnées ci-dessus dans le cas des variétés de Debarre–Voisin.

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Nous présentons maintenant une description détaillée de la structure de cette thèse, en soulignant les résultats obtenus.

Au **chapitre 1**, nous allons rappeler quelques résultats importants sur les variétés hyperkählériennes, y compris la définition de la forme de Beauville–Bogomolov–Fujiki et celle de l'algèbre de Looijenga–Lunts–Verbitsky, ainsi que les propriétés des applications des périodes et les théorèmes de type Torelli. Nous rappellerons aussi des autres invariants importants, comme les constantes de Fujiki généralisées et le polynôme de Riemann–Roch. La plupart des résultats ici sont déjà connus ; nous démontrons seulement quelques lemmes faciles qui nous seront utiles dans la suite.

Au **chapitre 2**, nous étudierons des aspects numériques d'une variété hyperkählérienne X. Par « numérique », on entend des propriétés de X liées à son anneau de cohomologie $H^*(X, \mathbf{Q})$ et sa forme de Beauville–Bogomolov–Fujiki q_X. Telles propriétés peuvent être étudiées sans connaitre la géométrie de la variété X elle-même.

La première question que l'on se pose est de déterminer si la deuxième classe de Chern $c_2(X)$ est contenue dans la composante de Verbitsky $\mathrm{SH}(X)$. Nous présenterons un critère qui donne une réponse complète à cette question. En plus, cela nous fournit une borne conditionnelle sur le deuxième nombre de Betti $b_2(X)$ en termes de certains invariants numériques de X : soit par les constantes de Fujiki généralisées $C(c_2^2)$ et $C(c_4)$ (théorème 2.1.1) ; soit par le polynôme de Riemann–Roch (théorème 2.1.6).

La deuxième question à laquelle on s'intéresse est de calculer la classe $[P] \in H^{2n}(X, \mathbf{Q})$ d'un n-plan lagrangien P contenu dans X. En général, l'expression pour $[P]$ peut être compliquée, mais on va montrer que lorsque l'on se restreint à la composante de Verbitsky, il existe une formule très simple pour la projection $\overline{[P]}$ (théorème 2.2.4). La formule est exprimée en termes de $L \in H^2(X, \mathbf{Q})$, la classe duale d'une droite contenue dans P. Dans certains cas, la projection $\overline{[P]}$ nous permet de retrouver la classe P entièrement. Par exemple, pour les variétés de type $\mathrm{K3}^{[3]}$, on retrouve ainsi le résultat principal de [**HHT12**] (exemple 2.2.7).

Nous proposerons aussi une formule conjecturale de la classe $[P]$ pour les variétés de type $\mathrm{K3}^{[n]}$ (conjecture 2.2.10). Cette question a été posée par Bakker dans [**Bak17**], où il a démontré que toutes les classes $[P]$ sont dans la même orbite pour le groupe des monodromies, et par conséquent une expression universelle pour $[P]$ existe forcément. En particulier, pour vérifier la conjecture, il suffit de trouver un seul couple (X, P) pour lequel la formule est vraie. En utilisant le calcul formel, on a vérifié la conjecture pour n jusqu'à 6.

Toutes les deux formules font intervenir la partie de degré n du vecteur de Mukai $v(L) := \exp(L)\,\mathrm{td}_X^{1/2}$. Nous discuterons quelques propriétés conjecturales de $v(L)$ ainsi que quelques indices pour les justifier (voir la conjecture 2.2.12 et la proposition 2.2.13).

Le chapitre se conclut par l'appendice 2.A, où nous examinerons les constantes de Fujiki généralisées pour les exemples connus.

Au **chapitre 3**, nous étudierons les espaces de modules des variétés hyperkählériennes polarisées et les applications des périodes correspondantes. Grâce au théorème de Torelli polarisé démontré par Markman, pour un type de polarisation fixé, l'espace de modules est plongé comme une sous-variété ouverte dans le domaine des périodes via l'application des périodes polarisée. Nous allons d'abord rappeler cette construction, suivant les travaux de Markman [**Mar11**], et nous expliquerons comment le groupe des monodromies y intervient. En particulier, nous verrons que l'espace de modules pour les variétés hyperkählériennes polarisées d'un type de polarisation donné n'est pas nécessairement connexe, un phénomène qui a été premièrement constaté et examiné par Apostolov dans [**Apo14**] pour le cas de $\mathrm{K3}^{[n]}$.

Ensuite, nous nous concentrons sur les types de déformation connus et nous déterminerons le nombre des composantes connexes pour un type de polarisation donné (voir la proposition 3.3.4). Pour les types $\mathrm{K3}^{[n]}$ et Kum_n, nous donnerons aussi une formule pour le nombre des types de polarisation avec le carré et la divisibilité fixés (voir la proposition 3.3.5).

Ces résultats raffinent et simplifient les résultats existants obtenus par Apostolov [**Apo14**] dans le cas de $\mathrm{K3}^{[n]}$ et par Onorati [**Ono16**] dans le cas de Kum_n.

Une fois que la formule pour le nombre des composantes connexes sera établie, nous analyserons l'image de l'application des périodes, de nouveau pour les types de déformation connus. Curieusement, lorsque l'on se restreint aux différentes composantes connexes, les images peuvent être différentes. Un exemple explicit dans le cas de type $\mathrm{K3}^{[n]}$ sera donné en section 3.5.

Au **chapitre 4**, nous étudierons la géométrie des variétés de Debarre–Voisin mentionnées plus haut. Une telle variété, que l'on appellera X_6^σ, est construite à partir de la donnée d'un trivecteur $\sigma \in \bigwedge^3 V_{10}^\vee$, où V_{10} est un espace vectoriel complexe de dimension 10. À ce trivecteur σ, deux autres variétés intéressantes X_1^σ et X_3^σ peuvent encore être associées (ainsi nommées car X_k^σ est une sous-variété de $\mathrm{Gr}(k, V_{10})$).

Nous étudierons d'abord l'espace de modules GIT pour les trivecteurs

$$\mathcal{M} \coloneqq \mathbf{P}(\textstyle\bigwedge^3 V_{10}^\vee) /\!\!/ \mathrm{SL}(V_{10}).$$

Trois diviseurs intéressants dans $\mathcal{M}$ seront définis par des conditions de dégénérescence $\mathrm{SL}(V_{10})$-invariants sur le trivecteur σ.

Ensuite, nous examinerons la lissité des variétés X_1^σ, X_3^σ et X_6^σ. Nous montrerons que les lieux dans $\mathcal{M}$ où elles deviennent singulières sont précisément les diviseurs introduits ci-dessus (proposition 4.1.1).

Puis nous donnerons un aperçu global de l'espace de modules et l'application des périodes. Notons $\mathcal{M}^{\mathrm{smooth}}$ le sous-espace ouvert de $\mathcal{M}$ où σ définit une variété de Debarre–Voisin X_6^σ lisse : c'est le complément d'un diviseur $\mathcal{D}^{3,3,10}$. On a le diagramme suivant (voir le théorème 4.4.11 et le lemme 4.4.13).

$$\begin{array}{ccccccc}
\mathbf{P}(\bigwedge^3 V_{10}^\vee) \smallsetminus \Delta^{3,3,10} & & \mathcal{M} & & & & \mathcal{P} \\
\sqcup & \overset{\pi}{\searrow} & \shortparallel & & & & \shortparallel \\
\Delta^{3,3,10} & & \mathcal{M}^{\mathrm{smooth}} & \overset{\mathfrak{m}}{\hookrightarrow} & \mathcal{M}_{22}^{(2)} & \xrightarrow[\sim]{\mathfrak{p}} & \mathrm{Im}(\mathfrak{p}) \\
 & \overset{\pi}{\searrow} & \sqcup & & & & \sqcup \\
 & & \mathcal{D}^{3,3,10} & & \overset{\widetilde{\mathfrak{p}}}{\underset{\text{bir.}}{\dashrightarrow}} & & \mathcal{D}_{22}.
\end{array}$$

Ici $\mathcal{M}$ est l'espace de modules GIT, $\mathcal{M}_{22}^{(2)}$ est l'espace de modules des variétés hyperkählériennes polarisées de type $\mathrm{K3}^{[2]}$ avec une polarisation de carré 22 et de divisibilité 2, et $\mathcal{P}$ est le domaine des périodes correspondant ; π est l'application de quotient GIT, $\mathfrak{m}$ est l'application modulaire donnée par la construction de Debarre–Voisin, et $\mathfrak{p}$ est l'application des périodes ; l'application rationnelle $\mathfrak{p} \circ \mathfrak{m}$ peut être étendue en codimension 1 : on notera cette extension $\widetilde{\mathfrak{p}}$ et on l'appellera l'application des périodes étendue.

Nous étudierons ensuite les structures de Hodge des trois variétés X_1^σ, X_3^σ et X_6^σ. Comme X_6^σ est une variété hyperkählérienne polarisée, nous nous intéressons particulièrement à son groupe de cohomologie primitif $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}}$ muni de la forme de Beauville–Bogomolov–Fujiki. Sur les variétés de Fano X_1^σ et X_3^σ, nous allons définir les groupes de cohomologie évanescents $H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}$ et $H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}}$, tous deux en degré milieu et sont ainsi munis d'un produit d'intersection. Le résultat principal est que ces trois pièces de structures de Hodge sont liées par des isométries de Hodge via des correspondances algébriques (voir le théorème 4.1.2). Comme un corollaire, nous établirons la conjecture de Hodge entière pour $H^*(X_1^\sigma, \mathbf{Z})$ et $H^*(X_3^\sigma, \mathbf{Z})$ (corollaire 4.1.3). La démonstration pour les isométries de Hodge est cependant basée sur une analyse de l'un des diviseurs de Heegner, qui fera le sujet de la section suivante.

Enfin, nous allons effectuer une étude en détail de la géométrie des trois variétés lorsque le trivecteur σ appartient à l'un des trois diviseurs. Nous montrerons que tous les trois diviseurs sont envoyés sur des diviseurs de Heegner par l'application des périodes, donc ils sont tous spéciaux dans le sens de Hassett. Le diviseur de Heegner $\mathcal{D}_{28}$ est étroitement lié aux plans lagrangiens contenus dans X_6^σ (voir le théorème 4.6.3), et nous nous servirons de cette géométrie pour établir les isométries de Hodge. Pour le diviseur de Heegner $\mathcal{D}_{24}$, nous allons construire une surface K3 tordue (S, β) avec une classe de Brauer β, et nous montrerons qu'un membre général X_6^σ dans cette famille est isomorphe à un espace de modules des faisceaux sur S. Nous analyserons aussi la singularité pour un membre général de la famille $\mathcal{D}_{22}$ en section 4.8.

Au **chapitre 5**, nous étudierons une variété de Debarre–Voisin spéciale avec un très grand groupe d'automorphismes, partant des résultats généraux obtenus au chapitre 4. Les résultats principaux sont rassemblés dans le théorème 5.1.1. Notamment, en utilisant la théorie des représentations du groupe simple $\mathbf{G} \coloneqq \mathrm{PSL}(2, \mathbf{F}_{11})$ d'ordre 660, on construira un trivecteur σ_0 qui est $\mathbf{G}$-invariant. On va vérifier que la variété de Debarre–Voisin $X_6^{\sigma_0}$ associée est lisse de dimension 4, alors que la variété $X_1^{\sigma_0}$ associée admet 55 points singuliers isolés. On construira ensuite 55 diviseurs distincts sur $X_6^{\sigma_0}$, engendrant le réseau de Picard qui est de rang 21. Cela nous permettra de conclure que le groupe $\mathrm{Aut}_H^s(X_6^{\sigma_0})$ des automorphismes symplectiques fixant la polarisation est en fait isomorphe à $\mathbf{G}$.

Introduction

We first give the definition of a compact hyperkähler manifold, the main object studied in this thesis.

Definition. A simply connected compact Kähler manifold X is called a *hyperkähler manifold* if the vector space $H^{2,0}(X) := H^0(X, \Omega_X^2)$ is generated by a nowhere degenerate holomorphic 2-form σ, in other words, a holomorphic symplectic form. Such manifolds are also known as *irreducible holomorphic symplectic manifolds.*

Compact hyperkähler manifolds are extremely important in the study of manifolds with trivial canonical bundle. Notably, by the Beauville–Bogomolov decomposition theorem [**Bea83**, Théorème 2], they are one of the three building blocks.

Theorem (Beauville–Bogomolov). *Let X be a compact Kähler manifold with trivial canonical bundle. Then there exists a finite étale cover*

$$T \times \prod_i Y_i \times \prod_j K_j \longrightarrow X,$$

where T is a complex torus, Y_i are strict Calabi–Yau manifolds,[2] *and K_j are hyperkähler manifolds.*

Moreover, one can prove some very strong structural results for compact hyperkähler manifolds. We give a brief overview here, and refer to Chapter 1 for more detailed treatments of the known results.

First, the cohomology ring $H^*(X, \mathbf{Q})$ of a hyperkähler manifold X enjoys many rich features: the second cohomology group $H^2(X, \mathbf{Z})$ carries a quadratic form q_X called the Beauville–Bogomolov–Fujiki form, and the full cohomology $H^*(X, \mathbf{Q})$ admits an action of a Lie algebra called the Looijenga–Lunts–Verbitsky algebra. Consequently, we have a natural decomposition of $H^*(X, \mathbf{Q})$ into irreducible subrepresentations. One particularly important component is the subalgebra $\mathrm{SH}(X)$ generated by $H^2(X, \mathbf{Q})$ which is called the Verbitsky component. We can deduce a lot of interesting conclusions by just studying the numerical properties of the cohomology ring $H^*(X, \mathbf{Q})$, without much knowledge of the actual geometry of X.

[2]A strict Calabi–Yau manifold is a simply connected Kähler manifold Y with trivial canonical bundle such that $H^{k,0}(Y) = 0$ for all $k \notin \{0, \dim(Y)\}$.

Second, a lot of information of a hyperkähler manifold X is encoded in its second cohomology group $\big(H^2(X,\mathbf{Z}),q_X\big)$ equipped with the Beauville–Bogomolov–Fujiki form, which gives a polarized Hodge structure. For example, one form of the global Torelli theorem by Verbitsky states that one can (almost) recover the hyperkähler manifold X just from the polarized Hodge structure on $H^2(X,\mathbf{Q})$. More concretely, one can construct a period map from the moduli space of hyperkähler manifolds (of a fixed deformation type) to the corresponding moduli space of Hodge structures. The local Torelli theorem asserts that this is a local isomorphism, and the global Torelli theorem states that the period map is surjective and generically injective. One can thus study the geometry of this period map to get a better understanding of these moduli spaces.

Compact hyperkähler manifolds are also mysterious in that only very few examples are known, in sharp contrast to strict Calabi–Yau manifolds. We give a list of all the known examples below.

- In dimension 2, these are precisely K3 surfaces. They all have the same topological type, and the second Betti number b_2 is equal to 22.
- $\mathrm{K3}^{[n]}$ for $n\geq 2$: for a K3 surface S, one can consider the Hilbert scheme of points $S^{[n]}$, which is a hyperkähler manifold of dimension $2n$. More generally, their deformations are also hyperkähler. They have $b_2=23$.
- Kum_n for $n\geq 2$: similarly, for an Abelian surface A, one can consider the Hilbert scheme of points; but to produce a simply connected manifold, one use the sum map $\Sigma\colon A^{[n+1]}\to A$ and take the preimage of a point $\mathrm{Kum}(A):=\Sigma^{-1}(0)$. When $n=1$ this gives the Kummer surface of A, which is why the higher dimensional analogues and their deformations are called generalized Kummer varieties. They have $b_2=7$.
- Two sporadic examples discovered by O'Grady, using desingularizations of moduli spaces of sheaves: one example OG_6 of dimension 6 and $b_2=8$, and another OG_{10} of dimension 10 and $b_2=24$.

In the first part of the thesis, we study some general properties of hyperkähler manifolds, focusing on the numerical aspects as well as the period maps mentioned above. We will also emphasize on applying these results to the known examples.

From the point of view of algebraic geometry, it is natural to study projective or *polarized* hyperkähler manifolds, that is, hyperkähler manifolds X equipped with a primitive ample class H. We are particularly interested in locally complete families: for such families, we have a description of the projective model of a general member. For example, for K3 surfaces, the projective models are known when the degree of the polarization is small: the results in degree $2,4,6,8$ are classical, while in higher degrees, they are studied by Mukai in a series of works (see the survey [**Deb18**, Section 2.3] for a list of known projective models).

The question is more difficult in higher dimensions: only a few locally complete families of polarized hyperkähler manifolds are known (again see [**Deb18**, Section 3.6] for a list of known examples). The most famous and most studied family is given by varieties of lines for cubic fourfolds: they are hyperkähler manifolds of $K3^{[2]}$-type. A lot of studies have been carried out on such varieties and on their relation with their corresponding cubic fourfold: there are many interesting phenomena in terms of their Hodge structures, Chow rings, derived categories, and rationality problems. Another well studied family is given by double EPW sextics, also of $K3^{[2]}$-type. This family shares a lot of similarities with varieties of lines for cubic fourfolds. For example, for a general member, there is also a Fano fourfold associated, called a Gushel–Mukai fourfold.

Debarre–Voisin constructed another locally complete family of polarized hyperkähler fourfolds of $K3^{[2]}$-type in [**DV10**]. This family is not as well studied as the other two families, partly due to the fact that the variety playing the role of the Fano fourfold is a 20-dimensional Fano variety. Although in recent years, it has also attracted much attention [**DHOV20, BFM21**].

In the second part of the thesis, we focus on Debarre–Voisin hyperkähler manifolds and study their explicit geometry. We shall see that many features for the first two families have their counterparts in the case of Debarre–Voisin varieties.

⁂

We now provide a more detailed outline of the thesis, highlighting the main results obtained.

In **Chapter 1**, we recall some important and well known results for hyperkähler manifolds. This includes the definitions of the Beauville–Bogomolov–Fujiki form and the Looijenga–Lunts–Verbitsky algebra that we mentioned above, as well as the properties of the period maps and the Torelli theorems. We will also recall the definitions of some other interesting invariants like the generalized Fujiki constants and the Riemann–Roch polynomial RR_X. Most results can already be found in the literature. Occasionally, we deduce some easy lemmas that will become useful in later chapters.

In **Chapter 2**, we study some numerical aspects of a hyperkähler manifold X. By "numerical", we refer to properties of X involving mainly the cohomology ring $H^*(X,\mathbf{Q})$ and the Beauville–Bogomolov–Fujiki form q_X. Notably, such properties can be studied without much knowledge of the actual geometry of X.

The first question we study is whether the second Chern class $c_2(X)$ lies in the Verbitsky component $\mathrm{SH}(X)$. We present a criterion that completely answers this question. Interestingly, the criterion provides an upper bound on the second Betti number $b_2(X)$ in terms of some numerical invariants of X: it can either be expressed in terms of the generalized Fujiki constants $C(c_2^2)$ and $C(c_4)$ (see Theorem 2.1.1), or in terms of the Riemann–Roch polynomial RR_X (see Theorem 2.1.6).

The second question of interest is to determine the cohomology class $[P]\in H^{2n}(X,\mathbf{Q})$ of a Lagrangian n-plane P contained in X. In general, the class $[P]$ can be quite complicated.

But when restricted to the Verbitsky component, we deduce a very simple formula for the projection $\overline{[P]}$ (see Theorem 2.2.4). The formula is expressed in terms of $L \in H^2(X, \mathbf{Q})$, the dual class of a line ℓ contained in P. In certain cases, the projection $\overline{[P]}$ alone allows us to determine the full class $[P]$. For example, for $\mathrm{K3}^{[3]}$-type, this very easily recovers the main result of [**HHT12**] (see Example 2.2.7).

We also propose a conjectural formula for the full class $[P]$ in the case of $\mathrm{K3}^{[n]}$-type for all n (see Conjecture 2.2.10). This question has been asked by Bakker in [**Bak17**]: under the condition on the line class ℓ being primitive, he showed that all such classes $[P]$ are in the same monodromy orbit, so a universal expression for $[P]$ must exist. In this case, to verify the conjecture, it suffices to exhibit a single pair (X, P) for which the formula holds. Using computer algebra, we can thus verify it for n up to 6.

Both the formula for $\overline{[P]}$ in the general case and the one for $[P]$ in the case of $\mathrm{K3}^{[n]}$-type involve the degree n part of the Mukai vector $v(L) \coloneqq \exp(L)\,\mathrm{td}_X^{1/2}$. We will discuss some conjectural vanishing behavior of the full Mukai vector $v(L)$ as well as some evidence (see Conjecture 2.2.12 and Proposition 2.2.13).

We finish the chapter by Appendix 2.A, where we give an account on the generalized Fujiki constants for the known examples.

In **Chapter 3**, we study the moduli spaces for polarized hyperkähler manifolds and the corresponding period maps. Thanks to the polarized Torelli theorem of Markman, the polarized period map gives an open immersion of the moduli space into the period domain. We will first recall the construction of the polarized period map following the work of Markman [**Mar11**], and explain how the monodromy group is involved in the construction. In particular, we shall see that the moduli space for polarized hyperkähler manifolds with a fixed polarization type is not necessarily connected, a phenomenon first studied by Apostolov in the $\mathrm{K3}^{[n]}$-case in [**Apo14**].

We then focus on the known deformation types and determine the number of connected components for a given polarization type (see Proposition 3.3.4). For $\mathrm{K3}^{[n]}$-type and Kum_n-type, we also deduce a formula for the number of polarization types with fixed numerical invariants, namely, with fixed square and divisibility (see Proposition 3.3.5). This provides a refined and simplified version of Apostolov's result [**Apo14**] for $\mathrm{K3}^{[n]}$-type and Onorati's result [**Ono16**] for Kum_n-type.

Once the number of connected components is determined, we analyze the image of the period map, again focusing on the known deformation types. Interestingly, the images of the period map restricted to different connected components can be different. We provide an explicit example in the case of $\mathrm{K3}^{[n]}$-type in Section 3.5.

In **Chapter 4**, we study the geometry of Debarre–Voisin varieties, which, as already mentioned above, form a family of locally complete hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type. Such varieties are defined from the data of a trivector $\sigma \in \bigwedge^3 V_{10}^\vee$, where V_{10} is a 10-dimensional complex vector space. We will denote by X_6^σ the corresponding Debarre–Voisin fourfold.

To a trivector σ we can associate two other interesting varieties, which are denoted by X_1^σ and X_3^σ (they are named this way because X_k^σ is a subvariety of $\mathrm{Gr}(k, V_{10})$).

We first study the GIT quotient moduli space for trivectors

$$\mathcal{M} \coloneqq \mathbf{P}(\textstyle\bigwedge^3 V_{10}^\vee) /\!\!/ \mathrm{SL}(V_{10}).$$

We define three interesting divisors in $\mathcal{M}$ given by some $\mathrm{SL}(V_{10})$-invariant vanishing condition on σ.

Next, we study the smoothness of the varieties X_1^σ, X_3^σ, and X_6^σ. We show that the loci in $\mathcal{M}$ where they become singular are given by the divisors that we introduced above (see Proposition 4.1.1).

Then we give a picture of the moduli spaces and period maps. We write $\mathcal{M}^{\mathrm{smooth}}$ for the open locus of $\mathcal{M}$ where σ defines a smooth Debarre–Voisin fourfold X_6^σ, which is the complement of a divisor $\mathcal{D}^{3,3,10}$. We have the following picture (see Theorem 4.4.11 and Lemma 4.4.13).

$$
\begin{array}{ccccccc}
\mathbf{P}(\bigwedge^3 V_{10}^\vee) \smallsetminus \Delta^{3,3,10} & & \mathcal{M} & & & & \mathcal{P} \\
\sqcup & \overset{\pi}{\searrow} & \shortparallel & & & & \shortparallel \\
\Delta^{3,3,10} & & \mathcal{M}^{\mathrm{smooth}} & \xhookrightarrow{\mathfrak{m}} & \mathcal{M}_{22}^{(2)} & \xrightarrow[\sim]{\mathfrak{p}} & \mathrm{Im}(\mathfrak{p}) \\
 & \overset{\pi}{\dashrightarrow} & \sqcup & & & & \sqcup \\
 & & \mathcal{D}^{3,3,10} & & \overset{\widetilde{\mathfrak{p}}}{\underset{\text{bir.}}{\dashrightarrow}} & & \mathcal{D}_{22}.
\end{array}
$$

Here $\mathcal{M}$ is the GIT quotient moduli space, $\mathcal{M}_{22}^{(2)}$ is the moduli space of polarized hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type with a polarization of square 22 and divisibility 2, and $\mathcal{P}$ is the corresponding period domain; π is the GIT quotient map, $\mathfrak{m}$ is the modular map given by the Debarre–Voisin construction, and $\mathfrak{p}$ is the period map; the rational map $\mathfrak{p} \circ \mathfrak{m}$ can be extended to codimension 1, which we denote by $\widetilde{\mathfrak{p}}$ and refer to as the extended period map.

We proceed to study the Hodge structures on the three varieties X_1^σ, X_3^σ, and X_6^σ. Since X_6^σ is a polarized hyperkähler fourfold, the primitive cohomology group $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}}$ equipped with the Beauville–Bogomolov–Fujiki form q is of great interest to us. On the two Fano varieties X_1^σ and X_3^σ, one can define the vanishing cohomologies $H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}$ and $H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}}$, both lying in the middle degree so are equipped with the intersection product. Our main result is that one can obtain Hodge isometries among these three pieces of Hodge structures using algebraic correspondences (see Theorem 4.1.2). And as a corollary, we obtain the integral Hodge conjecture for $H^*(X_1^\sigma, \mathbf{Z})$ and $H^*(X_3^\sigma, \mathbf{Z})$ (see Corollary 4.1.3). The proof of the Hodge isometries however involves the study of one of the Heegner divisors, which is discussed in the next section.

Finally, we study in detail the geometry of the three varieties when σ belongs to the three divisors. We show that all three divisors are mapped to some Heegner divisors via the period map so they are special in the sense of Hassett. The Heegner divisor $\mathcal{D}_{28}$ is closely related to Lagrangian planes contained in X_6^σ (see Theorem 4.6.3), and we will use the geometry to prove the Hodge isometries. For the Heegner divisor $\mathcal{D}_{24}$, we construct a Brauer-twisted K3 surface (S,β) of degree 22, and show that a general member X_6^σ in the family is isomorphic to a moduli space of sheaves on S. We will also analyze the singularity for a general member of $\mathcal{D}_{22}$ in Section 4.8.

In **Chapter 5**, we study a special Debarre–Voisin fourfold with a large automorphism group, using the general results obtained in Chapter 4. We have gathered the results in Theorem 5.1.1. Namely, using the representation theory of the simple group $\mathbf{G} := \mathrm{PSL}(2,\mathbf{F}_{11})$ of order 660, we construct a $\mathbf{G}$-invariant trivector σ_0. We verify that the associated Debarre–Voisin $X_6^{\sigma_0}$ is a smooth fourfold, while the associated $X_1^{\sigma_0}$ admits 55 isolated singular points. Then we construct 55 distinct divisors on $X_6^{\sigma_0}$ generating the Picard lattice of rank 21. This allows us to show that the group $\mathrm{Aut}^s_H(X_6^{\sigma_0})$ of symplectic automorphisms fixing the polarization is isomorphic to $\mathbf{G}$.

Part I

Generalities on hyperkähler manifolds

CHAPTER 1

Basic results

Let X be a compact hyperkähler manifold of dimension $2n$. We recall the definitions of some interesting objects that one can associate with X, as well as some important general results that we will need.

1.1. Beauville–Bogomolov–Fujiki form

1.1.1. One of the most important objects in the study of hyperkähler manifolds is the second integral cohomology group, which is equipped with a natural quadratic form.

THEOREM 1.1.1 (Beauville, Bogomolov, Fujiki). *Let X be a compact hyperkähler manifold of dimension $2n$. There exist a unique integral primitive quadratic form q_X on $H^2(X,\mathbf{Z})$ and a unique positive rational constant C_X such that*

$$\forall \beta \in H^2(X,\mathbf{Z}) \qquad \int_X \beta^{2n} = C_X \cdot q_X(\beta)^n.$$

The form q_X is called the Beauville–Bogomolov–Fujiki form *of X (BBF form for short). It is of signature $(3, b_2 - 3)$ and satisfies $q_X(\omega) > 0$ for any Kähler class $\omega \in H^{1,1}(X,\mathbf{R})$ on X. The constant C_X is called the* Fujiki constant *of X.*

The isomorphism class of the lattice $\big(H^2(X,\mathbf{Z}), q_X\big)$ and the Fujiki constant C_X are both deformation invariants of X. It is also common to normalize the Fujiki constant C_X by letting $C_X = (2n-1)!! \cdot c_X$, and we will refer to c_X as the *small* Fujiki constant. For example, it is known that $c_{\mathrm{K3}^{[n]}} = 1$ and $c_{\mathrm{Kum}_n} = n+1$. We will frequently write q instead of q_X for the Beauville–Bogomolov–Fujiki form if no confusion can arise. The form q satisfies the following basic properties.

PROPOSITION 1.1.2. *Let X be a hyperkähler manifold with a holomorphic symplectic form $\sigma \in H^{2,0}(X)$ and the Beauville–Bogomolov–Fujiki form $q = q_X$.*

- *The Hodge decomposition*

  $$H^2(X,\mathbf{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

  is orthogonal with respect to q.
- *We have $q(\sigma,\sigma) = 0$ and $q(\sigma,\bar{\sigma}) > 0$.*

As already mentioned in the introduction, there are only very few known examples of hyperkähler manifolds. Below we give the descriptions of the lattice structure on $H^2(X,\mathbf{Z})$ for all the known examples. The cases of $\mathrm{K3}^{[n]}$ and Kum_n are studied by Beauville in [**Bea83**], and OG_6 and OG_{10} by Rapagnetta in [**Rap08**].

	dim	$b_2(X)$	$\Lambda = H^2(X,\mathbf{Z})$	$D(\Lambda)$	c_X	C_X
K3	2	22	$U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$	0	1	1
$\mathrm{K3}^{[n]}$	$2n$	23	$\Lambda_{\mathrm{K3}} \oplus \langle -(2n-2) \rangle$	$\mathbf{Z}/(2n-2)\mathbf{Z}$	1	$\frac{(2n)!}{2^n n!}$
Kum_n	$2n$	7	$U^{\oplus 3} \oplus \langle -(2n+2) \rangle$	$\mathbf{Z}/(2n+2)\mathbf{Z}$	$n+1$	$\frac{(2n)!}{2^n n!}(n+1)$
OG_6	6	8	$U^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2}$	$(\mathbf{Z}/2\mathbf{Z})^2$	4	60
OG_{10}	10	24	$\Lambda_{\mathrm{K3}} \oplus \left(\begin{smallmatrix} -6 & 3 \\ 3 & -2 \end{smallmatrix}\right)$	$\mathbf{Z}/3\mathbf{Z}$	1	945

TABLE 1. Lattice structure on $H^2(X,\mathbf{Z})$ for known deformation types

It is conjectured that hyperkähler manifolds are *bounded*, that is, in each dimension there are only finitely many deformation types (see [**Bea11**]). Since a lot of information for a hyperkähler manifold is encoded in its second cohomology group, a first step towards boundedness results for hyperkähler manifolds would be to find an upper bound for the second Betti number. In dimension 4, we have such a bound by results of Guan [**Gua01**].

THEOREM 1.1.3 (Guan). *We have $b_2(X) \leq 23$ for all hyperkähler fourfolds X. The bound is sharp and is attained by* $\mathrm{K3}^{[2]}$. *Moreover, if $b_2(X) < 23$, then in fact $b_2(X) \leq 8$.*

Starting from dimension 6, no upper bounds for b_2 are currently known (see [**Saw15, KL20**] for some conjectural results). In Section 2.1, we will present a conditional bound.

1.1.2. Generalized Fujiki constants. The following result was obtained by Fujiki [**Fuj87**] and Huybrechts [**Huy99**]. It is a generalization of the property of the Beauville–Bogomolov–Fujiki form (compare with Theorem 1.1.1).

THEOREM 1.1.4 (Fujiki, Huybrechts). *Let X be a compact hyperkähler manifold of dimension $2n$, and let $\alpha \in H^{4k}(X,\mathbf{R})$ be a class that remains of type $(2k,2k)$ on all small deformations of X (for example, all characteristic classes satisfy this condition). Then there exists a constant $C(\alpha) \in \mathbf{R}$, called the* generalized Fujiki constant of α, *such that*

$$\forall \beta \in H^2(X,\mathbf{R}) \qquad \int_X \alpha \cdot \beta^{2n-2k} = C(\alpha) \cdot q_X(\beta)^{n-k}.$$

REMARK 1.1.5.

- This generalizes the usual Fujiki constant C_X: we have $C(1_X) = C_X$.
- It also generalizes characteristic numbers. In fact, for any class α of top degree, we have $C(\alpha) = \int_X \alpha$. This in particular includes all products of Chern classes in top degree, and similarly for products of Chern characters.
- We have $C(c_2) > 0$. This positivity is explained by results from differential geometry. Namely, for a Kähler manifold X of dimension m with trivial canonical bundle, one can choose a Ricci flat metric and obtain the following pointwise relation
$$8\pi^2\, c_2\, \omega^{m-2} = \frac{\|R\|^2}{m(m-1)}\, \omega^m,$$
where ω is the Kähler form and R is the curvature tensor (see for example [**Tos17**, Lemma 2.6] for a detailed proof of this identity). By taking $\frac{\omega^m}{m!}$ as the volume form and integrating over X, we get
$$\int_X c_2 \cdot \omega^{m-2} = \frac{(m-2)!\,\|R\|^2}{8\pi^2}.$$
Hence for a hyperkähler manifold X, using the Fujiki relations and the fact that R cannot be everywhere zero, we see that $C(c_2) > 0$.[1] Equivalently, we have $C(\mathrm{ch}_2) = C(-c_2) < 0$.
- There are extra relations that the generalized Fujiki constants of a hyperkähler manifold must satisfy. We will see such examples in Section 1.1.3.

Denote by $\mathfrak{q} \in H^4(X, \mathbf{Q})$ the dual of the Beauville–Bogomolov–Fujiki form. We compute the generalized Fujiki constants for the powers $\mathfrak{q}^k$.

Proposition 1.1.6. *Let X be a hyperkähler manifold of dimension $2n$ with second Betti number $b := b_2(X)$. For each $\alpha \in H^{4k}(X, \mathbf{R})$ that remains of type $(2k, 2k)$ on all small deformations of X, we have*
$$C(\mathfrak{q} \cdot \alpha) = \frac{b + 2n - 2k - 2}{2n - 2k - 1} C(\alpha).$$
In particular, we get
$$C(\mathfrak{q}^k) = \frac{b + 2n - 2k}{1 + 2n - 2k} C(\mathfrak{q}^{k-1}) = \prod_{i=1}^{k} \frac{b + 2n - 2i}{1 + 2n - 2i} \cdot C(1).$$

Proof. Take a basis $(e_1, \ldots, e_b)$ of $H^2(X, \mathbf{R})$ such that
$$\mathfrak{q} = e_1^2 + e_2^2 + e_3^2 - e_4^2 - \cdots - e_b^2.$$
Writing $s_i := q_X(e_i) \in \{\pm 1\}$, we have
$$\begin{aligned} C(\mathfrak{q} \cdot \alpha) = \int_X \mathfrak{q} \cdot \alpha \cdot e_1^{2n-2k-2} &= \int_X \alpha \cdot (e_1^{2n-2k} + e_1^{2n-2k-2} e_2^2 + \cdots - e_1^{2n-2k-2} e_b^2) \\ &= C(\alpha) + \sum_{i>1} s_i \int_X \alpha \cdot e_1^{2n-2k-2} e_i^2. \end{aligned}$$

[1] More precisely, we have in this case $C(1) = \frac{\int_X \omega^{2n}}{q(\omega)^n} = \frac{(2n)!\,\mathrm{vol}\,X}{q(\omega)^n}$ and $C(c_2) = \frac{(2n-2)!\,\|R\|^2}{8\pi^2 q(\omega)^{n-1}}$.

For each term $e_1^{2n-2k-2}e_i^2$, consider the function

$$t \longmapsto \int_X \alpha \cdot (e_1 + te_i)^{2n-2k} = C(\alpha) \cdot (1 + t^2 s_i)^{n-k},$$

which is a polynomial in t. Comparing the coefficient before t^2, we get the polarized Fujiki relation

$$\binom{2n-2k}{2} \int_X \alpha \cdot e_1^{2n-2k-2} e_i^2 = C(\alpha) \cdot (n-k) s_i.$$

So we have

$$C(\mathfrak{q} \cdot \alpha) = C(\alpha) + \sum_{i>1} s_i \frac{C(\alpha) s_i}{2n-2k-1} = C(\alpha) + (b-1) \frac{C(\alpha)}{2n-2k-1} = \frac{b + 2n - 2k - 2}{2n-2k-1} C(\alpha),$$

where we used the fact that $s_i^2 = 1$. □

EXAMPLE 1.1.7.

- For X of K3$^{[2]}$-type, we have $C(1) = 3$ and $b_2 = 23$, hence $C(\mathfrak{q}) = 25$ and $C(\mathfrak{q}^2) = 575$ (these are first computed by O'Grady in [**O'G08**]).
- For X of Kum_2-type, we have $C(1) = 9$ and $b_2 = 7$, hence $C(\mathfrak{q}) = 27$ and $C(\mathfrak{q}^2) = 189$.

1.1.3. Riemann–Roch polynomial. Let $\mathrm{td} \coloneqq \mathrm{td}_X$ be the Todd class of X and let $\mathrm{td}_{2k} \in H^{4k}(X, \mathbf{Q})$ be its degree-$2k$ part (here we are referring to the degree in the Chow ring to be consistent with the notation for Chern classes). The *Riemann–Roch polynomial* of X is defined as

$$\begin{aligned}\mathrm{RR}_X(q) &\coloneqq \sum_{i=0}^{n} \frac{C(\mathrm{td}_{2n-2i})}{(2i)!} q^i \\ &= \frac{C(1)}{(2n)!} q^n + \frac{C(\mathrm{td}_2)}{(2n-2)!} q^{n-1} + \cdots + \frac{C(\mathrm{td}_{2n})}{1}.\end{aligned}$$

The Hirzebruch–Riemann–Roch theorem, whence the name, together with the property of the generalized Fujiki constants assert that this polynomial satisfies

$$\mathrm{RR}_X(q_X(c_1(L))) = \chi(X, L)$$

for all line bundles L on X.

The Riemann–Roch polynomial is a very strong invariant for a hyperkähler manifold. Among all known examples, there are only two types of Riemann–Roch polynomials

$$\mathrm{RR}_{\mathrm{K3}^{[n]}}(q) = \binom{q/2 + n + 1}{n}, \quad \mathrm{RR}_{\mathrm{Kum}_n}(q) = (n+1) \binom{q/2 + n}{n},$$

see [**EGL01**, Lemma 5.1] and [**NW03**, Lemma 5.2]. Ríos Ortiz showed in [**RO20**] that O'Grady's sporadic examples satisfy $\mathrm{RR}_{\mathrm{OG}_{10}} = \mathrm{RR}_{\mathrm{K3}^{[5]}}$ and $\mathrm{RR}_{\mathrm{OG}_6} = \mathrm{RR}_{\mathrm{Kum}_3}$.

We also recall the following result of Nieper-Wißkirchen [**NW03**], which replaces the Todd class td_X with its square root $\mathrm{td}_X^{1/2}$. This is a generalization of the Hitchin–Sawon formula [**HS01**]. In particular, it produces many extra relations among generalized Fujiki constants. The proof uses the theory of Rozansky–Witten invariants (see also [**BS22**, Section 5.3] for a conceptual proof).

Theorem 1.1.8 (Nieper-Wißkirchen). *Let X be a hyperkähler manifold of dimension $2n$. Consider the polynomial*

$$\begin{aligned}\mathrm{RR}_{X,1/2}(q) &:= \sum_{i=0}^{n} \frac{C(\mathrm{td}^{1/2}_{2n-2i})}{(2i)!} q^i \\ &= \frac{C(1)}{(2n)!}q^n + \frac{C(\frac{1}{24}c_2)}{(2n-2)!}q^{n-1} + \frac{C(\frac{7}{5760}c_2^2 - \frac{1}{1440}c_4)}{(2n-4)!}q^{n-2} + \cdots + \frac{C(\mathrm{td}^{1/2}_{2n})}{1}.\end{aligned}$$

There exists a positive constant r_X such that this polynomial factorizes as

$$\mathrm{RR}_{X,1/2}(q) = C(\mathrm{td}^{1/2}_{2n})\left(1 + \frac{1}{2r_X}q\right)^n.$$

By comparing the first two coefficients, we can obtain the value of the constant r_X

$$r_X = \frac{(2n-1)C(c_2)}{24C(1)} = \frac{(2n-1)2^n n! C(c_2)}{24(2n)! c_X} \tag{1.1}$$

and also the Hitchin–Sawon formula[2]

$$C(\mathrm{td}^{1/2}_{2n}) = \frac{C(1)(2r_X)^n}{(2n)!} = c_X \frac{r_X^n}{n!},$$

where c_X is the small Fujiki constant. In the literature, the normalized quadratic form $\frac{1}{2r_X}q_X$ is usually denoted by λ_X.

By comparing the rest of the coefficients, we get the following equivalent result, which generalizes the Hitchin–Sawon formula to all degrees (*cf.* [**Bec21**, Lemma 3.3]).

Corollary 1.1.9. *Let X be a hyperkähler manifold of dimension $2n$ with $n \geq 2$. Let c_X be the small Fujiki constant and let r_X be the constant defined in* (1.1)*. Then the following holds for all $0 \leq k \leq n$*

$$C(\mathrm{td}^{1/2}_{2k}) = \frac{(2n-2k)!}{2^{n-k}(n-k)!} \cdot c_X \frac{r_X^k}{k!}.$$

1.2. Period maps and Torelli theorems

1.2.1. Local Torelli theorem. The deformation theory for a Kähler manifold X with trivial canonical bundle works very well thanks to the Bogomolov–Tian–Todorov theorem, which states that the deformations of X are unobstructed. More precisely, when X is hyperkähler, this means that the Kuranishi space $\mathrm{Def}(X)$ of deformations of X is smooth of dimension $h^{1,1}(X) = b_2(X) - 1$, and there is a universal family $\pi\colon \mathcal{X} \to \mathrm{Def}(X)$ where $\mathcal{X}_0 \simeq X$ (see [**Huy99**, Section 1.12]). Note that up to replacing $\mathrm{Def}(X)$ with a simply connected open neighborhood of 0, we may assume that the family π is a trivial topological fibration, so all fibers $\mathcal{X}_t$ have the same topology. In particular, the local

[2]We may use Remark 1.1.5 to recover the original Hitchin–Sawon formula $C(\mathrm{td}^{1/2}_{2n}) = \frac{\|R\|^{2n}}{(192n\pi^2)^n(\mathrm{vol}\,X)^{n-1}}$.

system $R^2\pi_*\mathbf{Z}$ is trivial, and we may use parallel transports to identify each $H^2(\mathcal{X}_t,\mathbf{Z})$ with $H^2(\mathcal{X}_0,\mathbf{Z}) \simeq H^2(X,\mathbf{Z})$.

We have the following local Torelli theorem, proved by Beauville in [**Bea83**, Théorème 5].

Theorem 1.2.1 (Local Torelli theorem). *Let X be a hyperkähler manifold and let $\mathcal{X} \to \mathrm{Def}(X)$ be the Kuranishi family of X. Consider the local period map*

$$\begin{aligned}\wp\colon \mathrm{Def}(X) &\longrightarrow \mathbf{P}(H^2(X,\mathbf{C}))\\ t &\longmapsto [H^{2,0}(\mathcal{X}_t)]\end{aligned}$$

Its image is contained in the period domain

$$\Omega := \left\{x \in \mathbf{P}(H^2(X,\mathbf{C})) \;\middle|\; q(x) = 0, q(x,\bar{x}) > 0\right\},$$

and the period map $\wp\colon \mathrm{Def}(X) \to \Omega$ is a local isomorphism.

Using the local Torelli theorem, we prove the following useful lemma concerning the uniqueness of the Beauville–Bogomolov–Fujiki form q_X.

Lemma 1.2.2. *Let X be a hyperkähler manifold. Using parallel transport operators, a quadratic form on $H^2(X,\mathbf{Q})$ can be transported to $H^2(X',\mathbf{Q})$ for any small deformation X' of X. Then up to multiplying by a scalar, the Beauville–Bogomolov–Fujiki form q_X is the unique quadratic form q on $H^2(X,\mathbf{Q})$ satisfying the following property:*

for any small deformation X' of X, $H^{2,0}(X')$ is orthogonal to $H^{2,0}(X') \oplus H^{1,1}(X')$ with respect to q.

We have a similar uniqueness result for quadratic forms on $H^2(X,\mathbf{Q})_{\mathrm{prim}}$ for a polarized hyperkähler manifold (X,H), where we consider all small deformations of the pair (X,H) instead.

Proof. Clearly the Beauville–Bogomolov–Fujiki form q itself satisfies the desired property by Proposition 1.1.2.

We prove the first uniqueness statement. Let q' be another quadratic form satisfying the desired property. We need to show that there exists λ such that $q' = \lambda q$. Let $\sigma \in H^{2,0}(X)$ be the class of a holomorphic symplectic form on X. Then σ is orthogonal to $H^{2,0}(X)\oplus H^{1,1}(X)$ with respect to both q and q' by assumption. Since we also have $q(\sigma,\bar{\sigma}) > 0$, we can consider the following number

$$\lambda = \frac{q'(\sigma,\bar{\sigma})}{q(\sigma,\bar{\sigma})}.$$

We see that σ is also orthogonal to $\bar{\sigma}$ with respect to the form $q' - \lambda q$, therefore σ lies in the kernel $\ker(q' - \lambda q)$. In the projective setting, we write K for the kernel of the form $q' - \lambda q$, which is a linear subspace of $\mathbf{P}(H^2(X,\mathbf{C}))$. We see that K contains the period point $[H^{2,0}(X)]$. Using parallel transport operators, the value of λ remains constant on all small deformations of X. Since the properties of q and q' also hold for all small deformations of X, the subspace K must contain all the period points $[H^{2,0}(X')]$. By the local Torelli theorem, these period points form an analytic open subset U of the period domain Ω, which is itself a dense open subset inside the smooth quadric in $\mathbf{P}(H^2(X,\mathbf{C}))$ defined by q. Since the quadratic form q is non-degenerate, the open subset U cannot be contained in a hyperplane, so K coincides with the entire $\mathbf{P}(H^2(X,\mathbf{C}))$. In other words, the form $q' - \lambda q$ is identically zero on $H^2(X,\mathbf{C})$, so we may conclude that $q' = \lambda q$.

The uniqueness in the polarized case follows from the same argument. □

1.2.2. Global Torelli theorem. We can glue together the local period maps introduced above to construct a global period map. More precisely, we know that the isomorphism class of the lattice $\big(H^2(X,\mathbf{Z}), q_X\big)$ is a deformation invariant of X. So for a fixed deformation type, we pick one fixed lattice Λ and consider *marked pairs* (X,η) consisting of a hyperkähler manifold X of the given deformation type, and an isometry $\eta\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} \Lambda$, which is called a *marking* of X. Denote by $\mathcal{M}_{\text{marked}}$ the moduli space for marked hyperkähler manifolds (X,η) of the given deformation type. On each connected component $\mathcal{M}^0_{\text{marked}}$ of the moduli space $\mathcal{M}_{\text{marked}}$, we obtain the following period map

$$\begin{aligned} \wp^0_{\text{marked}}\colon \mathcal{M}^0_{\text{marked}} &\longrightarrow \Omega_{\text{marked}} \\ (X,\eta) &\longmapsto [\eta(H^{2,0}(X))] \end{aligned}$$

where

$$\Omega_{\text{marked}} \coloneqq \{[x] \in \mathbf{P}(\Lambda_{\mathbf{C}}) \mid q(x) = 0, q(x,\bar{x}) > 0\}$$

is the same period domain that we have seen above. We have the following global Torelli theorem, proved by Verbitsky in [**Ver13**] (see also [**Mar11**, Theorem 2.2]).

THEOREM 1.2.3 (Verbitsky). *The period map $\wp^0_{\text{marked}}$ is surjective, generically injective, and identifies pairwise inseparable points.*

There is also a version of the period map and the global Torelli theorem for polarized hyperkähler manifolds. We will discuss them in details in Chapter 3.

1.3. Looijenga–Lunts–Verbitsky algebra

The rational second cohomology group $H^2(X,\mathbf{Q})$ is also equipped with the Beauville–Bogomolov–Fujiki form. Following [**GKLR21**], we consider the quadratic vector space

$$(V,\widetilde{q}) \coloneqq \big(H^2(X,\mathbf{Q}) \oplus \mathbf{Q}^2, q_X \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\big).$$

This quadratic space is known as the *Mukai completion*, and is often also denoted by $\widetilde{H}(X,\mathbf{Q})$. Let $h \in \operatorname{End} H^*(X,\mathbf{Q})$ be the cohomological degree operator defined by

$$h|_{H^k(X,\mathbf{Q})} = (k - \dim X)\operatorname{Id},$$

such that the degrees are centered at the middle cohomology.

The *Looijenga–Lunts–Verbitsky algebra* $\mathfrak{g} := \mathfrak{g}(X)$ is the subalgebra of $\operatorname{End} H^*(X, \mathbf{Q})$ generated by $\mathfrak{sl}_2$-triples (L_a, h, Λ_a) for all classes $a \in H^2(X, \mathbf{Q})$ satisfying the hard Lefschetz property. Looijenga–Lunts [**LL97**] and Verbitsky [**Ver96**] determined the structure of this Lie algebra.

Theorem 1.3.1 (Looijenga–Lunts, Verbitsky).

(1) The Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(V, \widetilde{q})$.
(2) Consider the adjoint action of the operator h on $\mathfrak{g}$, then we have an eigenspace decomposition $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$. In particular, the action of an element in $\mathfrak{g}_0$ preserves the cohomological degree.
(3) We have a decomposition $\mathfrak{g}_0 = \mathfrak{g}_0' \oplus \mathbf{Q}h$ where $\mathfrak{g}_0' := [\mathfrak{g}_0, \mathfrak{g}_0]$. The Lie subalgebra $\mathfrak{g}_0'$ is isomorphic to $\mathfrak{so}(H^2(X, \mathbf{Q}), q_X)$.

The cohomology $H^*(X, \mathbf{Q})$ is naturally a $\mathfrak{g}$-module by construction. Due to the semisimplicity of the Lie algebra $\mathfrak{g}$, the cohomology splits into a direct sum of irreducible $\mathfrak{g}$-submodules V_λ

$$H^*(X, \mathbf{Q}) \simeq \bigoplus_\lambda V_\lambda^{\oplus m_\lambda},$$

where $m_\lambda \in \mathbf{N}$ are the multiplicities of the components. We call this the *LLV decomposition*. The LLV decompositions for all known examples are determined in [**GKLR21**].

One natural component of the LLV decomposition is the subalgebra of $H^*(X, \mathbf{Q})$ generated by $H^2(X, \mathbf{Q})$. It is denoted by $\mathrm{SH}(X, \mathbf{Q})$ and is referred to as the *Verbitsky component*. Verbitsky [**Ver96**] and Bogomolov [**Bog96**] have determined its ring structure.

Theorem 1.3.2 (Verbitsky, Bogomolov). *Consider the subalgebra $\mathrm{SH}(X, \mathbf{Q})$ of $H^*(X, \mathbf{Q})$ generated by $H^2(X, \mathbf{Q})$. It is an irreducible $\mathfrak{g}$-module and we have an isomorphism of algebras*

$$\mathrm{SH}(X, \mathbf{Q}) \simeq \operatorname{Sym}^* H^2(X, \mathbf{Q}) / \langle \alpha^{n+1} \mid q_X(\alpha) = 0 \rangle.$$

Example 1.3.3. In the $\mathrm{K3}^{[2]}$-type case, by comparing the dimensions, we see that the Verbitsky component coincides with the entire cohomology $\mathrm{SH}(X, \mathbf{Q}) = H^*(X, \mathbf{Q})$.

Recall that $\mathfrak{q} \in H^4(X, \mathbf{Q})$ is the dual of the Beauville–Bogomolov–Fujiki form, so it lies in $\operatorname{Sym}^2 H^2(X, \mathbf{Q}) \simeq \mathrm{SH}^4(X, \mathbf{Q})$. For a very general compact hyperkähler manifold X, the special Mumford–Tate algebra of the Hodge structure on the cohomology $H^*(X, \mathbf{Q})$ is isomorphic to $\mathfrak{g}_0' \simeq \mathfrak{so}(H^2(X, \mathbf{Q}), q_X)$ [**GKLR21**, Proposition 2.38]. Using the branching rules from $\mathfrak{g}$ to $\mathfrak{g}_0'$ (see for example Appendix B.2.1 of *loc. cit.*), one can verify that for such X, the only Hodge classes in $\mathrm{SH}(X, \mathbf{Q})$ are multiples of the powers $\mathfrak{q}^k \in \mathrm{SH}^{4k}(X, \mathbf{Q})$ for $0 \leq k \leq n$.

CHAPTER 2

Numerical aspects

In this chapter, we study some numerical aspects of hyperkähler manifolds. First, we produce a conditional upper bound on the second Betti number, in terms of the *generalized Fujiki constants*, or equivalently, in terms of the *Riemann–Roch polynomial*. Then we study the cohomology class of a Lagrangian plane. We deduce a formula for the projection of the cohomology class to the Verbitsky component. We also propose a conjectural formula for the full class in the $\mathrm{K3}^{[n]}$-type case, which is verified for n up to 6.

2.1. Second Chern class and Riemann–Roch polynomial

In this section, we consider the question of whether the second Chern class $c_2 \in H^4(X, \mathbf{Z})$ lies in the Verbitsky component $\mathrm{SH}^4(X)$. Since c_2 is necessarily a Hodge class of type $(2,2)$, this is the same as asking whether c_2 is a multiple of $\mathfrak{q} \in H^4(X, \mathbf{Q})$, the dual of the Beauville–Bogomolov–Fujiki form.

It turns out that the answer to this question provides an inequality involving the second Betti number b_2, which leads to a conditional bound on b_2.

This is a joint work with Thorsten Beckmann, and has appeared in [**BS22**].

2.1.1. Second Chern class.

We state the main result.

Theorem 2.1.1. *Let X be a hyperkähler manifold of dimension $2n$ for $n \geq 2$, with second Betti number $b_2(X)$. If $C(c_2^2) > 2C(c_4)$ or equivalently, $C(\mathrm{ch}_4) > 0$, then we have the inequality*

$$b_2(X) \leq \frac{10}{\dfrac{C(c_2^2)}{C(c_4)} - 2} - 2n + 9, \tag{2.1}$$

with equality holds if and only if c_2 lies in the Verbitsky component $\mathrm{SH}^4(X)$, or equivalently, if c_2 is a multiple of $\mathfrak{q}$. If $C(c_2^2) \leq C(c_4)$, then c_2 is not contained in $\mathrm{SH}^4(X)$.

Note that when $n = 2$, the inequality $C(c_2^2) > 2C(c_4)$ holds *a posteriori* by the bound on b_2 of Guan (see [**OSV21**, Lemma 4.6] and [**Saw21**, Theorem 7]). It is conjectured that this inequality holds in all dimensions.

Proposition 2.1.2. *For a hyperkähler fourfold X, the inequality $C(c_2^2) > 2C(c_4)$ holds. Equivalently, we have $C(\mathrm{ch}_4) > 0$.*

PROOF. We first note that $C(c_2^2)$ and $C(c_4)$ are just Chern numbers, and $C(c_4) = \chi_{\mathrm{top}}(X)$ gives the topological Euler characteristic of X. By Hirzebruch–Riemann–Roch formula, we have

$$3 = \chi(\mathcal{O}_X) = \int_X \mathrm{td}_X = \frac{1}{240}C(c_2^2) - \frac{1}{720}C(c_4).$$

Therefore the desired inequality is equivalent to $C(c_4) < 432$. By Salamon's relation on the Betti numbers [**Sal96**], we have

$$b_3 + b_4 = 46 + 10b_2.$$

So we may compute that

$$C(c_4) = \chi_{\mathrm{top}}(X) = 2 + 2b_2 - 2b_3 + b_4 \leq 2 + 2b_2 + b_3 + b_4 = 48 + 12b_2 \leq 324,$$

where in the last inequality we used the bound $b_2 \leq 23$ by Guan. □

The key step to Theorem 2.1.1 is the following inequality.

PROPOSITION 2.1.3. *Let X be a hyperkähler manifold of dimension $2n$ for $n \geq 2$. We have the following inequality*

$$C(c_2^2) \geq \frac{C(c_2)^2}{C(\mathfrak{q})^2}C(\mathfrak{q}^2), \tag{2.2}$$

where equality holds if and only if $c_2 \in \mathrm{SH}^4(X)$.

PROOF. We write

$$c_2 = a\mathfrak{q} + z \quad \text{where } a \in \mathbf{R}, z \in \mathrm{SH}(X, \mathbf{R})^{\perp}.$$

In other words, we project c_2 orthogonally to the Verbitsky component and let $a\mathfrak{q}$ be its image. Then we have

$$C(c_2) = C(a\mathfrak{q}), \quad \text{so } a = \frac{C(c_2)}{C(\mathfrak{q})}.$$

Now we consider the square $c_2^2 = a^2\mathfrak{q}^2 + 2a\mathfrak{q}z + z^2 \in H^8(X, \mathbf{R})$. Since the class z is in $\mathrm{SH}(X, \mathbf{R})^{\perp}$, it is orthogonal to the image of $\mathrm{Sym}^{2n-2} H^2(X, \mathbf{R})$, so the class $\mathfrak{q}z$ is orthogonal to the image of $\mathrm{Sym}^{2n-4} H^2(X, \mathbf{R})$ and also lies in $\mathrm{SH}(X, \mathbf{R})^{\perp}$.

On the other hand, for any Kähler class $\omega \in H^2(X, \mathbf{R})$, since z lies $\mathrm{SH}(X, \mathbf{R})^{\perp}$, the class $z \cdot \omega^{2n-3} \in H^{4n-2}(X, \mathbf{R})$ is orthogonal to the entire $H^2(X, \mathbf{R})$ hence must vanish. So the class z is primitive of type $(2, 2)$ with respect to all Kähler classes on X. By the Hodge–Riemann bilinear relations, for a Kähler class $\omega \in H^2(X, \mathbf{R})$ we have

$$\int_X z^2 \cdot \omega^{2n-4} \geq 0, \quad \text{hence } C(z^2) \geq 0,$$

where equality holds if and only if $z = 0$. In other words, the projection of z^2 to the Verbitsky component is non-trivial, unless z is itself trivial. Therefore we obtain the desired inequality

$$C(c_2^2) = a^2C(\mathfrak{q}^2) + C(z^2) \geq a^2C(\mathfrak{q}^2) = \frac{C(c_2)^2}{C(\mathfrak{q})^2}C(\mathfrak{q}^2),$$

where equality holds if and only if $c_2 \in \mathrm{SH}^4(X)$. □

Note that the inequality (2.2) implies that $C(c_2^2)$ is always positive.

PROOF OF THEOREM 2.1.1. Using Proposition 1.1.6, we can compute the values of $C(\mathfrak{q})$ and $C(\mathfrak{q}^2)$ and insert them into (2.2). We get

$$C(c_2^2) \geq \frac{(2n-1)(b_2+2n-4)C(c_2)^2}{(2n-3)(b_2+2n-2)C(1)}. \tag{2.3}$$

On the other hand, by Corollary 1.1.9, we have the following relation in degree 4

$$7C(c_2^2) - 4C(c_4) = \frac{5(2n-1)C(c_2)^2}{(2n-3)C(1)}. \tag{2.4}$$

We combine (2.3) and (2.4) to obtain

$$C(c_2^2) \geq \frac{b_2+2n-4}{5(b_2+2n-2)}\big(7C(c_2^2) - 4C(c_4)\big),$$

which is equivalent to

$$\big(C(c_2^2) - 2C(c_4)\big)(b_2+2n-9) \leq 10C(c_4).$$

Hence if $C(c_2^2) > 2C(c_4)$, then we can divide by $C(c_2^2) - 2C(c_4)$ and obtain the bound in (2.1).

By Proposition 2.1.3, we know that $c_2 \in \mathrm{SH}^4(X)$ if and only if equality holds in the above inequality. We show that in this case we must have $C(c_2^2) > 2C(c_4)$. By Proposition 2.1.2, we may assume that $n \geq 3$.

- Suppose that $C(c_2^2) = 2C(c_4)$, then we get $C(c_4) = 0$ so $C(c_2^2) = 0$ as well, which would contradict the positivity of $C(c_2^2)$ shown in (2.2).
- Suppose that $C(c_2^2) < 2C(c_4)$, then again by the positivity of $C(c_2^2)$, we get $C(c_4) > 0$. This means that $b_2 + 2n - 9 < 0$, which is impossible since $b_2 \geq 3$ and $n \geq 3$.

Therefore we may conclude that $c_2 \in \mathrm{SH}^4(X)$ if and only if $C(c_2^2) > 2C(c_4)$ and the upper bound for b_2 is attained. □

The condition for c_2 to be contained inside the Verbitsky component also gives an equivalent condition for $\mathrm{td}^{1/2}_{2n-2}$ to lie inside $\mathrm{SH}(X, \mathbf{R})$, by the following result.

PROPOSITION 2.1.4. *For a hyperkähler manifold X of dimension $2n$, we have* $\mathrm{td}^{1/2}_{2k} \in \mathrm{SH}(X, \mathbf{R})$ *if and only if* $\mathrm{td}^{1/2}_{2n-2k} \in \mathrm{SH}(X, \mathbf{R})$. *Moreover, for $k' < k \leq n$, the fact* $\mathrm{td}^{1/2}_{2k} \in \mathrm{SH}(X, \mathbf{R})$ *implies that* $\mathrm{td}^{1/2}_{2k'} \in \mathrm{SH}(X, \mathbf{R})$.

Proof. For a class $\alpha \in H^2(X, \mathbf{C})$, denote by $L_\alpha \in \mathfrak{g}(X)_{\mathbf{C}}$ the operator $x \mapsto x \cdot \alpha$. Define h_p to be the holomorphic grading operator that acts on $H^{p,q}(X)$ as $(p-n)\,\mathrm{Id}$ (which is denoted by Π in [**Jia20**]). Recall that for the class σ of a symplectic form, the operator L_σ has the Lefschetz property with respect to the grading given by h_p: there exists a dual Lefschetz operator $\Lambda_\sigma \in \mathfrak{g}(X)_{\mathbf{C}}$, such that together with the operator h_p, we get an $\mathfrak{sl}_2$-triple $(L_\sigma, h_p, \Lambda_\sigma)$ in the LLV algebra $\mathfrak{g}(X)_{\mathbf{C}}$.

Jiang showed in [**Jia20**, Corollary 3.19] that there exists a constant $\mu = \mu_\sigma \in \mathbf{R}_{>0}$ such that

$$\Lambda_\sigma(\mathrm{td}^{1/2}_{2k}) = \mu\,\mathrm{td}^{1/2}_{2k-2}\cdot[\bar\sigma] = \mu L_{\bar\sigma}(\mathrm{td}^{1/2}_{2k-2}). \tag{2.5}$$

Furthermore, the operators $L_{\bar\sigma}$ and Λ_σ commute for degree reasons. Applying (2.5) repeatedly, we see that the following holds for all $k \le n/2$

$$\Lambda_\sigma^{n-2k}(\mathrm{td}^{1/2}_{2n-2k}) = \mu^{n-2k} L_{\bar\sigma}^{n-2k}(\mathrm{td}^{1/2}_{2k}) \in H^{2k,2n-2k}(X). \tag{2.6}$$

On the other hand, Fujiki showed in [**Fuj87**] that the operators $L_{\bar\sigma}$ and Λ_σ yield isomorphisms

$$L^a_{\bar\sigma}\colon H^{p,n-a}(X) \xrightarrow{\sim} H^{p,n+a}(X), \quad \Lambda^a_\sigma\colon H^{n+a,q}(X) \xrightarrow{\sim} H^{n-a,q}(X).$$

In particular, we have the following isomorphisms

$$H^{2n-2k,2n-2k}(X) \xrightarrow[\sim]{L_{\bar\sigma}^{n-2k}} H^{2k,2n-2k}(X) \xleftarrow[\sim]{\Lambda_\sigma^{n-2k}} H^{2k,2k}(X).$$

Moreover, these isomorphisms are compatible with the decomposition of $H^*(X, \mathbf{C})$ into irreducible $\mathfrak{g}(X)_{\mathbf{C}}$-representations. In other words, for each irreducible representation $V \subset H^*(X, \mathbf{C})$, the isomorphism $L^a_{\bar\sigma}$ restricts to an isomorphism

$$L^a_{\bar\sigma}\colon H^{p,n-a}(X) \cap V \xrightarrow{\sim} H^{p,n+a}(X) \cap V,$$

and similar for Λ^a_σ. Combining this with (2.6), we get the first assertion.

The second statement follows from a similar argument. If $\mathrm{td}^{1/2}_{2k}$ lies in the Verbitsky component, then so is the class

$$\Lambda_\sigma^{k-k'}(\mathrm{td}^{1/2}_{2k}) = \mu L_{\bar\sigma}^{k-k'}(\mathrm{td}^{1/2}_{2k'}) \in H^{2k',2k}(X).$$

The map $L_{\bar\sigma}^{k-k'}\colon H^{2k',2k'}(X) \to H^{2k',2k}(X)$ composed with $L_{\bar\sigma}^{n-k-k'}$ gives an isomorphism, so $L_{\bar\sigma}^{k-k'}$ itself is at least injective. Thus we may conclude that $\mathrm{td}^{1/2}_{2k'}$ also lies in the Verbitsky component. □

Corollary 2.1.5. *For a hyperkähler manifold X of dimension $2n$, the class $\mathrm{td}^{1/2}_{2n-2}$ lies in the Verbitsky component if and only if $C(c_2^2) > 2C(c_4)$ and equality holds in (2.1).*

2.1.2. Riemann–Roch polynomial.

We give an alternative formulation of the bound on b_2 in Theorem 2.1.1, in terms of the Riemann–Roch polynomial RR_X.

THEOREM 2.1.6. *Let X be a hyperkähler manifold of dimension $2n$ for $n \geq 2$, with second Betti number $b_2(X)$. Write the Riemann–Roch polynomial of X as*

$$\mathrm{RR}_X(q) = A_0 q^n + A_1 q^{n-1} + A_2 q^{n-2} + \cdots$$

Then $C(\mathrm{ch}_4) > 0$ if and only if $2nA_0A_2 < (n-1)A_1^2$. In this case, we have the inequality

$$b_2(X) \leq \left(1 - \frac{2nA_0A_2}{(n-1)A_1^2}\right)^{-1} - (2n-2), \tag{2.7}$$

with equality holds if and only if c_2 lies in the Verbitsky component $\mathrm{SH}^4(X)$.

This is an immediate consequence of the following lemma.

LEMMA 2.1.7. *All generalized Fujiki constants for characteristic classes of degree ≤ 4 are determined by the Riemann–Roch polynomial, or more precisely, by its first three coefficients*

$$\mathrm{RR}_X(q) = A_0 q^n + A_1 q^{n-1} + A_2 q^{n-2} + \cdots$$

Namely, we have

$$\begin{aligned} C(1) &= (2n)!A_0, \quad C(c_2) = 12(2n-2)!A_1, \\ C(c_2^2) &= 144(2n-4)!\left(4A_2 - \frac{(n-1)A_1^2}{nA_0}\right), \\ C(c_4) &= 144(2n-4)!\left(7A_2 - \frac{3(n-1)A_1^2}{nA_0}\right). \end{aligned}$$

PROOF. Clearly $C(1)$ and $C(c_2)$ appear as coefficients of the Riemann–Roch polynomial. For $C(c_2^2)$ and $C(c_4)$, we already have one linear relation (2.4)

$$7C(c_2^2) - 4C(c_4) = \frac{5(2n-1)C(c_2)^2}{(2n-3)C(1)} = 720(2n-4)!\frac{(n-1)A_1^2}{nA_0}.$$

The third coefficient of the Riemann–Roch polynomial gives another one

$$3C(c_2^2) - C(c_4) = 720(2n-4)!A_2,$$

which allows us to uniquely determine their values. Hence we get all four generalized Fujiki constants of degree ≤ 4. □

REMARK 2.1.8. Suppose that the Riemann–Roch polynomial factorizes as a product of linear factors

$$\mathrm{RR}_X(q) = A_0 \prod_i (q + \lambda_i).$$

It was shown in [**Jia20**] that all the coefficients of $\mathrm{RR}_X(q)$ are positive. Hence the λ_i must all be positive. If, moreover, we assume that the λ_i are not all equal, then the condition $2nA_0A_2 < (n-1)A_1^2$ is satisfied by Cauchy–Schwarz, and the inequality (2.7) can be written as

$$b_2(X) \leq \frac{n-1}{\dfrac{n\sum \lambda_i^2}{(\sum \lambda_i)^2} - 1} - (2n-2).$$

This is homogeneous with respect to the λ_i and measures in a certain sense the dispersion of the roots.

We now examine the bound (2.7) for the known deformation types of smooth hyperkähler manifolds. Recall from Section 1.1.3 that there are only two types of Riemann–Roch polynomial known.

EXAMPLE 2.1.9 ($\mathrm{K3}^{[n]}$-type). We compute the first three coefficients

$$\begin{aligned}\mathrm{RR}_{\mathrm{K3}^{[n]}}(q) &= \binom{q/2+n+1}{n}\\ &= \frac{1}{2^n n!}q^n + \frac{n+3}{2^n(n-1)!}q^{n-1} + \frac{3n^2+17n+26}{3\cdot 2^{n+1}(n-2)!}q^{n-2} + \cdots\end{aligned}$$

Then by inserting the values A_0, A_1, A_2 into (2.7), we get the following upper bound

$$b_2(X) \le n + 17 + \frac{12}{n+1}.$$

Alternatively, we could also have used Remark 2.1.8 to obtain the expression. When $n = 2$ or $n = 3$, it evaluates to 23 and is attained by $\mathrm{K3}^{[n]}$; when $n = 5$, it evaluates to 24 and is attained by OG_{10}.

EXAMPLE 2.1.10 (Kum_n-type). We compute similarly the first three coefficients

$$\begin{aligned}\mathrm{RR}_{\mathrm{Kum}_n}(q) &= (n+1)\binom{q/2+n}{n}\\ &= \frac{n+1}{2^n n!}q^n + \frac{(n+1)^2}{2^n(n-1)!}q^{n-1} + \frac{(n+1)^2(3n+2)}{3\cdot 2^{n+1}(n-2)!}q^{n-2} + \cdots\end{aligned}$$

and insert these three coefficients into (2.7). In this case, the upper bound we get is

$$b_2(X) \le n + 5.$$

When $n = 2$, it is attained by Kum_2; when $n = 3$ it is attained by OG_6.

We gather these examples in the following corollary.

COROLLARY 2.1.11. *Among all known examples of hyperkähler manifolds of dimension* $2n \ge 4$ *(that is,* $\mathrm{K3}^{[n]}$, Kum_n, OG_6, *and* OG_{10}*), the second Chern class* c_2 *of* X *lies in the Verbitsky component* $\mathrm{SH}^4(X)$ *if and only if* X *is one of the following types*

$$\mathrm{K3}^{[2]},\quad \mathrm{K3}^{[3]},\quad \mathrm{Kum}_2,\quad \mathrm{OG}_6,\quad \mathrm{OG}_{10}.$$

In these cases, we have the following relations

$$c_2(\mathrm{K3}^{[2]}) = \tfrac{6}{5}\mathfrak{q},\quad c_2(\mathrm{K3}^{[3]}) = \tfrac{4}{3}\mathfrak{q},$$
$$c_2(\mathrm{Kum}_2) = 2\mathfrak{q},\quad c_2(\mathrm{OG}_6) = 2\mathfrak{q},\quad c_2(\mathrm{OG}_{10}) = \tfrac{3}{2}\mathfrak{q}.$$

PROOF. The first statement is the direct consequence of Theorem 2.1.6. To obtain the coefficients, it suffices to compare the generalized Fujiki constants for c_2 and $\mathfrak{q}$, which are known for all these examples. (Alternatively, one can also compare the Chern number $\int_X c_2^n$ and the value of $C(\mathfrak{q}^n)$.) □

2.2. Lagrangian plane

One important question on hyperkähler manifolds asked by Hassett–Tschinkel is to determine the value of $q(\ell)$ when ℓ is the class of a line in a Lagrangian n-plane. In [**HT09a**, Thesis 1.1], they proposed that this value is a universal constant that only depends on the deformation type. Moreover, it should provide the lower bound for $q(\rho)$, where ρ is any primitive generator of an extremal ray of the cone of effective curves. Dually, the dual class $L \in H^2(X, \mathbf{Q})$ (more precisely, an integral multiple of it) gives the example of a wall divisor (see Section 3.4 for this notion). In the $\mathrm{K3}^{[n]}$-type case, Bakker gave a positive answer in [**Bak17**]: we always have $q(\ell) = -\frac{n+3}{2}$.

More generally, one can ask for a description of the cohomology class $[P]$ of the Lagrangian n-plane P. In the $\mathrm{K3}^{[n]}$-type case, this problem has been studied by several authors and was answered in lower dimensions for $n \leq 4$ [**HT09b, HHT12, BJ14**]. Moreover, Bakker showed that all the classes $[P]$ with primitive line class ℓ are in the same monodromy orbit, so there should exist a universal formula for $[P]$, but there were no candidates available starting from $n = 5$. The Kum_2-type case was treated in [**HT13**].

In this section,[1] we will first deduce a formula for the projection $\overline{[P]}$ of the class $[P]$ to the Verbitsky component (Theorem 2.2.4). Then we propose a conjectural formula for the full class $[P]$ in the $\mathrm{K3}^{[n]}$-type case (Conjecture 2.2.10). In the case where the line class ℓ is primitive, this formula can be verified via computer algebra for $n \leq 6$. Finally, we propose a conjectural behavior of the Mukai vector $v(L) := \exp(L)\,\mathrm{td}_X^{1/2}$ (Conjecture 2.2.12), which in particular would imply that $q(\ell) = -2r_X$, where r_X is the constant given by (1.1).

2.2.1. Setup. Let X be a hyperkähler manifold of dimension $2n$. By Poincaré duality, we have $H_2(X, \mathbf{Z}) \simeq H^2(X, \mathbf{Z})^\vee$. On the other hand, since $H^2(X, \mathbf{Z})$ is equipped with the Beauville–Bogomolov–Fujiki form q which is non-degenerate, we may identify $H^2(X, \mathbf{Z})^\vee$ as a subgroup of $H^2(X, \mathbf{Q})$. In particular, each curve class $\ell \in H_2(X, \mathbf{Z})$ can be seen as an element of $H^2(X, \mathbf{Q})$. To make the distinction, we will denote this class by L and refer to it as the *dual class* of ℓ. In other words, it is the unique class in $H^2(X, \mathbf{Q})$ satisfying

$$\forall \alpha \in H^2(X, \mathbf{Q}) \quad q(L, \alpha) = \ell \cdot \alpha.$$

We define $q(\ell)$ to be $q(L)$, the Beauville–Bogomolov–Fujiki square of the class L.

LEMMA 2.2.1. *Let X be a hyperkähler manifold of dimension $2n$ containing a k-plane $\mathbf{P}^k$. Let ℓ be the class of a line contained in $\mathbf{P}^k$ and let $L \in H^2(X, \mathbf{Q})$ be the dual class. For any $\alpha_i \in H^2(X, \mathbf{Q})$, we have*

$$\alpha_1 \cdots \alpha_k \cdot [\mathbf{P}^k] = q(\alpha_1, L) \cdots q(\alpha_k, L).$$

[1]The results of this section have benefited from inspiring discussions with Thorsten Beckmann and Georg Oberdieck, to whom I express my sincere gratitude.

PROOF. Denote by ι the inclusion $\iota\colon \mathbf{P}^k \hookrightarrow X$. Each restriction $\iota^*\alpha_i \in H^2(\mathbf{P}^k, \mathbf{Q})$ will be a multiple of the hyperplane class, and the degree is precisely the intersection number $\alpha_i \cdot \ell$, hence we have

$$\alpha_1 \cdots \alpha_k \cdot [\mathbf{P}^k] = \iota^*\alpha_1 \cdots \iota^*\alpha_k = (\alpha_1 \cdot \ell) \cdots (\alpha_k \cdot \ell) = q(\alpha_1, L) \cdots q(\alpha_k, L),$$

so we get the desired relation. □

From now on, we let $P \subset X$ be a Lagrangian n-plane and let ℓ be the class of a line in P and L the dual class. If c_2 lies in the Verbitsky component, we can use this lemma to deduce the value of $q(L)$.

COROLLARY 2.2.2. *Let $P \subset X$ be a Lagrangian n-plane and let ℓ be the class of a line in P and L the dual class. If we assume moreover that c_2 lies in the Verbitsky component so it is a multiple of $\mathfrak{q}$, then we have*

$$q(L) = -(n+1)\frac{\mathfrak{q}}{c_2}.$$

In particular, we may compute the value of $q(L)$ in the following cases.

	$\mathrm{K3}^{[2]}$	$\mathrm{K3}^{[3]}$	Kum_2	OG_6	OG_{10}
$q(L)$	$-\frac{5}{2}$	-3	$-\frac{3}{2}$	-2	-4

PROOF. By Lemma 2.2.1, for any $\alpha_i \in H^2(X, \mathbf{Q})$, we have the relation

$$\alpha_1 \cdots \alpha_n \cdot [P] = q(\alpha_1, L) \cdots q(\alpha_n, L).$$

If we replace α_{n-1} and α_n by the elements of a orthonormal basis of q, this gives

$$\alpha_1 \cdots \alpha_{n-2} \cdot \mathfrak{q} \cdot [P] = q(L) \cdot q(\alpha_1, L) \cdots q(\alpha_{n-2}, L).$$

On the other hand, using the normal sequence of $P \subset X$

$$0 \longrightarrow \mathcal{T}_P \longrightarrow \mathcal{T}_X|_P \longrightarrow \mathcal{N}_{P/X} \simeq \Omega_P \longrightarrow 0,$$

we get $c_2(\mathcal{T}_X)|_P = -(n+1)h^2$, so by a similar argument to Lemma 2.2.1, we have

$$\alpha_1 \cdots \alpha_{n-2} \cdot c_2 \cdot [P] = -(n+1) \cdot q(\alpha_1, L) \cdots q(\alpha_{n-2}, L).$$

Comparing the two relations, we get

$$q(L) = -(n+1)\frac{\mathfrak{q}}{c_2},$$

where the quotient makes sense if and only if c_2 is a multiple of $\mathfrak{q}$. To conclude for the five cases, we use the values from Corollary 2.1.11. □

We also state the following useful fact.

LEMMA 2.2.3. *Let $P \subset X$ be a Lagrangian n-plane, then we have*

$$[P]^2 = (-1)^n(n+1).$$

PROOF. Since P is Lagrangian, the normal bundle $\mathcal{N}_{P/X}$ is isomorphic to the cotangent bundle Ω_P, so

$$[P]^2 = \int_P c_n(\mathcal{N}_{P/X}) = \int_P c_n(\Omega_P) = (-1)^n(n+1),$$

which gives the self-intersection number of P. □

2.2.2. Projection to the Verbitsky component. We deduce an explicit formula for the projection $\overline{[P]}$ of the class of P to the Verbitsky component SH. The advantage of restricting to the Verbitsky component is that all information of a class can be retrieved by pairing it with classes in $H^2(X,\mathbf{Q})$, and everything can be explicitly computed using the Fujiki relations. Similar results have been obtained by Beckmann in [**Bec21**, Example 4.17].

THEOREM 2.2.4. *Let X be a hyperkähler manifold of dimension $2n$ containing a Lagrangian n-plane $P \simeq \mathbf{P}^n$. Let ℓ be the class of a line contained in P and let L be the dual class. Let $\overline{[P]} \in \mathrm{SH}^{2n}(X,\mathbf{Q})$ be the orthogonal projection of $[P]$ to the Verbitsky component. Then we have the formula*

$$\overline{[P]} = \overline{\left[\frac{\mu^n}{c_X}\exp(L/\mu)\,\mathrm{td}_X^{1/2}\right]_n} \quad \text{with } \mu = \sqrt{\frac{-q(L)}{2r_X}}. \tag{2.8}$$

Here $[-]_n$ means the degree-n (cohomological degree-$2n$) part of a class.

PROOF. By the result of Voisin [**Voi92**], since P is a Lagrangian subvariety, one may deform $P \subset X$ and assume that X is of Picard rank 1. In this case, all Hodge classes in the Verbitsky component are generated by L and $\mathfrak{q}$. In particular, the classes $L^n, \mathfrak{q}L^{n-2}, \mathfrak{q}^2L^{n-4}, \ldots$ form a $\mathbf{Q}$-linear basis for Hodge classes of type (n,n). Hence we can write $\overline{[P]}$ as a linear combination

$$\overline{[P]} = \mathfrak{a}_0 L^n + \mathfrak{a}_2 L^{n-2} + \mathfrak{a}_4 L^{n-4} + \cdots + \mathfrak{a}_{2k} L^{n-2k} + \cdots$$

where $\mathfrak{a}_{2k} \in \mathbf{Q}\mathfrak{q}^k \subset \mathrm{SH}^{4k}(X,\mathbf{Q})$. Moreover, each class $\mathfrak{a}_{2k}$ is uniquely determined by its generalized Fujiki constant $C(\mathfrak{a}_{2k})$.

We will show that the generalized Fujiki constants $C(\mathfrak{a}_{2k})$ are given by the following formula

$$C(\mathfrak{a}_{2k}) = \frac{(2n-2k)!}{2^n k!(n-2k)!(n-k)!}(-q(L))^k.$$

We will prove the formula by induction on k. Pick one class $\alpha \in H^2(X,\mathbf{R})$ such that $q(\alpha) = 0$ but $q(\alpha, L) \neq 0$. By Lemma 2.2.1, we have

$$q(\alpha, L)^n = \alpha^n \cdot [P] = \alpha^n(\mathfrak{a}_0 L^n + \mathfrak{a}_2 L^{n-2} + \cdots).$$

Using the polarized Fujiki relations and the fact that $q(\alpha) = 0$, we get

$$\alpha^n \cdot L^n = \frac{2^n}{\binom{2n}{n}} C(1) q(\alpha, L)^n,$$

while for $j \geq 1$ we have $\alpha^n \cdot \mathfrak{a}_{2j} L^{n-2j} = 0$. This shows that $C(\mathfrak{a}_0) = \frac{\binom{2n}{n}}{2^n}$, so we have proved the case for $k = 0$.

Now we assume the cases of $1, \ldots, k-1$ and prove the case of k. We again use Lemma 2.2.1

$$q(\alpha, L)^{n-k} q(L)^k = \alpha^{n-k} L^k \cdot [P] = \alpha^{n-k} L^k (\mathfrak{a}_0 L^n + \mathfrak{a}_2 L^{n-2} + \cdots + \mathfrak{a}_{2k} L^{n-2k} + \cdots).$$

Again, using the polarized Fujiki relations and $q(\alpha) = 0$, for $0 \leq j < k$ we get from the induction hypothesis

$$\begin{aligned}
\alpha^{n-k} L^k \cdot \mathfrak{a}_{2j} L^{n-2j} &= \mathfrak{a}_{2j} \cdot \alpha^{n-k} L^{n-k+2(k-j)} \\
&= \frac{2^{n-k} \binom{n-j}{n-k}}{\binom{2n-2j}{n-k}} \cdot C(\mathfrak{a}_{2j}) q(\alpha, L)^{n-k} q(L)^{k-j} \\
&= \frac{2^{n-k} \binom{n-j}{n-k}}{\binom{2n-2j}{n-k}} \cdot \frac{(2n-2j)!}{2^n j!(n-2j)!(n-j)!} (-1)^j q(L)^k q(\alpha, L)^{n-k} \\
&= \frac{(n-2j+k)!}{2^k j!(k-j)!(n-2j)!} (-1)^j q(L)^k q(\alpha, L)^{n-k},
\end{aligned}$$

for $j = k$ we get

$$\alpha^{n-k} L^k \cdot \mathfrak{a}_{2k} L^{n-2k} = \frac{2^{n-k}}{\binom{2n-2k}{n-k}} \cdot C(\mathfrak{a}_{2k}) q(\alpha, L)^{n-k},$$

and for $j > k$ the terms $\alpha^{n-k} L^k \cdot \mathfrak{a}_{2j} L^{n-2j}$ vanish. Putting all the terms together, we get

$$\begin{aligned}
q(\alpha, L)^{n-k} q(L)^k = &\sum_{j=0}^{k-1} \frac{(n-2j+k)!}{2^k j!(k-j)!(n-2j)!} (-1)^j q(L)^k q(\alpha, L)^{n-k} \\
&+ \frac{2^{n-k}}{\binom{2n-2k}{n-k}} \cdot C(\mathfrak{a}_{2k}) q(\alpha, L)^{n-k},
\end{aligned}$$

so dividing out $q(\alpha, L)^{n-k}$, which we have assumed to be non-zero, we get

$$\begin{aligned}
C(\mathfrak{a}_{2k}) &= \frac{\binom{2n-2k}{n-k}}{2^{n-k}} q(L)^k \left(1 - \frac{1}{2^k} \sum_{j=0}^{k-1} \frac{(n-2j+k)!}{j!(k-j)!(n-2j)!} (-1)^j \right) \\
&= \frac{\binom{2n-2k}{n-k}}{2^{n-k}} q(L)^k \frac{(-1)^k \binom{n-k}{k}}{2^k},
\end{aligned}$$

which proves the case of k. Here, the last equality follows from the combinatorial identity

$$2^k = \sum_{j=0}^{k} \frac{(n-2j+k)!}{j!(k-j)!(n-2j)!} (-1)^j = \sum_{j=0}^{k} \binom{k}{j} \binom{n-2j+k}{k} (-1)^j,$$

which, by expanding $\binom{n-2j+k}{k}$ as a polynomial in j, can be obtained from the well-known identities

$$\sum_{j=0}^{k} \binom{k}{j} j^a (-1)^j = \begin{cases} 0 & \text{if } 0 \leq a \leq k-1, \\ (-1)^k k! & \text{if } a = k. \end{cases}$$

Finally, recall from Corollary 1.1.9 that we have the following relation

$$C\left(\frac{\mathrm{td}_{2k}^{1/2}}{(n-2k)!}\right) = \frac{(2n-2k)!c_X}{2^n k!(n-2k)!(n-k)!}(2r_X)^k.$$

Comparing with the generalized Fujiki constant $C(\mathfrak{a}_{2k})$, we see that

$$\mathfrak{a}_{2k} = \frac{1}{c_X}\left(\frac{-q(L)}{2r_X}\right)^k \cdot \frac{1}{(n-2k)!}\overline{\mathrm{td}_{2k}^{1/2}},$$

where $\overline{\mathrm{td}_{2k}^{1/2}}$ is the projection of $\mathrm{td}_{2k}^{1/2}$ to the Verbitsky component. We have thus obtained the desired description of $\overline{[P]}$. □

We will present another proof in Remark 2.2.14. Note that one usually replaces the class L with a suitable multiple that is integral and primitive.

In certain cases, the knowledge of $\overline{[P]}$ provides enough information for us to determine the class $[P]$. In particular, we have the following result by Beckmann, which helps eliminate some of the extra LLV components. It follows from the much stronger statement in [**Bec22**, Proposition 3.5].

PROPOSITION 2.2.5 (Beckmann). *Let X be a hyperkähler manifold of dimension $2n$ containing a Lagrangian n-plane P. Assuming that $q(L) \neq 0$, where L is the dual class of a line in P as usual. Then the class $[P] \in H^{2n}(X, \mathbf{Q})$ (or more generally, the Mukai vector $v(\mathcal{O}_P)$) only has non-zero terms in LLV components of type $V_{(k)}$ for $k \geq 0$.*

EXAMPLE 2.2.6. In the $\mathrm{K3}^{[2]}$-type case, the Verbitsky component coincides with the entire cohomology, hence we have

$$[P] = \overline{[P]} = \tfrac{1}{2}L^2 + \tfrac{1}{20}\mathfrak{q} = \tfrac{1}{2}L^2 + \tfrac{1}{24}c_2,$$

where $q(L) = -\frac{5}{2}$. Using the values of $C(\mathfrak{q})$ and $C(\mathfrak{q}^2)$ from Example 1.1.7, we may verify that the self-intersection number $[P]^2$ is indeed equal to 3.

The class L is not integral since $q(L)$ is not an integer. On the other hand, the discriminant of $H^2(X, \mathbf{Z})$ is 2, so the class $\lambda := 2L$ is integral and primitive, and we may also write

$$[P] = \tfrac{1}{8}\lambda^2 + \tfrac{1}{20}\mathfrak{q} = \tfrac{1}{8}\lambda^2 + \tfrac{1}{24}c_2.$$

This result was originally obtained by Hassett–Tschinkel in [**HT09b**, Section 5].

EXAMPLE 2.2.7. In the $\mathrm{K3}^{[3]}$-type case, we obtain

$$\overline{[P]} = \tfrac{1}{6}L^3 + \tfrac{1}{24}c_2 L.$$

The LLV decomposition of the cohomology is given by

$$H^*(\mathrm{K3}^{[3]}, \mathbf{Q}) = V_{(3)} \oplus V_{(1,1)} \quad \text{as } \mathfrak{so}(3,22)\text{-modules}.$$

Therefore by Proposition 2.2.5, we may conclude that $[P] = \overline{[P]}$. Again, one can easily verify that $[P]^2 = -4$ using the generalized Fujiki constants. This recovers the main result of [**HHT12**], avoiding all the difficult Diophantine analysis.

More generally, when X is of $\mathrm{K3}^{[n]}$-type, we have $c_X = 1$ and $r_X = \frac{n+3}{4}$, while $q(L) = -\frac{n+3}{2}$ by the result of Bakker [**Bak17**]. So Theorem 2.2.4 simplifies to $\overline{[P]} = \big[\exp(L)\,\mathrm{td}_X^{1/2}\big]_n$. Note that starting from $n = 4$, the second Chern class hence also the class $\mathrm{td}_2^{1/2}$ no longer lie in the Verbitsky component, so the information of $\overline{[P]}$ alone does not suffice to recover $[P]$. We will discuss the full cohomology class $[P]$ in the next section.

Example 2.2.8. In the Kum_2-type case, we have

$$[P] = \tfrac{1}{6}L^2 + \tfrac{1}{72}c_2 + z,$$

where z is some class in $\mathrm{SH}(X,\mathbf{Q})^\perp$. Using the self-intersection number $[P]^2 = 3$, we may deduce that $z^2 = \frac{8}{3}$. Similarly, the class $2L$ is integral and primitive.

The class z is however not a monodromy invariant and can be different for different planes. Namely, let X be the generalized Kummer variety of $E_1 \times E_2$, a product of two elliptic curves. Effective divisors of degree 3 on E_1 together with a 3-torsion point on E_2 provides a plane in X. In this case, the class z has been explicitly described in [**HT13**, Proposition 7.1] (where it is denoted by $\widehat{Z}$). We note that there are 81 Kummer surfaces contained in X, parametrized by all the 3-torsion points of $E_1 \times E_2$. Their classes Z_i are linearly independent in the cohomology and we have $c_2 = \frac{1}{3}\sum Z_i$. The class z is a linear combination of the form $\frac{1}{27}(-8\sum_{i=1}^{9} Z_i + \sum_{i=10}^{81} Z_i)$, where the 9 distinguished Kummer surfaces are those given by the chosen 3-torsion point on E_2. So the class z depends on the choice of the 3-torsion point on E_2 and is therefore not a monodromy invariant.

Example 2.2.9. For the 10-dimensional example OG_{10}, since $c_X = 1$, the situation is very similar to the $\mathrm{K3}^{[n]}$-type case: using Theorem 2.2.4 and the value $q(L) = -4$, we obtain

$$\overline{[P]} = \tfrac{1}{120}L^5 + \tfrac{1}{96}\mathfrak{q}L^3 + \tfrac{1}{480}\mathfrak{q}^2 L.$$

The LLV decomposition of the cohomology was determined in [**GKLR21**, Theorem 3.26]

$$H^*(\mathrm{OG}_{10},\mathbf{Q}) = V_{(5)} \oplus V_{(2,2)} \quad \text{as } \mathfrak{so}(4,22)\text{-modules.}$$

So by Proposition 2.2.5, we may conclude that $[P] = \overline{[P]}$.

The class L is integral and primitive in this case. To show this, we use the description of the Beauville–Bogomolov–Fujiki form: $H^2(X,\mathbf{Z}) \simeq \Lambda_{\mathrm{K3}} \oplus \left(\begin{smallmatrix} -6 & 3 \\ 3 & -2 \end{smallmatrix}\right)$. Write u and v for the generators of the rank-2 sublattice, then we have $H^2(X,\mathbf{Z})^\vee \simeq \Lambda_{\mathrm{K3}} \oplus \frac{1}{3}\mathbf{Z}u \oplus \mathbf{Z}v$. Write $L = L_0 + \frac{a}{3}u + bv$ for $L_0 \in \Lambda_{\mathrm{K3}}$ and $a, b \in \mathbf{Z}$, we have $-4 = q(L) = q(L_0) - \frac{2}{3}a^2 + 2ab - 2b^2$. Hence $3 \mid a$ and L is integral. Since $q(L) = -4$ so $q(L/2) = -1$ is not an even number, we see that L is also primitive.

2.2.3. $\mathrm{K3}^{[n]}$-type. When X is of $\mathrm{K3}^{[n]}$-type, the formula (2.8) simplifies to $\overline{[P]} = \big[\exp(L)\,\mathrm{td}_X^{1/2}\big]_n$, which motivates the following conjecture on the full cohomology class $[P]$ of a Lagrangian n-plane.

Conjecture 2.2.10. *Let X be a hyperkähler manifold of* $\mathrm{K3}^{[n]}$*-type containing a Lagrangian n-plane P. Let ℓ be the class of a line contained in P and let $L \in H^2(X,\mathbf{Q})$ be the dual class. Then the cohomology class of P is given by the degree-n part of* $\exp(L)\,\mathrm{td}_X^{1/2}$

$$[P] = \Big[\exp(L)\,\mathrm{td}_X^{1/2}\Big]_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{L^{n-2k}}{(n-2k)!}\,\mathrm{td}_{2k}^{1/2}\,. \tag{2.9}$$

Bakker proved in [**Bak17**] that for $\mathrm{K3}^{[n]}$-type, there is a unique monodromy orbit for the class $[P]$ with *primitive* line class ℓ, that is, ℓ is indivisible in $H_2(X,\mathbf{Z})$. Hence in this case, a universal formula as above must exist. In lower dimensions, the class $[P]$ has been explicitly determined: $n=2$ by Hassett–Tschinkel [**HT09b**], $n=3$ by Harvey–Hassett–Tschinkel [**HHT12**], and $n=4$ by Bakker–Jorza [**BJ14**]. So the conjecture is verified for $n \le 4$, although the above general formula has not been guessed before. Note that for $n=3,4$, the proofs are done by considering linear combination of all possible Hodge classes and then determining the coefficients through hardcore Diophantine analysis. If we use the same strategy for larger dimensions, there will be more Hodge classes hence more variables appearing, and each dimension would require a separate treatment.

We focus on the case where the line class ℓ is primitive. Since in this case the monodromy orbit of $[P]$ is unique, to prove the conjecture, it suffices to exhibit a *single* pair (X,P) for which the formula (2.9) holds. We thus look for the simplest example of a Lagrangian n-plane. Consider a K3 surface S containing a smooth rational curve C. Then the inclusion $C^{[n]} \hookrightarrow S^{[n]}$ provides a $\mathbf{P}^n$ in $X := S^{[n]}$. The cohomology group $H^*(S^{[n]},\mathbf{Q})$ has a nice description in terms of Nakajima operators (see [**Nak99**]), and Lehn–Sorger have constructed in [**LS03**] an explicit algebraic model for the ring structure.[2] Therefore one could use computer algebra to explicitly compute the class $[P]$ and verify the formula. We have done this for n up to 6.

Proposition 2.2.11. *The Conjecture 2.2.10 holds for $n \le 6$ when the line class ℓ is primitive.*

Proof. We reduce the problem to the case of $C^{[n]} \hookrightarrow S^{[n]}$ using the uniqueness of the monodromy orbit. Then we verify (2.9) in the Nakajima basis using computer algebra. □

It seems hard to generalize this method to all dimensions, hence we do not provide further details. See Section 2.2.4 for some further discussions on the Mukai vector $v(L) := \exp(L)\,\mathrm{td}_X^{1/2}$ that might lead to a proof. One could also ask if the same relation holds in the Chow ring as well.

Because the formula (2.8) for the projection $\overline{[P]}$ does not depend on the primitiveness of the line class ℓ, we formulate the Conjecture 2.2.10 without the primitiveness condition. It appears that one cannot exclude the possibility of a Lagrangian n-plane with an imprimitive line class, see [**Bak17**, Remark 28].

[2] An implementation in `Haskell` is available (see [**Kap16**]), although it contains an error: the Euler class of the algebra $H^*(S,\mathbf{Z})$ as defined in [**LS03**] should be equal to $-24\mathrm{pt}$, the *minus* of the actual Euler class of S.

2.2.4. Mukai vector. We propose a generalization for the results of previous sections by considering the Mukai vector $v(L) \coloneqq \exp(L)\,\mathrm{td}_X^{1/2}$ in its entirety. Note that the class L is not necessarily integral, so this may not be the Mukai vector of a genuine line bundle (unless we consider twisted sheaves).

CONJECTURE 2.2.12. *Let X be a hyperkähler manifold containing a Lagrangian n-plane P, and let L be the dual class of a line in P. The Mukai vector $v(L) = \exp(L)\,\mathrm{td}_X^{1/2}$ only has non-zero terms in degree 0 to n. In other words, $[v(L)]_k \in H^{2k}(X, \mathbf{Q})$ vanishes for $k \geq n+1$.*

Note that this behavior closely resembles the case of a non-zero class $\alpha \in H^2(X, \mathbf{Q})$ with $q(\alpha) = 0$: by Theorem 1.3.2, we have $\alpha^k \neq 0$ if and only if $0 \leq k \leq n$.

We prove some partial results on the projection of the Mukai vector to the Verbitsky component, which provide evidence to the conjecture. In particular, the conjecture implies that $q(L) = -2r_X$, which is indeed a value depending only on the deformation type of X, confirming the thesis of Hassett–Tschinkel. Moreover, the constant μ appearing in Theorem 2.2.4 would simplify to 1.

PROPOSITION 2.2.13. *Let X be a hyperkähler manifold of dimension $2n$ and let α be a class in $H^2(X, \mathbf{Q})$. Consider the Mukai vector $v(\alpha) \coloneqq \exp(\alpha)\,\mathrm{td}_X^{1/2}$.*

(1) The top-degree part $[v(\alpha)]_{2n} \in H^{4n}(X, \mathbf{Q})$ vanishes if and only if $q(\alpha) = -2r_X$.
(2) If $q(\alpha) = -2r_X$, then the projection $\overline{[v(\alpha)]_k} \in \mathrm{SH}^{2k}(X, \mathbf{Q})$ vanishes for $k \geq n+1$.

In particular, when $n = 2$ and $q(L) = -2r_X$, since the Verbitsky component spans $H^6(X, \mathbf{Q})$, we have $[v(L)]_3 = \overline{[v(L)]_3} = 0$ and $[v(L)]_4 = 0$. Hence for $n = 2$, Conjecture 2.2.12 is equivalent to the numerical condition $q(L) = -2r_X$.

PROOF. The first point follows from Theorem 1.1.8, since we have

$$\int_X [v(\alpha)]_{2n} = \mathrm{RR}_{X,1/2}\big(q(\alpha)\big) = C(\mathrm{td}_X^{1/2}) \left(1 + \frac{1}{2r_X} q(\alpha)\right)^2,$$

and $C(\mathrm{td}_X^{1/2})$ is non zero.

For the second point, we need to show that the class $[v(\alpha)]_k$ lies in $\mathrm{SH}(X)^\perp$ for $k \geq n+1$, or equivalently, the following symmetric $(2n-k)$-form on $H^2(X, \mathbf{Q})$ vanishes

$$(\beta_1, \ldots, \beta_{2n-k}) \longmapsto [v(\alpha)]_k \cdot \beta_1 \cdots \beta_{2n-k}.$$

It suffices to show that for $k \geq n+1$,

$$\forall \beta \in H^2(X, \mathbf{Q}) \quad [v(\alpha)]_k \cdot \beta^{2n-k} = 0.$$

We introduce a formal variable t and consider the sum

$$\sum_{k=0}^{2n} [v(\alpha)]_k \cdot \beta^{2n-k} \cdot \frac{t^k}{(2n-k)!}.$$

It suffices to show that this is a polynomial in t of degree $\leq n$. We compute

$$\begin{aligned}\sum_{k=0}^{2n}[v(\alpha)]_k\cdot\beta^{2n-k}\cdot\frac{t^k}{(2n-k)!}&=\sum_{k=0}^{2n}\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor}\frac{\alpha^{k-2j}}{(k-2j)!}\operatorname{td}_{2j}^{1/2}\cdot\beta^{2n-k}\cdot\frac{t^k}{(2n-k)!}\\&=\sum_{j=0}^{n}\sum_{k=2j}^{2n}\frac{\alpha^{k-2j}}{(k-2j)!}\operatorname{td}_{2j}^{1/2}\cdot\beta^{2n-k}\cdot\frac{t^k}{(2n-k)!}\\&=\sum_{j=0}^{n}\sum_{i=0}^{2n-2j}\frac{\alpha^{i}}{i!}\operatorname{td}_{2j}^{1/2}\cdot\beta^{2n-2j-i}\cdot\frac{t^{2j+i}}{(2n-2j-i)!},\end{aligned}$$

where we exchanged the two summations and took a change of variables by letting $i=k-2j$. The factorials $i!$ and $(2n-2j-i)!$ suggest that they come from the binomial coefficient $\binom{2n-2j}{i}$, so we may rewrite the sum as

$$\sum_{j=0}^{n}\operatorname{td}_{2j}^{1/2}t^{2j}\frac{1}{(2n-2j)!}(t\alpha+\beta)^{2n-2j}.$$

Then we use the Fujiki relations as well as the formula for $C(\operatorname{td}_{2k}^{1/2})$ from Corollary 1.1.9 to obtain

$$\begin{aligned}\sum_{j=0}^{n}\frac{t^{2j}}{(2n-2j)!}C(\operatorname{td}_{2j}^{1/2})q(t\alpha+\beta)^{n-j}&=\sum_{j=0}^{n}\frac{t^{2j}c_Xr_X^j}{2^{n-j}j!(n-j)!}q(t\alpha+\beta)^{n-j}\\&=\frac{c_X}{2^nn!}\big(2r_Xt^2+q(t\alpha+\beta)\big)^n\\&=\frac{c_X}{2^nn!}\big(2q(\alpha,\beta)t+q(\beta)\big)^n,\end{aligned}$$

where in the last equality we used the fact that $q(\alpha)=-2r_X$. Thus the polynomial is indeed of degree $\leq n$ and this concludes the proof. □

REMARK 2.2.14. The above method also provides an alternative proof for Theorem 2.2.4: namely, by letting $\alpha=L/\mu$, the computation above shows that the class $[v(L/\mu)]_n$ satisfies the property

$$\forall\beta\in H^2(X,\mathbf{Q})\quad[v(L/\mu)]_n\cdot\beta^n=c_X\cdot q(L/\mu,\beta)^n=\frac{c_X}{\mu^n}q(L,\beta)^n,$$

which by Lemma 2.2.1 is equivalent to

$$[v(L/\mu)]_n\cdot\beta^n=\frac{c_X}{\mu^n}[P]\cdot\beta^n.$$

So the difference $[v(L/\mu)]_n-\frac{c_X}{\mu^n}[P]$ lies in $\mathrm{SH}^{2n}(X,\mathbf{Q})^\perp$, and we have

$$\overline{[P]}=\frac{\mu^n}{c_X}\overline{[v(L/\mu)]_n}=\overline{\left[\frac{\mu^n}{c_X}\exp(L/\mu)\operatorname{td}_X^{1/2}\right]_n}.$$

We also note that some similar analysis for α with $q(\alpha)=0$ using the Fujiki relations can be found in [**Mat99**].

EXAMPLE 2.2.15.

- When X is a K3 surface containing a smooth rational curve C, L is just the class $[C]$ and $L^2 = -2$. We have
$$v(L) = (1 + L + \tfrac{1}{2}L^2)(1 + \mathrm{pt}) = 1 + L,$$
so $[v(L)]_2$ indeed vanishes.
- When X is of K3$^{[2]}$-type, we have $q(L) = -\frac{5}{2} = -2r_X$. By Proposition 2.2.13, both $[v(L)]_3$ and $[v(L)]_4$ vanish so the conjecture holds.
- Similarly, when X is of Kum$_2$-type, we have $q(L) = -\frac{3}{2} = -2r_X$ so the conjecture holds.
- For K3$^{[n]}$-type, since the class L satisfies $q(L) = -\frac{n+3}{2} = -2r_X$ by the result of Bakker [**Bak17**], the projections $\overline{[v(L)]_k}$ for $k \geq n+1$ all vanish by Proposition 2.2.13. Moreover, by assuming that the line class ℓ is primitive, all classes L are in the same monodromy orbit. Therefore, to verify the conjecture on the full Mukai vector in this case, it suffices to provide one single example where it holds. So again we may study the Lagrangian n-plane $C^{[n]} \hookrightarrow S^{[n]}$ for a K3 surface S containing a smooth rational curve C. Using the Nakajima basis and computer algebra, we have verified the conjecture for K3$^{[n]}$ with $n \leq 5$.
- For the two O'Grady examples, Conjecture 2.2.12 also holds: on the one hand, we have computed the value of $q(L)$ in Corollary 2.2.2, which is indeed equal to $-2r_X$ in both cases, so the projection $\overline{v(L)}$ has the desired form by Proposition 2.2.13; on the other hand, all characteristic classes lie in the Verbitsky component for these two examples (see Appendix 2.A), so we may conclude that $v(L) = \overline{v(L)}$ also has the desired form.

2.A. Generalized Fujiki constants of known examples

In this appendix, we give an account for the generalized Fujiki constants $C(c_\lambda)$ of characteristic classes $c_\lambda \coloneqq c_2^{\lambda_2} c_4^{\lambda_4} \cdots c_{2n}^{\lambda_{2n}}$ for all known deformation types of hyperkähler manifolds.

2.A.1. K3$^{[\mathrm{n}]}$ **and** Kum$_n$**.** The results are classical for the two infinite families. In the K3$^{[n]}$-case, the method in Ellingsrud–Göttsche–Lehn [**EGL01**] can be used to compute all the generalized Fujiki constants using a computer for small n. A similar algorithmic method can be used to treat the Kum$_n$-case, with some slight modifications based on the work of Nieper-Wißkirchen [**NW02**].[3] Closed formulae for the values $C(c_{2k})$ for both families were recently established in [**COT22**, Theorem 4.2].

[3]An implementation for both these algorithms in `Sage` is available here.

2.A.2. OG_6**.** By Lemma 2.1.7, the generalized Fujiki constants for characteristic classes of degree ≤ 4 for OG_6 are the same as those for Kum_3, since they share the same Riemann–Roch polynomial. Since the Chern numbers of OG_6 are also known by [**MRS18**, Proposition 6.8], we can obtain all of them:

α	1	c_2	c_4	c_2^2	c_6	c_4c_2	c_2^3
$C(\alpha)$	60	288	480	1920	1920	7680	30720

Alternatively, since for OG_6-type the second Chern class c_2 lies in the Verbitsky component (namely, $c_2(\mathrm{OG}_6) = 2\mathfrak{q}$), Corollary 2.1.5 shows that the class $\mathrm{td}_4^{1/2}$ also lies in $\mathrm{SH}(X, \mathbf{R})$. Now $\mathrm{td}_4^{1/2}$ is a linear combination of c_2^2 and c_4, so the same may be said for the class c_4. Then we can use Proposition 1.1.6 to determine that $c_4(\mathrm{OG}_6) = \mathfrak{q}^2$, which then allows us to also compute $C(c_4c_2)$ and $C(c_2^3)$. Finally we can use $C(\mathrm{td}_6) = 4$ to solve the Euler characteristic $C(c_6)$.

Proposition 2.A.1. *For hyperkähler manifolds of* OG_6*-type, all Chern classes* c_2, c_4, c_6 *lie in the Verbitsky component. We have*

$$c_2(\mathrm{OG}_6) = 2\mathfrak{q}, \quad c_4(\mathrm{OG}_6) = \mathfrak{q}^2, \quad c_6(\mathrm{OG}_6) = \tfrac{1}{2}\mathfrak{q}^3.$$

2.A.3. OG_{10}**.** The question for OG_{10} might seem difficult at first, as there are many more unknown Fujiki constants to determine. It turns out to be quite easy, due to the following observation.

Proposition 2.A.2. *For hyperkähler manifolds of* OG_{10}*-type, all Chern classes* c_{2k} *lie in the Verbitsky component. We have*

$$c_2(\mathrm{OG}_{10}) = \tfrac{3}{2}\mathfrak{q}, \quad c_4(\mathrm{OG}_{10}) = \tfrac{15}{16}\mathfrak{q}^2, \quad c_6(\mathrm{OG}_{10}) = \tfrac{21}{64}\mathfrak{q}^3,$$
$$c_8(\mathrm{OG}_{10}) = \tfrac{237}{3328}\mathfrak{q}^4, \quad c_{10}(\mathrm{OG}_{10}) = \tfrac{27}{2560}\mathfrak{q}^5.$$

Proof. We use the LLV decomposition of the cohomology obtained in [**GKLR21**, Theorem 3.26]

$$H^*(\mathrm{OG}_{10}, \mathbf{Q}) = V_{(5)} \oplus V_{(2,2)} \quad \text{as } \mathfrak{so}(4,22)\text{-modules.}$$

We are interested in the second component, which only contributes to cohomological degree k for $k \in \{6, 8, 10, 12, 14\}$.

For a generic X in the moduli space, the (special) Mumford–Tate algebra is the maximal possible and is isomorphic to $\mathfrak{so}(3, 21)$. Using the branching rules, we get the following decompositions of $\mathfrak{so}(3, 21)$-modules/Hodge structures (H^{12} and H^{14} are omitted by symmetry)

$$\begin{aligned} H^6(X, \mathbf{Q}) &= \mathrm{SH}^6(X, \mathbf{Q}) \oplus V_{(2)}, \\ H^8(X, \mathbf{Q}) &= \mathrm{SH}^8(X, \mathbf{Q}) \oplus V_{(2,1)} \oplus V_{(1)}, \\ H^{10}(X, \mathbf{Q}) &= \mathrm{SH}^{10}(X, \mathbf{Q}) \oplus V_{(2,2)} \oplus V_{(2)} \oplus V_{(1,1)} \oplus \mathbf{Q}. \end{aligned}$$

In other words, up to multiplying by a non-zero scalar, there is only one Hodge class $\eta \in H^{10}(X, \mathbf{Q})$ that lies in $\mathrm{SH}(X, \mathbf{Q})^{\perp}$ for a generic X. In particular, this means that all the Chern classes $c_2, \ldots, c_{10}$ lie in the Verbitsky component.

For a generic X, the only Hodge classes in the Verbitsky components are multiples of powers of $\mathfrak{q}$, so each Chern class c_{2k} is a multiple of $\mathfrak{q}^k$. We explain how to determine the scalars, starting from smaller k: we use Lemma 2.1.7 to determine $C(c_2)$ and $C(c_4)$. Since the values of $C(\mathfrak{q}^k)$ are known by Proposition 1.1.6, we have therefore determined c_2 and c_4. Once all c_{2i} for $i < k$ are known, we study the class $\mathrm{td}^{1/2}_{2k}$, whose generalized Fujiki constant $C(\mathrm{td}^{1/2}_{2k})$ is known by Corollary 1.1.9 and whose only unknown term is a given multiple of c_{2k}. Therefore we will be able to uniquely determine $C(c_{2k})$ and thus c_{2k} itself. ☐

It is then straightforward to compute the generalized Fujiki constants, which we include for the reader's convenience.

α	1	c_2	c_4	c_2^2	c_6	c_4c_2	c_2^3	c_8	c_6c_2	c_4^2	$c_4c_2^2$	c_2^4
$C(\alpha)$	945	5040	13500	32400	26460	113400	272160	49770	343980	614250	1474200	3538080

c_{10}	c_8c_2	c_6c_4	$c_6c_2^2$	$c_4^2c_2$	$c_4c_2^3$	c_2^5
176904	1791720	5159700	12383280	22113000	53071200	127370880

Note that the Chern numbers for OG_{10} have already been computed by Cao–Jiang in the appendix of [**RO20**].

It is remarkable that the knowledge of the Riemann–Roch polynomial together with the assumption that all Chern classes lie in the Verbitsky component allow us to completely determine the second Betti number as well as all the generalized Fujiki constants, in particular all the Chern numbers including the Euler characteristic $C(c_{2n}) = \int_X c_{2n}$.

CHAPTER 3

Image of the period map

In this chapter, we study the moduli spaces and period maps for projective hyperkähler manifolds. In general, the moduli space for polarized hyperkähler manifolds with a fixed polarization type is not necessarily connected. For $\mathrm{K3}^{[m]}$-type and Kum_m-type, we deduce a precise formula for the number of connected components, as well as the number of polarization types with fixed square and divisibility. Then we study the image of the polarized period map, focusing on the known examples. We show that when the moduli space is not connected, the images of the period map restricted to different connected components can be different.

The results of this chapter have appeared in [**Son22**].

3.1. Introduction

We fix a deformation type of compact hyperkähler manifolds. For a hyperkähler manifold X, the integral cohomology group $\big(H^2(X,\mathbf{Z}), q_X\big)$ equipped with the Beauville–Bogomolov–Fujiki form is a lattice of signature $(3, b_2-3)$. The isomorphism class of this lattice depends only on the deformation type of X, so we will fix one such lattice and denote it by Λ. We call an isometry $\eta\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} \Lambda$ a *marking* of X. Denote by $\mathcal{M}_{\mathrm{marked}}$ the moduli space for marked hyperkähler manifolds (X,η) of the given deformation type. On each connected component $\mathcal{M}^0_{\mathrm{marked}}$ of the moduli space $\mathcal{M}_{\mathrm{marked}}$, the Hodge structures provide a *period map*

$$\wp^0_{\mathrm{marked}}\colon \mathcal{M}^0_{\mathrm{marked}} \longrightarrow \Omega_{\mathrm{marked}},$$

where

$$\Omega_{\mathrm{marked}} \coloneqq \{[x] \in \mathbf{P}(\Lambda_{\mathbf{C}}) \mid (x,x) = 0, (x,\bar{x}) > 0\}$$

is a complex manifold called the *period domain*. The global Torelli theorem, proved by Verbitsky, states that $\wp^0_{\mathrm{marked}}$ is surjective, generically injective, and identifies pairwise inseparable points.

On a projective hyperkähler manifold X, we may consider the extra datum of a *polarization*, that is, a primitive ample class $H \in H^2(X,\mathbf{Z})$. Any marking η maps H to a vector $\eta(H) \in \Lambda$, so it is reasonable to define the *polarization type* T of (X,H) as the $\mathrm{O}(\Lambda)$-orbit of $\eta(H)$ in Λ, which does not depend on the choice of the marking η. There is a quasi-projective moduli space $\mathcal{M}_T$ for polarized hyperkähler manifolds (X,H) of fixed polarization type T. For K3 surfaces, each polarization type T is uniquely determined by its square $2d$ and each moduli space $\mathcal{M}_{2d}$ is an irreducible quasi-projective variety of dimension 19. However, for their higher-dimensional analogues, the polarization types are more complicated to describe: apart from the square, there is another invariant, the *divisibility*. Moreover, Apostolov showed in [**Apo14**] that for some polarization types T on manifolds of $\mathrm{K3}^{[m]}$-type, the moduli space $\mathcal{M}_T$ may have several connected components. Onorati obtained similar results for Kum_m-type in [**Ono16**]. We shall review their results and give a simplified expression for the exact number of components in Section 3.3 (Proposition 3.3.4 and Proposition 3.3.5).

One can also consider the period map for polarized hyperkähler manifolds and its restriction to each connected component $\mathcal{M}_T^0$ of the polarized moduli space $\mathcal{M}_T$. We will use the letter τ to denote a *deformation type* of polarizations of type T. Such deformation types are in bijection with the connected components of $\mathcal{M}_T$, so we will write $\mathcal{M}_\tau$ instead of $\mathcal{M}_T^0$. In order to get rid of the choice of a marking, we consider the quotient of the corresponding period domain Ω, which is a hyperplane section inside Ω_{marked}, by the action of the elements in the orthogonal group $\mathrm{O}(\Lambda)$ that stabilize Ω. In this way, we get a period domain $\mathcal{P}_T$, depending only on the polarization type T. But the global Torelli theorem no longer holds in this case, as the map from $\mathcal{M}_\tau$ to $\mathcal{P}_T$ might not be generically injective. In fact, it factors through $\mathcal{P}_\tau$, the quotient of Ω by a smaller group $\mathrm{Mon}(\Lambda)$, the *monodromy group*, which is a normal subgroup of $\mathrm{O}(\Lambda)$ for all the known deformation types (see Table 2). Thus the correct global Torelli theorem says that the polarized period map

$$\begin{array}{ccc} \wp_\tau \colon \mathcal{M}_\tau & \hookrightarrow & \mathcal{P}_\tau \\ & & \big\downarrow /G \\ & & \mathcal{P}_T \end{array}$$

is an open immersion, where $\mathcal{P}_\tau$ is a covering space of $\mathcal{P}_T$ with finite deck transformation group G. The complement of the image of this open immersion is a finite union of divisors in $\mathcal{P}_\tau$. Intuitively, when the periods of the manifolds X in the family move towards the boundary of the image, the polarization H on X will move towards the boundary of the ample cone. Therefore, the determination of the divisors in the complement of the image is intimately related to the geometry of the ample cone for manifolds X in the family.

In the $\mathrm{K3}^{[m]}$-type case, the description of the ample cone was given by Bayer–Hassett–Tschinkel [**BHT15**], using the theory of Bayer–Macrì [**BM14**]. The description is based on a canonical embedding of $H^2(X, \mathbf{Z})$ into a larger lattice $\widetilde{\Lambda}$, known as the *Mukai lattice*. The ample cone can then be described using some numerical conditions. The analogous result for Kum_m-type was obtained by Yoshioka [**Yos16**]. We will review this in Section 3.4 and give a simplified description, without explicitly referring to the larger Mukai lattice (Proposition 3.4.5). We will use this description to characterize the divisors in the complement of the image of the period map. Note that the $\mathrm{K3}^{[2]}$-type case was completely treated in [**DM19**] (see also [**Deb18**, Appendix B]).

A natural question arises of whether for a given polarization type T, different connected components $\mathcal{M}_\tau$ of $\mathcal{M}_T$ have the same image in $\mathcal{P}_\tau$ under their corresponding period map.

This question in general is not well-posed, as there is no canonical way to identify the period domains $\mathcal{P}_\tau$ for different components, due to the action of the deck transformation group G. Nevertheless, there is no problem of identification when G is trivial, and we provide a negative answer in the $\mathrm{K3}^{[m]}$-type case: by using our numerical description of the image, we construct in Section 3.5 an example where two connected components of the same $\mathcal{M}_T$ have different images in $\mathcal{P}_T$. We will also give another example where the group G is non-trivial and the image of the period map in $\mathcal{P}_\tau$ is not G-invariant above $\mathcal{P}_T$.

Notation. In this chapter, we will use the letter $2m$ for the dimension of the hyperkähler manifold and $2n$ for the Beauville–Bogomolov–Fujiki square $q_X(H)$ of the ample class H.

For a fixed deformation type of hyperkähler manifolds, we use $\mathcal{M}_{\mathrm{marked}}$ (resp. $\mathcal{M}_T$) to denote the marked (resp. polarized) moduli space. The notation $\mathcal{M}^0$ will be used to denote a connected component of the corresponding moduli space $\mathcal{M}$.

For a positive integer n, we denote by $\rho(n)$ the number of distinct prime divisors of n and by $\widetilde{\rho}(n)$ the number $\rho(n)$ if n is odd and $\rho(n/2)$ if n is even. For a prime number p, we write $v_p(n)$ for the p-adic valuation of n.

To treat $\mathrm{K3}^{[m]}$-type and Kum_m-type manifolds simultaneously, we let $\widetilde{m} = m-1$ for $\mathrm{K3}^{[m]}$-type and $\widetilde{m} = m+1$ for Kum_m-type.

3.2. Setup

In this section, we review the construction of the polarized period map and its relation with the monodromy group, following the work of Markman [**Mar11**, Section 4,7, and 8]. We reformulate some of the results to give a simpler presentation and to better suit our needs for later sections. We will consider a fixed deformation type of hyperkähler manifolds and denote by Λ the lattice defined by the Beauville–Bogomolov–Fujiki form on the second cohomology group, which has signature $(3, b_2 - 3)$.

First we recall the following definitions (*cf.* [**Mar11**, Definition 1.1]).

Definition 3.2.1. Let X and X' be hyperkähler manifolds of the given deformation type.

(i) An isomorphism $f\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} H^2(X',\mathbf{Z})$ is called a *parallel transport operator* if there exist a smooth and proper family $\pi\colon \mathcal{X} \to B$ of hyperkähler manifolds, with points $b, b' \in B$ and a path $\gamma\colon [0,1] \to B$ connecting b and b', such that $X \simeq \mathcal{X}_b$, $X' \simeq \mathcal{X}_{b'}$, and f is given as the parallel transport in the local system $R^2\pi_*\mathbf{Z}$ along γ.

(ii) An automorphism $f\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} H^2(X,\mathbf{Z})$ that is a parallel transport operator is called a *monodromy operator*. The subgroup of $\mathrm{O}(H^2(X,\mathbf{Z}))$ generated by monodromy operators is called the *monodromy group* of X and denoted by $\mathrm{Mon}(X)$.

(iii) If (X,H) and (X',H') are polarized hyperkähler manifolds, we define similarly a *polarized parallel transport operator* $f\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} H^2(X',\mathbf{Z})$ to be one induced by a path γ in a family of polarized hyperkähler manifolds. In other words, the local system $R^2\pi_*\mathbf{Z}$ admits a section h of ample classes, such that $h(b)=H$ and $h(b')=H'$.

In this paper, we will make the assumption that the monodromy group $\mathrm{Mon}(X)$ is a normal subgroup of $\mathrm{O}(H^2(X,\mathbf{Z}))$, in which case it can be identified as a subgroup $\mathrm{Mon}(\Lambda)$ of $\mathrm{O}(\Lambda)$. This holds for all known deformation types of hyperkähler manifolds.

A first property of the monodromy group $\mathrm{Mon}(\Lambda)$ can be given in terms of the *spinor norm*, which is the following homomorphism of groups

$$\sigma\colon\ \mathrm{O}(\Lambda_{\mathbf{R}}) \simeq \mathrm{O}(3, b_2-3) \longrightarrow \{\pm 1\},$$

given by the action on the orientation of a positive three-space W_3 of $\Lambda_{\mathbf{R}}$. In a more canonical way, we may consider the positive cone

$$\widetilde{\mathcal{C}}_\Lambda \coloneqq \{x \in \Lambda_{\mathbf{R}} \mid (x,x) > 0\}.$$

For any positive three-space W_3 in $\Lambda_{\mathbf{R}}$, $W_3 \smallsetminus \{0\}$ is a deformation retract of $\widetilde{\mathcal{C}}_\Lambda$. So an orientation of W_3 determines a generator of $H^2(W_3 \smallsetminus \{0\}, \mathbf{Z}) \simeq H^2(\widetilde{\mathcal{C}}_\Lambda, \mathbf{Z}) \simeq \mathbf{Z}$. The two generators of $H^2(\widetilde{\mathcal{C}}_\Lambda, \mathbf{Z})$ are called *orientation classes* of the positive cone $\widetilde{\mathcal{C}}_\Lambda$ and the spinor norm can be defined by the action on them (*cf.* [**Mar11**, Section 4]). For any subgroup G of $\mathrm{O}(\Lambda)$, we write G^+ for the subgroup of G consisting of elements of trivial spinor norm.

PROPOSITION 3.2.2. *The monodromy group* $\mathrm{Mon}(\Lambda)$ *is contained in* $\mathrm{O}^+(\Lambda)$.

PROOF. For a marked pair (X,η) with period $[x] \in \Omega_{\mathrm{marked}}$, we can take a Kähler class H on X and consider the orientation on the positive three-space $\mathbf{C}x \oplus \mathbf{R}\eta(H)$ given by the basis $\{\mathrm{Re}\,x, \mathrm{Im}\,x, \eta(H)\}$. This gives a distinguished orientation class of $\widetilde{\mathcal{C}}_\Lambda$, which is constant on each connected component $\mathcal{M}^0_{\mathrm{marked}}$ of the marked moduli space $\mathcal{M}_{\mathrm{marked}}$. Therefore every monodromy operator must have trivial spinor norm. □

From now on, we pick one connected component $\mathcal{M}^0_{\mathrm{marked}}$ of the marked moduli space $\mathcal{M}_{\mathrm{marked}}$. Recall from the introduction that we have the period map

$$\wp = \wp^0_{\mathrm{marked}}\colon \mathcal{M}^0_{\mathrm{marked}} \longrightarrow \Omega_{\mathrm{marked}}, \tag{3.1}$$

which is surjective by the global Torelli theorem. Let $h \in \Lambda$ be a primitive element of positive square. Consider the hyperplane section

$$\begin{aligned}\Omega_{\mathrm{marked}} \cap h^\perp &= \{[x] \in \Omega_{\mathrm{marked}} \mid (x,h) = 0\}\\ &= \{[x] \in \mathbf{P}(\Lambda_{\mathbf{C}}) \mid (x,x) = (x,h) = 0, (x,\bar{x}) > 0\}\end{aligned}$$

inside the marked period domain Ω_{marked}. It has two connected components denoted by Ω_h and Ω_{-h}. For any $[x] \in \Omega_h \sqcup \Omega_{-h}$, the real vector space $\mathbf{C}x \oplus \mathbf{R}h$ is a positive three-space in $\Lambda_{\mathbf{R}}$, but the orientation classes given by the basis $\{\operatorname{Re} x, \operatorname{Im} x, h\}$ are opposite on the two connected components. Since there is a distinguished orientation class for the connected component $\mathcal{M}^0_{\text{marked}}$, up to interchanging Ω_h and Ω_{-h}, we may suppose that it coincides with $\{\operatorname{Re} x, \operatorname{Im} x, h\}$ for $[x] \in \Omega_h$ (and consequently it also coincides with $\{\operatorname{Re} x, \operatorname{Im} x, -h\}$ for $[x] \in \Omega_{-h}$).

Consider the preimages under the period map (3.1) of each of these two connected components. We denote them by $\mathcal{M}_h$ and $\mathcal{M}_{-h}$. Due to the surjectivity of the period map, both are non-empty divisors in $\mathcal{M}^0_{\text{marked}}$. In fact, the union $\mathcal{M}_h \sqcup \mathcal{M}_{-h}$ is exactly the locus where the class $\eta^{-1}(h)$ is algebraic.

Proposition 3.2.3. *For a very general (X, η) in $\mathcal{M}_h$, the class $\eta^{-1}(h)$ is ample, while for a very general (X, η) in $\mathcal{M}_{-h}$, the class $\eta^{-1}(-h)$ is ample.*

Proof. For a very general element (X, η) in $\mathcal{M}_h$ with period $[x] \in \Omega_h$, the Néron–Severi group is generated by the class $H \coloneqq \eta^{-1}(h)$. In this case the Kähler cone coincides with the positive cone [**Huy99**, Corollary 7.2]. Since h is primitive of positive square, this implies that either H or $-H$ is ample. On the other hand, $[x]$ lies in Ω_h, so the orientation class given by $\{\operatorname{Re} x, \operatorname{Im} x, h\}$ coincides with the distinguished one, which can be given by $\{\operatorname{Re} x, \operatorname{Im} x, \eta(H')\}$ for some Kähler class H'. This implies that only H can be ample. By symmetry, we get the result for $-h$. □

By removing the locus inside $\mathcal{M}_h$ where $\eta^{-1}(h)$ is not ample, which is a (possibly infinite) union of subvarieties, we get the following result [**Mar11**, Corollary 7.3].

Proposition 3.2.4 (Markman). *Let $\mathcal{M}_h^{\text{amp}}$ be the locus in $\mathcal{M}_h$ that consists of marked pairs (X, η) such that $\eta^{-1}(h)$ is ample. Then $\mathcal{M}_h^{\text{amp}}$ is connected and Hausdorff, and the marked period map $\wp$ restricts to an injective map from $\mathcal{M}_h^{\text{amp}}$ onto a dense open subset of Ω_h (in the analytic topology).*

Remark 3.2.5. In Markman's survey, the domains Ω_h, $\mathcal{M}_h$, and $\mathcal{M}_h^{\text{amp}}$ are denoted as $\Omega^+_{h^\perp}$, $\mathfrak{M}^+_{h^\perp}$, and $\mathfrak{M}^a_{h^\perp}$. We believe our notation is simpler and better reflects the symmetry between h and $-h$: we may identify $\Omega^-_{h^\perp} = \Omega^+_{(-h)^\perp}$ as Ω_{-h}, and $\mathfrak{M}^-_{h^\perp} = \mathfrak{M}^+_{(-h)^\perp}$ as $\mathcal{M}_{-h}$.

The connectedness of the locus $\mathcal{M}_h^{\text{amp}}$ implies the following result [**Mar11**, Corollary 7.4], which determines whether two polarized hyperkähler manifolds lie in the same connected component of the polarized moduli space.

Proposition 3.2.6 (Markman). *A parallel transport operator*

$$f \colon H^2(X, \mathbf{Z}) \xrightarrow{\sim} H^2(X', \mathbf{Z})$$

is a polarized parallel transport operator from (X, H) to (X', H') if and only if $f(H) = H'$.

Definition 3.2.7. We fix one connected component $\mathcal{M}^0_{\mathrm{marked}}$ of the marked moduli space $\mathcal{M}_{\mathrm{marked}}$ as before. Given a polarized pair (X, H), choose a marking η such that (X, η) lies in $\mathcal{M}^0_{\mathrm{marked}}$. We define the *polarization type* T of (X, H) to be the $\mathrm{O}(\Lambda)$-orbit of $\eta(H)$ in Λ. We also denote by τ the $\mathrm{Mon}(\Lambda)$-orbit of $\eta(H)$ in Λ, which is contained in T. This orbit is clearly constant on each connected component $\mathcal{M}^0_T$ of $\mathcal{M}_T$, so we have a map

$$\{\text{connected components of } \mathcal{M}_T\} \longrightarrow \{\mathrm{Mon}(\Lambda)\text{-orbits contained in } T\}, \tag{3.2}$$

which may depend on the initial choice of the connected component $\mathcal{M}^0_{\mathrm{marked}}$. We will call the orbit τ the *deformation type* of (X, H).

Proposition 3.2.6 can be used to show that the deformation type defined here is the good notion. More precisely, we have the following result.

Proposition 3.2.8. *Let T be a polarization type, in other words, an* $\mathrm{O}(\Lambda)$*-orbit of a primitive element of positive square. The map* (3.2) *above gives a bijection from the set of connected components of $\mathcal{M}_T$ to the set of* $\mathrm{Mon}(\Lambda)$*-orbits contained in T.*

Proof. For the injectivity, suppose that two polarized pairs (X, H) and (X', H') have the same deformation type, which means that we may choose markings η and η' such that (X, η) and (X', η') both lie in the fixed connected component $\mathcal{M}^0_{\mathrm{marked}}$, and $\eta(H)$ and $\eta'(H')$ have the same $\mathrm{Mon}(\Lambda)$-orbit in Λ. We want to show that (X, H) and (X', H') lie in the same connected component of $\mathcal{M}_T$.

Suppose that there exists some $\phi \in \mathrm{Mon}(\Lambda)$ such that $\phi \circ \eta(H) = \eta'(H')$. By the definition of $\mathrm{Mon}(\Lambda)$, the marking $(X, \phi \circ \eta)$ is also in $\mathcal{M}^0_{\mathrm{marked}}$. The isomorphism $\eta'^{-1} \circ \phi \circ \eta$ is a parallel transport operator that takes H to H' so, by Proposition 3.2.6, it is a polarized one, that is, (X, H) and (X', H') are indeed connected by some path in the polarized moduli space $\mathcal{M}_T$.

For the surjectivity, since the locus $\mathcal{M}^{\mathrm{amp}}_h$ is non-empty for every $h \in T$, the class h can always be realized as the image $\eta(H)$ for some polarized pair (X, H) and a marking η with (X, η) lying in the fixed connected component $\mathcal{M}^0_{\mathrm{marked}}$. This in particular means that every $\mathrm{Mon}(\Lambda)$-orbit can be realized as the deformation type of some polarized pair. □

So for a given polarization type T, once we picked a connected component $\mathcal{M}^0_{\mathrm{marked}}$, we can distinguish each connected component $\mathcal{M}^0_T$ of $\mathcal{M}_T$ by its deformation type τ. We can thus write $\mathcal{M}_\tau$ instead of $\mathcal{M}^0_T$.

A first observation is that, if the group $\mathrm{Mon}(\Lambda)$ is a proper subgroup of $\mathrm{O}(\Lambda)$, an $\mathrm{O}(\Lambda)$-orbit may contain several $\mathrm{Mon}(\Lambda)$-orbits and consequently, the corresponding polarized moduli space $\mathcal{M}_T$ may have several components. As the result of Apostolov [**Apo14**] shows, this is indeed the case for certain polarization types of $\mathrm{K3}^{[m]}$-type manifolds. We will give a simplified expression for the exact number of components in Proposition 3.3.4.

Finally, we explain the construction of the polarized period map and the statement of the polarized global Torelli theorem, as mentioned in the introduction. For a polarization type T, we consider the connected component $\mathcal{M}_T^0 = \mathcal{M}_\tau$ of the polarized moduli space $\mathcal{M}_T$ corresponding to a $\mathrm{Mon}(\Lambda)$-orbit τ and pick some $h \in \tau$. We consider the stabilizer groups

$$\mathrm{O}(\Lambda, h) \coloneqq \{\phi \in \mathrm{O}(\Lambda) \mid \phi(h) = h\} \quad \text{and} \quad \mathrm{Mon}(\Lambda, h) \coloneqq \mathrm{Mon}(\Lambda) \cap \mathrm{O}(\Lambda, h).$$

For a polarized pair (X, H) of deformation type τ, if we pick a suitable marking η in the connected component $\mathcal{M}^0_{\text{marked}}$ such that $\eta(H) = h$ then, by the ampleness of the class H, the marked pair (X, η) must lie in $\mathcal{M}_h^{\text{amp}}$. By quotienting out the action of the monodromy group, we get the following result [**Mar11**, Lemma 8.1, Lemma 8.3, and Theorem 8.4].

Theorem 3.2.9 (Markman).

(i) The marked period map (3.1) *descends to an open embedding of analytic spaces*

$$\mathcal{M}_h^{\text{amp}} / \mathrm{Mon}(\Lambda, h) \hookrightarrow \Omega_h / \mathrm{Mon}(\Lambda, h),$$

where the second quotient $\Omega_h / \mathrm{Mon}(\Lambda, h)$ is a normal quasi-projective variety by Baily–Borel theory. We denote this quotient by $\mathcal{P}_\tau$, since if we choose another $h' \in \tau$, the two quotients are canonically isomorphic.

(ii) For each $h \in \tau$, there is an isomorphism of analytic spaces

$$\mathcal{M}_\tau \xrightarrow{\sim} \mathcal{M}_h^{\text{amp}} / \mathrm{Mon}(\Lambda, h).$$

The composition with the above embedding gives the polarized period map

$$\wp_\tau \colon \mathcal{M}_\tau \hookrightarrow \mathcal{P}_\tau,$$

which is an open immersion of algebraic varieties.

Notice that if τ and τ' are different $\mathrm{Mon}(\Lambda)$-orbits contained in T, the quotients $\mathcal{P}_\tau$ and $\mathcal{P}_{\tau'}$ are isomorphic but in general not canonically. This can be seen as follows. We consider the quotient $(\Omega_h \sqcup \Omega_{-h}) / \mathrm{O}(\Lambda, h) \simeq \Omega_h / \mathrm{O}^+(\Lambda, h)$, which is again a normal quasi-projective variety. This quotient can be denoted by $\mathcal{P}_T$, since if another $h' \in T$ is chosen, the two quotients are canonically isomorphic. We see that $\mathcal{P}_\tau$ is a covering space of $\mathcal{P}_T$ and it admits an action of the group $\mathrm{O}^+(\Lambda, h) / \mathrm{Mon}(\Lambda, h)$, not necessarily free. The deck transformation group G will be some quotient of this group. Thus we have a diagram

$$\begin{array}{cccc} \wp_\tau \colon \mathcal{M}_\tau \hookrightarrow & \mathcal{P}_\tau & = & \Omega_h / \mathrm{Mon}(\Lambda, h) \\ & \big\downarrow /G & & \\ & \mathcal{P}_T & = & \Omega_h / \mathrm{O}^+(\Lambda, h) \end{array} \tag{3.3}$$

In particular, when G is non-trivial, for two deformation types τ and τ', there is no canonical isomorphism between the period domains $\mathcal{P}_\tau$ and $\mathcal{P}_{\tau'}$: any two such isomorphisms differ by the action of an element in G (to be more precise, in this case we have two groups G_τ and $G_{\tau'}$ that are non-canonically isomorphic).

Remark 3.2.10. For K3 surfaces, the monodromy group $\mathrm{Mon}(\Lambda)$ coincides with $\mathrm{O}^+(\Lambda)$, and each polarization is characterized by its square $2d$. Each period domain $\mathcal{P}_T = \mathcal{P}_{2d}$ is given above as the quotient $(\Omega_h \sqcup \Omega_{-h})/\operatorname{O}(\Lambda, h)$. This is usually formulated in terms of the orthogonal lattice $h^\perp$: the hyperplane section $(\Omega_h \sqcup \Omega_{-h})$ can be identified as the following space

$$\Omega_{h^\perp} := \Big\{[x] \in \mathbf{P}\big((h^\perp)_{\mathbf{C}}\big) \;\Big|\; (x,x) = 0, (x,\bar{x}) > 0\Big\},$$

and by Proposition 3.2.13 below, the group $\operatorname{O}(\Lambda, h)$ restricts to a subgroup $\widetilde{\operatorname{O}}(h^\perp)$ of $\operatorname{O}(h^\perp)$, so $\mathcal{P}_{2d}$ can also be given as the quotient $\Omega_{h^\perp}/\widetilde{\operatorname{O}}(h^\perp)$.

Remark 3.2.11. Another subtlety is that the polarized period map depends on the initial choice of the connected component $\mathcal{M}^0_{\text{marked}}$ for the definition of deformation types: if we choose another connected component by acting on the marking using an element in $\operatorname{Mon}(\Lambda) \cdot \operatorname{O}(\Lambda, h)$, the deformation type—the $\operatorname{Mon}(\Lambda)$-orbit—of $\mathcal{M}^0_T$ is still τ, but the period map is acted on by some element in G; if we choose another connected component by acting on the marking using an element in the larger group $\operatorname{O}(\Lambda)$, the deformation type of $\mathcal{M}^0_T$ may change to an entirely different τ', in which case the period map maps the component $\mathcal{M}^0_T$ to a different $\mathcal{P}_{\tau'}$ that, as we already stated, can only be identified with $\mathcal{P}_\tau$ up to the action of some element in G. In Markman's survey, this subtlety is handled by taking disjoint copies of $\mathcal{M}^{\text{amp}}_h$ (resp. Ω_h) and by quotienting out by the action of $\operatorname{O}(\Lambda)$ to get a canonically defined polarized moduli space (resp. polarized period domain). This approach is certainly more canonical as it does not depend on the particular choice of a connected component $\mathcal{M}^0_{\text{marked}}$. However, it is more difficult to describe the connected components of $\mathcal{M}_T$ in this setting.

Before ending this section, we review some lattice theoretical results that will be used later. We first recall some basic definitions. Let Λ be a lattice with isometry group $\operatorname{O}(\Lambda)$. The *divisibility* $\operatorname{div}(x)$ of a primitive element x in Λ is the positive generator γ of the subgroup (x, Λ) of $\mathbf{Z}$. The *discriminant group* of Λ is the finite abelian group $D(\Lambda) := \Lambda^\vee/\Lambda$. We define $x_* := [x/\operatorname{div}(x)]$, which is an element of $D(\Lambda)$ of order $\operatorname{div}(x)$. When Λ is even, the quadratic form on Λ induces a $(\mathbf{Q}/2\mathbf{Z})$-valued quadratic form on $D(\Lambda)$, and there is a natural homomorphism $\chi\colon \operatorname{O}(\Lambda) \to \operatorname{O}(D(\Lambda))$. In this case, we let $\widetilde{\operatorname{O}}(\Lambda)$ and $\widehat{\operatorname{O}}(\Lambda)$ be the respective preimages of $\{1\}$ and $\{\pm 1\}$ by χ. We have the following results from lattice theory.

Proposition 3.2.12 ([**Nik79**, Theorem 1.14.2]). *For any even indefinite lattice Λ of rank larger than or equal to the minimal number of generators of $D(\Lambda)$ plus 2, the homomorphism $\chi\colon \operatorname{O}(\Lambda) \to \operatorname{O}(D(\Lambda))$ is surjective.*

Proposition 3.2.13 ([**GHS10**, Lemma 3.2]). *Let Λ be an even unimodular lattice and let x be an element of Λ with non-zero square. Denote by $x^\perp$ the orthogonal of x in Λ. We have*

$$\operatorname{O}(\Lambda, x)|_{x^\perp} = \widetilde{\operatorname{O}}(x^\perp),$$

where $\operatorname{O}(\Lambda, x)$ is the stabilizer group of x in $\operatorname{O}(\Lambda)$.

PROPOSITION 3.2.14 (Eichler's criterion, [**GHS10**, Lemma 3.5]). *Let Λ be an even lattice which contains at least two orthogonal copies of the hyperbolic plane U. The $\widetilde{\mathrm{O}}(\Lambda)$-orbit of a primitive element x is determined by its square x^2 and the class $x_* = [x/\operatorname{div}(x)]$ in $D(\Lambda)$.*

The Eichler's criterion can be slightly strengthened by replacing $\widetilde{\mathrm{O}}(\Lambda)$ with smaller subgroups.

PROPOSITION 3.2.15. *Under the same assumption for Λ as above, for a primitive element x, the following three orbits coincide*

$$\widetilde{\mathrm{O}}(\Lambda)x = \widetilde{\mathrm{SO}}(\Lambda)x = \widetilde{\mathrm{SO}}^+(\Lambda)x.$$

In particular, all three orbits are determined by the square x^2 and the class x_* in $D(\Lambda)$.

PROOF. Write $\Lambda = U_1 \oplus U_2 \oplus \Lambda_0$ where U_1 and U_2 are two copies of the hyperbolic plane U. Since U is unimodular, by Eichler's criterion, we may find $\phi \in \widetilde{\mathrm{O}}(\Lambda)$ such that $\phi(x) \in U_2 \oplus \Lambda_0$. Take $u, v \in U_1$ with $u^2 = 2$ and $v^2 = -2$, then the reflections R_u, R_v lie in $\mathrm{O}(\Lambda, \phi(x))$ and they satisfy $\sigma(R_u) = -1$, $\sigma(R_v) = 1$, $\chi(R_u) = \chi(R_v) = 1$, and $\det(R_u) = \det(R_v) = -1$.

Now for $\varphi \in \widetilde{\mathrm{O}}(\Lambda)$ with $\det(\varphi) = -1$, we have $\varphi(x) = \varphi \circ \phi^{-1} \circ \phi(x) = \varphi \circ \phi^{-1} \circ R_u \circ \phi(x)$, and the element $\varphi \circ \phi^{-1} \circ R_u \circ \phi$ has determinant 1, so $\varphi(x)$ lies in the same $\widetilde{\mathrm{SO}}(\Lambda)$-orbit as x and we get $\widetilde{\mathrm{O}}(\Lambda)x = \widetilde{\mathrm{SO}}(\Lambda)x$.

Similarly, for $\varphi \in \widetilde{\mathrm{SO}}(\Lambda)$ with $\sigma(\varphi) = -1$, we have $\varphi(x) = \varphi \circ \phi^{-1} \circ \phi(x) = \varphi \circ \phi^{-1} \circ R_u \circ R_v \circ \phi(x)$, and the element $\varphi \circ \phi^{-1} \circ R_u \circ R_v \circ \phi$ lies in $\widetilde{\mathrm{SO}}^+(\Lambda)$, so we get $\widetilde{\mathrm{SO}}(\Lambda)x = \widetilde{\mathrm{SO}}^+(\Lambda)x$. □

3.3. Monodromy group and number of components

In this section, we will calculate the number of components of the moduli space $\mathcal{M}_T$ of a given polarization type T, for all known deformation types. The polarization type determines the square and the divisibility of its elements, but the converse is in general not true: we will calculate the number of T with given square and divisibility.

First we recollect the descriptions for the lattice $\Lambda = H^2(X, \mathbf{Z})$ and the monodromy group $\mathrm{Mon}(\Lambda)$ for all known deformation types. The lattice structures for $\mathrm{K3}^{[m]}$ and Kum_m are known by Beauville [**Bea83**], and for OG_6 and OG_{10} they are computed by Rapagnetta [**Rap08**]. The monodromy group is computed by Markman in the $\mathrm{K3}^{[m]}$-case, Markman and Mongardi in the Kum_m-case [**Mar22, Mon16**], Mongardi–Rapagnetta for OG_6 [**MR21**], and Onorati for OG_{10} [**Ono22**].

Theorem 3.3.1. *The descriptions for the lattice* $\Lambda = H^2(X, \mathbf{Z})$ *and the monodromy group* $\mathrm{Mon}(\Lambda)$ *for all known deformation types are as follows.*

	$\Lambda = H^2(X, \mathbf{Z})$	$D(\Lambda)$	$\mathrm{Mon}(\Lambda)$
K3	$U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$	0	$\mathrm{O}^+(\Lambda)$
$\mathrm{K3}^{[m]}$	$\Lambda_{\mathrm{K3}} \oplus \langle -(2m-2) \rangle$	$\mathbf{Z}/(2m-2)\mathbf{Z}$	$\widehat{\mathrm{O}}^+(\Lambda)$
Kum_m	$U^{\oplus 3} \oplus \langle -(2m+2) \rangle$	$\mathbf{Z}/(2m+2)\mathbf{Z}$	$\left\{ g \in \widehat{\mathrm{O}}^+(\Lambda) \,\middle\vert\, \chi(g) \cdot \det(g) = 1 \right\}$
OG_6	$U^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2}$	$(\mathbf{Z}/2\mathbf{Z})^2$	$\mathrm{O}^+(\Lambda)$
OG_{10}	$\Lambda_{\mathrm{K3}} \oplus \left(\begin{smallmatrix} -6 & 3 \\ 3 & -2 \end{smallmatrix}\right)$	$\mathbf{Z}/3\mathbf{Z}$	$\mathrm{O}^+(\Lambda)$

Table 2. Lattice and monodromy group for known deformation types

Here U *is the hyperbolic plane,* $E_8(-1)$ *is the* E_8*-lattice with negative definite form, and* $\langle k \rangle$ *is the lattice generated by one primitive element with square* k*.*

We may compute the number of components for a given polarization type T using Proposition 3.2.8. We first prove a lemma concerning the orthogonal group of the discriminant group $D(\Lambda)$.

Lemma 3.3.2. *Let* D *be a cyclic group of order* $2n$ *with a quadratic form* $q\colon D \to \mathbf{Q}/2\mathbf{Z}$*. If there is a generator* $g \in D$ *with* $q(g) = \frac{1}{2n}$*, then*

$$\mathrm{O}(D) = \left\{ g \longmapsto ag \,\middle\vert\, \begin{array}{c} a \in \mathbf{Z}/2n\mathbf{Z} \\ a^2 \equiv 1 \pmod{4n} \end{array} \right\} \simeq (\mathbf{Z}/2\mathbf{Z})^{\rho(n)},$$

where $\rho(n)$ *denotes the number of distinct prime divisors of* n*.*

Proof. Write $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $r = \rho(n)$. If n is odd, a is determined by the conditions $a \equiv 1 \pmod 2$ and $a \equiv \pm 1 \pmod{p_i^{\alpha_i}}$; if n is even, we let $p_1 = 2$, then a is determined by the conditions $a \equiv \pm 1 \pmod{2^{\alpha_1 + 1}}$ and $a \equiv \pm 1 \pmod{p_i^{\alpha_i}}$ for $i \geq 2$. In both cases, we have $\mathrm{O}(D) \simeq (\mathbf{Z}/2\mathbf{Z})^{\rho(n)}$. □

The Eichler's criterion allows us to compute the number of $\widetilde{\mathrm{O}}(\Lambda)$-orbits.

Lemma 3.3.3. *Let* Λ *be an even lattice containing at least two orthogonal copies of the hyperbolic plane* U*, such that the discriminant group* $D(\Lambda)$ *is cyclic of order* $2n$*. Then for each* $\mathrm{O}(\Lambda)$*-orbit* T *of a primitive element with divisibility* γ*, the number of* $\widetilde{\mathrm{O}}(\Lambda)$*-orbits contained in* T *is equal to* $2^{\widetilde{\rho}(\gamma)}$*, where* $\widetilde{\rho}(n)$ *is equal to* $\rho(n)$*—the number of distinct prime divisors of* n*—if* n *is odd, and* $\rho(n/2)$ *if* n *is even.*

Proof. Fix one element $h \in T$ so T is the set $\{\phi(h) \mid \phi \in \mathrm{O}(\Lambda)\}$. By Eichler's criterion (Proposition 3.2.14), as the square is fixed, the number of $\widetilde{\mathrm{O}}(\Lambda)$-orbits is the same as the number of possible values of $\big(\phi(h)\big)_* = \chi(\phi)(h_*) \in D(\Lambda)$ for all $\phi \in \mathrm{O}(\Lambda)$. The lattice Λ satisfies the condition in Proposition 3.2.12, so the homomorphism $\chi\colon \mathrm{O}(\Lambda) \to \mathrm{O}(D(\Lambda))$ is surjective. Therefore it suffices to count the number of possible $ah_* \in D(\Lambda)$ for all $a \in \mathrm{O}(D(\Lambda))$. Since h is primitive of divisibility γ, the class $h_* = [h/\gamma]$ is of order γ. Viewing the isometry a as an element of $\mathbf{Z}/2n\mathbf{Z}$, we therefore need to count the number of possible remainders of a modulo γ under the quotient map $\mathbf{Z}/2n\mathbf{Z} \to \mathbf{Z}/\gamma\mathbf{Z}$.

Using a similar argument as in the proof of Lemma 3.3.2, we write $\gamma = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $r = \rho(\gamma)$. If γ is odd, then a modulo γ can take all the values satisfying $a \equiv \pm 1 \pmod{p_i^{\alpha_i}}$. If γ is even, let $p_1 = 2$; if γ is not divisible by 4, that is, $\alpha_1 = 1$, then a modulo γ can take all the values satisfying $a \equiv 1 \pmod 2$ and $a \equiv \pm 1 \pmod{p_i^{\alpha_i}}$; if $\alpha_1 \geq 2$, a modulo γ can take all the values satisfying $a \equiv \pm 1 \pmod{2^{\alpha_1+1}}$ and $a \equiv \pm 1 \pmod{p_i^{\alpha_i}}$ for $i \geq 2$. Combining all three cases, the number of $\widetilde{\mathrm{O}}(\Lambda)$-orbits is equal to $2^{\widetilde{\rho}(\gamma)}$. □

Now we can compute the number of connected components.

Proposition 3.3.4. *Let X be a hyperkähler manifold and T be a polarization type of divisibility γ on X.*

- *If X is of $\mathrm{K3}^{[m]}$-type or Kum_m-type, the number of connected components of the polarized moduli space $\mathcal{M}_T$ is equal to $2^{\max(\widetilde{\rho}(\gamma)-1,0)}$.*
- *If X is of OG_6-type or OG_{10}-type, the polarized moduli space $\mathcal{M}_T$ is connected.*

Proof. As Proposition 3.2.8 shows, the number of connected components of $\mathcal{M}_T$ is equal to the number of $\mathrm{Mon}(\Lambda)$-orbits contained in the $\mathrm{O}(\Lambda)$-orbit T. We fix one element $h \in T$, so T is the set $\{\phi(h) \mid \phi \in \mathrm{O}(\Lambda)\}$.

Case $\mathrm{K3}^{[m]}$**:** The discriminant group $D(\Lambda)$ is cyclic of order $2m-2$, so Lemma 3.3.3 applies and the number of $\widetilde{\mathrm{O}}(\Lambda)$-orbits contained in T is equal to $2^{\widetilde{\rho}(\gamma)}$.

Since the subgroup $\widehat{\mathrm{O}}(\Lambda)$ is generated by $\widetilde{\mathrm{O}}(\Lambda)$ and $-\mathrm{Id}$, we see that if h and $-h$ are in the same $\widetilde{\mathrm{O}}(\Lambda)$-orbit, that is, when γ is 1 or 2, the number of $\widehat{\mathrm{O}}(\Lambda)$-orbits is the same as the number of $\widetilde{\mathrm{O}}(\Lambda)$-orbits; otherwise it should be divided by 2. So this gives $2^{\max(\widetilde{\rho}(\gamma)-1,0)}$ as the number of $\widehat{\mathrm{O}}(\Lambda)$-orbits.

To conclude, we show that the $\widehat{\mathrm{O}}(\Lambda)$-orbits and the $\widehat{\mathrm{O}}^+(\Lambda)$-orbits are the same. Following the proof of Proposition 3.2.15, there is an element $R \in \mathrm{O}(\Lambda, h)$ (namely R_u) with $\sigma(R) = -1$ and $\chi(R) = 1$. Now for $\phi \in \widehat{\mathrm{O}}(\Lambda)$ with $\sigma(\phi) = -1$, we have $\phi(h) = \phi \circ R(h)$, where $\phi \circ R$ lies in $\widehat{\mathrm{O}}^+(\Lambda)$. So $\phi(h)$ lies in the same $\widehat{\mathrm{O}}^+(\Lambda)$-orbit as h and therefore $\widehat{\mathrm{O}}(\Lambda)h = \widehat{\mathrm{O}}^+(\Lambda)h$.

Case Kum_m**:** The discriminant group $D(\Lambda)$ is cyclic of order $2m+2$, so again Lemma 3.3.3 applies and we get $2^{\widetilde{\rho}(\gamma)}$ as the number of $\widetilde{\mathrm{O}}(\Lambda)$-orbits. By Proposition 3.2.15, this is also the number of $\widetilde{\mathrm{SO}}^+(\Lambda)$-orbits.

Moreover, following the proof of Proposition 3.2.15, there exists an element $R \in O(\Lambda, h)$ (namely $R_u \circ R_v$) such that $\sigma(R) = -1$, $\chi(R) = 1$, $\det(R) = 1$. On the other hand, we note that $\sigma(-\operatorname{Id}) = -1$, $\chi(-\operatorname{Id}) = -1$, $\det(-\operatorname{Id}) = -1$. This shows that $\operatorname{Mon}(\Lambda)$ is generated by $\widetilde{\mathrm{SO}}^+(\Lambda)$ and $-R$. If h and $-h = -R(h)$ are in the same $\widetilde{\mathrm{SO}}^+(\Lambda)$-orbit, that is, when γ is 1 or 2, then the number of $\operatorname{Mon}(\Lambda)$-orbits is the same as the number of $\widetilde{\mathrm{SO}}^+(\Lambda)$-orbits; otherwise it should be divided by 2. So again we obtain $2^{\max(\widetilde{\rho}(\gamma)-1,0)}$ as the number of $\operatorname{Mon}(\Lambda)$-orbits.

Case OG_6 **and** OG_{10}**:** In these two cases, the monodromy group is equal to $\mathrm{O}^+(\Lambda)$. Again, following the proof of Proposition 3.2.15, there exists a reflection $R \in O(\Lambda, h)$ (namely R_u) such that $\sigma(R) = -1$. So one may conclude that the $\mathrm{O}^+(\Lambda)$-orbit of h coincides with the entire $\mathrm{O}(\Lambda)$-orbit T. □

We also have the following result on the number of polarization types with given square and divisibility on a hyperkähler manifold of $\mathrm{K3}^{[m]}$-type or Kum_m-type. Together with Proposition 3.3.4, this gives a refined version of Apostolov's result [**Apo14**] for $\mathrm{K3}^{[m]}$ and Onorati's [**Ono16**] result for Kum_m (*cf.* also [**GHS10**, Proposition 3.6]).

PROPOSITION 3.3.5. *Let m, n, and γ be positive integers with $m \geq 2$. Let $\widetilde{m}$ be $m-1$ for the $\mathrm{K3}^{[m]}$-type and $m+1$ for the Kum_m-type, so in both cases we have $D(\Lambda) \simeq \mathbf{Z}/2\widetilde{m}\mathbf{Z}$. Moreover we assume that $\gamma \mid \gcd(2\widetilde{m}, 2n)$. For a prime divisor p of γ, set $\alpha := \min(v_p(\widetilde{m}), v_p(n))$ and $\beta := v_p(\gamma)$, where v_p is the p-adic valuation. Then there exists a polarization type T of square $2n$ and of divisibility γ, if and only if the following conditions are satisfied for all prime divisors p of γ:*

- *if $v_p(\widetilde{m}) \neq v_p(n)$, then $\beta \leq \alpha/2$;*
- *if $v_p(\widetilde{m}) = v_p(n) = \alpha$, then either $\beta \leq \alpha/2$, or $\beta > \alpha/2$ and $-n/\widetilde{m}$ is a square modulo $p^{2\beta-\alpha}$.*

The total number of these T is given by the product $\prod_{p|\gamma} N_p$, where for $p \geq 3$

$$N_p := \begin{cases} \frac{1}{2}(p-1)p^{\beta-1} & \text{if } \beta \leq \alpha/2; \\ p^{\alpha-\beta} & \text{if } \beta > \alpha/2; \end{cases}$$

and for $p = 2$

$$N_2 := \begin{cases} 1 & \text{if } \beta = 1; \\ 2^{\beta-2} & \text{if } \beta \geq 2, \beta \leq \alpha/2+1; \\ 2^{\alpha+1-\beta} & \text{if } \beta > \alpha/2+1. \end{cases}$$

PROOF. For the $\mathrm{K3}^{[m]}$-type and the Kum_m-type, we have $\Lambda = \Lambda_0 \oplus \mathbf{Z}\delta$, where Λ_0 is an even unimodular lattice containing three orthogonal copies of the hyperbolic plane U, and δ is of square $-2\widetilde{m}$. The discriminant group is cyclic of order $2\widetilde{m}$, generated by δ_*.

We first study the existence of a polarization type with given square and divisibility. Let $h \in \Lambda$ be a primitive element of divisibility γ. If $\gamma = 1$, since Λ_0 contains orthogonal copies of U, it is clear that a polarization type of square $2n$ exists for all $n > 0$. So we will look at $\gamma \geq 2$. We write

$$h = \gamma a x + b\delta,$$

where $x \in \Lambda_0$ is primitive of square $x^2 = 2c$, with $a, b, c \in \mathbf{Z}$ such that $\gcd(\gamma a, 2\widetilde{m}) = \gamma$ and $\gcd(\gamma a, b) = 1$. Suppose that h is of square $2n$. We obtain the relation

$$2n = h^2 = 2ca^2\gamma^2 - b^2 \cdot 2\widetilde{m}.$$

For such an h to exist, it is necessary and sufficient that there exist some integer b satisfying

$$\gamma^2 \mid b^2\widetilde{m} + n.$$

For each prime divisor p of γ, since $\gcd(\gamma, b) = 1$, we see that b is not divisible by p. So if $v_p(\widetilde{m}) \neq v_p(n)$, then $v_p(b^2\widetilde{m} + n) = \min(v_p(\widetilde{m}), v_p(n))$ and we obtain the first condition; if $v_p(\widetilde{m}) = v_p(n) = \alpha$, then for $p^{2\beta} \mid b^2\widetilde{m} + n$ to hold we obtain the second condition.

Given the square and the divisibility, to count the number of such $\mathrm{O}(\Lambda)$-orbits T, we first count the number of $\widetilde{\mathrm{O}}(\Lambda)$-orbits. Any such element h can again be expressed as $\gamma a x + b\delta$. By Eichler's criterion, since the square is fixed, the number of $\widetilde{\mathrm{O}}(\Lambda)$-orbits is just the number of possible $h_* = \frac{b \cdot 2\widetilde{m}}{\gamma}\delta_*$, or equivalently, the number of possible remainders of b modulo γ. We thus express this number as the product $\prod_{p|\gamma} M_p$, where M_p is the number of possible remainders of b modulo p^β.

For $p \geq 3$, if $\beta \leq \alpha/2$, then we only need $\gcd(b, p) = 1$, thus M_p is equal to $(p-1)p^{\beta-1}$; if $\beta > \alpha/2$, then the equation $b^2 \equiv -n/\widetilde{m} \pmod{p^{2\beta-\alpha}}$ has two solutions, thus M_p is equal to $2p^{\alpha-\beta}$.

For $p = 2$, as $\gcd(b, p) = 1$, we see first that b is necessarily odd. If $\beta \leq \alpha/2 + 1$, we will show that this is also sufficient, so M_2 is equal to $2^{\beta-1}$. To prove this, we distinguish three cases: if $\beta \leq \alpha/2$, it is clear that $b^2\widetilde{m} + n$ is a multiple of $2^{2\beta}$; if $\beta = \alpha/2 + 1/2$, then $v_p(\widetilde{m}) = v_p(n) = \alpha$ and $b^2\widetilde{m} + n$ is a multiple of $2^{\alpha+1} = 2^{2\beta}$; if $\beta = \alpha/2 + 1$, then $v_p(\widetilde{m}) = v_p(n) = \alpha$ and $-n/\widetilde{m} \equiv 1 \pmod 4$, so $b^2\widetilde{m} + n$ is a multiple of $2^{\alpha+2} = 2^{2\beta}$. If $\beta > \alpha/2 + 1$, the equation $b^2 \equiv 1 \pmod{2^{2\beta-\alpha}}$ has two solutions modulo $2^{2\beta-\alpha-1}$, so M_2 is equal to $2 \times 2^{\alpha+1-\beta}$.

To conclude, as Lemma 3.3.3 shows that each $\mathrm{O}(\Lambda)$-orbit T contains $2^{\widetilde{\rho}(\gamma)}$ different $\widetilde{\mathrm{O}}(\Lambda)$-orbits, the number of T is given by $\prod_{p|\gamma} M_P$ divided by $2^{\widetilde{\rho}(\gamma)}$. We let $N_p = M_p/2$ for $p \geq 3$, $N_2 = M_2/2$ if $v_2(\gamma) \geq 2$, and $N_2 = M_2 = 1$ if $v_2(\gamma) = 1$. This gives the desired formula. □

For completeness, we also provide the results for OG_6 and OG_{10}.

Proposition 3.3.6. *Let n and γ be positive integers. For the OG_6-type and the OG_{10}-type, a polarization type T is uniquely determined by its square $2n$ and divisibility γ.*

- *For the OG_6-type, such T exists if and only if $\gamma = 1$, or $\gamma = 2$ and $n \equiv 2, 3 \pmod 4$;*
- *for the OG_{10}-type, such T exists if and only if $\gamma = 1$, or $\gamma = 3$ and $n \equiv 6 \pmod 9$.*

Proof. In both cases, since the lattice Λ contains orthogonal copies of U, the existence of a polarization type of square $2n$ and divisibility 1 is clear, and the uniqueness follows from Eichler's criterion.

For the OG_6-type, we write u and v for the two generators with square -2 so $\Lambda = \Lambda_0 \oplus \mathbf{Z}u \oplus \mathbf{Z}v$. Each primitive element h of divisibility 2 can be written as

$$h = 2ax + bu + cv,$$

where $x \in \Lambda_0$ is primitive with $x^2 = 2d$ and $a, b, c, d \in \mathbf{Z}$, such that $\gcd(2a, b, c) = 1$. In particular, b and c cannot be both even, and the class h_* is given by $(\bar{b}, \bar{c}) \in (\mathbf{Z}/2\mathbf{Z})^2$. Suppose that h is of square $2n$. We obtain the relation

$$2n = h^2 = 8a^2d - 2b^2 - 2c^2,$$

and we may deduce that $4 \mid n + b^2 + c^2$. If $n \not\equiv 2, 3 \pmod 4$ there are no integer solutions. If $n \equiv 2 \pmod 4$, then b and c must both be odd, so $h_* = (\bar{1}, \bar{1})$ and by Eichler's criterion all such h lie in the same $\widetilde{\mathrm{O}}(\Lambda)$-orbit, so the $\mathrm{O}(\Lambda)$-orbit is also unique. If $n \equiv 3 \pmod 4$, then b and c must be one odd one even, so h_* can either be $(\bar{1}, \bar{0})$ or $(\bar{0}, \bar{1})$, and by Eichler's criterion there are two $\widetilde{\mathrm{O}}(\Lambda)$-orbits, but the map that interchanges the coordinates u and v is an isometry, so these two lie in the same $\mathrm{O}(\Lambda)$-orbit, and again we get the uniqueness.

For the OG_{10}-type, we similarly write u and v for the two generators with matrix $\left(\begin{smallmatrix} -6 & 3 \\ 3 & -2 \end{smallmatrix}\right)$, so $\Lambda = \Lambda_0 \oplus \mathbf{Z}u \oplus \mathbf{Z}v$. Each primitive element h of divisibility 3 can be written as

$$h = 3ax + bu + 3cv,$$

where $x \in \Lambda_0$ is primitive with $x^2 = 2d$ and $a, b, c, d \in \mathbf{Z}$, such that $\gcd(3a, b, 3c) = 1$. In particular, b is not divisible by 3, and the class h_* is given by $\bar{b} \in \mathbf{Z}/3\mathbf{Z}$. Suppose that h is of square $2n$. We obtain the relation

$$2n = h^2 = 18a^d - 6b^2 + 18bc - 18c^2,$$

and we may deduce that $9 \mid n + 3b^2$, so we must have $n \equiv 6 \pmod 9$. By Eichler's criterion there are two $\widetilde{\mathrm{O}}(\Lambda)$-orbits depending on the value $h_* \in D(\Lambda) = \mathbf{Z}/3\mathbf{Z}$, but $-\mathrm{Id}$ interchanges the two non-zero classes in $D(\Lambda)$ so again the two lie in the same $\mathrm{O}(\Lambda)$-orbit. □

3.4. Image of the period map

We will now study the image of the polarized period map. For all known deformation types, the complement of the image in the period domain can be shown to be a finite union of divisors: we will give explicit numerical conditions describing these divisors. The image of the period map is closely related to the determination of the ample cone, which has been settled for all known deformation types, so we first review the results.

Recall that on $H^{1,1}(X, \mathbf{R})$ the Beauville–Bogomolov–Fujiki form induces a quadratic form of signature $(1, b_2 - 3)$, so the cone of positive classes has two connected components, and we call the one containing a Kähler class the *positive cone* and denote it by $\mathcal{C}_X$.

The cone of all Kähler classes sits inside $\mathcal{C}_X$ and is denoted by $\mathcal{K}_X$. We also consider the *birational Kähler cone* $\mathcal{BK}_X$, which is the union $\bigcup f^{-1}\mathcal{K}_{X'}$ over all birational maps f from X to some other hyperkähler manifold X'. The Néron–Severi group $\mathrm{NS}(X)$ is a sublattice $H^2(X,\mathbf{Z}) \cap H^{1,1}(X,\mathbf{R})$ inside $H^2(X,\mathbf{Z})$.

We have the following crucial notion: a divisor D on X is called a *wall divisor*, if $D^2 < 0$ and $f(D^\perp) \cap \mathcal{BK}_X = \varnothing$ for all monodromy operators f (*cf.* [**Mon15**, Definition 1.2] and [**AV15**, Definition 1.13]). The property of being a wall divisor is stable under parallel transport operators [**Mon15**, Theorem 3.1].

THEOREM 3.4.1 (Mongardi). *Let (X,η) and (X',η') be two marked hyperkähler manifolds lying in the same connected component $\mathcal{M}^0_{\mathrm{marked}}$ of the marked moduli space. Let $D \in \mathrm{NS}(X)$ and $D' \in \mathrm{NS}(X')$ be divisors such that $\eta^{-1} \circ \eta(D) = D'$. Then D is a wall divisor on X if and only if D' is a wall divisor on X'.*

Once we picked a connected component $\mathcal{M}^0_{\mathrm{marked}}$, we may extend this notion to elements of the lattice Λ: a class $\kappa \in \Lambda$ with $\kappa^2 < 0$ is called a *wall class*, if for all $(X,\eta) \in \mathcal{M}^0_{\mathrm{marked}}$ such that the class $\eta^{-1}(\kappa)$ is of type $(1,1)$, it gives a wall divisor on X. Clearly the property only depends on the $\mathrm{Mon}(\Lambda)$-orbit of κ. Wall divisors give a chamber decomposition on the positive cone $\mathcal{C}_X$, and the Kähler cone $\mathcal{K}_X$ is given by one of the chambers.

For $\mathrm{K3}^{[m]}$-type and Kum_m-type, a numerical characterization for wall divisors is known. Write as before $\widetilde{m} = m-1$ for $\mathrm{K3}^{[m]}$-type and $\widetilde{m} = m+1$ for Kum_m-type. Recall that in these two cases, the lattice Λ has the form $\Lambda = \Lambda_0 \oplus \mathbf{Z}\delta$, where Λ_0 is an even unimodular lattice containing three orthogonal copies of U, and δ is of square $-2\widetilde{m}$. We also consider the *Mukai lattice*

$$\widetilde{\Lambda} \coloneqq \Lambda_0 \oplus U,$$

which is even and unimodular. For any vector $v \in \widetilde{\Lambda}$ of square $2\widetilde{m}$, the sublattice $v^\perp$ is isomorphic to Λ. Since all such v are in the same $\mathrm{O}(\widetilde{\Lambda})$-orbit due to the unimodularity of $\widetilde{\Lambda}$, we may fix $v = u_1 + \widetilde{m}u_2$, where $\langle u_1, u_2\rangle$ is a copy of the hyperbolic plane U, and identify Λ as the sublattice $v^\perp$. In particular we set $\delta = u_1 - \widetilde{m}u_2$.

When X is of $\mathrm{K3}^{[m]}$-type or Kum_m-type, there is an embedding of $H^2(X,\mathbf{Z})$ into $\widetilde{\Lambda}$, canonical up to the action of $\mathrm{O}(\widetilde{\Lambda})$ (see [**Mar11**, Corollary 9.5] for $\mathrm{K3}^{[m]}$-type, and [**Wie18**, Theorem 4.9] for Kum_m-type). For any such embedding, the orthogonal of its image is generated by a vector of square $2\widetilde{m}$. So we can assume that the image is exactly Λ, by mapping one of these generators to the fixed v using some element in $\mathrm{O}(\widetilde{\Lambda})$. In this way, we get a distinguished marking $\eta\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} \Lambda$, canonical up to the action of $\{\pm\mathrm{Id}\}\cdot \mathrm{O}(\widetilde{\Lambda},v)|_\Lambda$. By Proposition 3.2.13, this group is equal to $\{\pm\mathrm{Id}\}\cdot\widetilde{\mathrm{O}}(\Lambda) = \widehat{\mathrm{O}}(\Lambda)$. Therefore we get the following result.

PROPOSITION 3.4.2. *Let X be a hyperkähler manifold of $\mathrm{K3}^{[m]}$-type or Kum_m-type. There is a distinguished marking*

$$\eta\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} \Lambda \subset \widetilde{\Lambda},$$

canonical up to the action of $\widehat{\mathrm{O}}(\Lambda)$*. It induces an isometry between the two discriminant groups* $D(H^2(X,\mathbf{Z}))$ *and* $D(\Lambda) \simeq \mathbf{Z}/2\widetilde{m}\mathbf{Z}$*, canonical up to a sign. In other words, there is a canonical choice of a pair of generators* $\pm g$ *for* $D(H^2(X,\mathbf{Z}))$*, mapped to* $\pm\delta_*$ *under the isometry.*

Any monodromy operator must respect the choice of the pair of generators $\pm g$, so the monodromy group $\mathrm{Mon}(\Lambda)$ must lie in the subgroup $\widehat{\mathrm{O}}(\Lambda)$, which is indeed the case.

We now give the description of the Kähler cone $\mathcal{K}_X$ for these two cases. The $\mathrm{K3}^{[m]}$-case is due to the results of Bayer–Macrì, Bayer–Hassett–Tschinkel, and Mongardi (note that in [**BHT15**], the manifold X is assumed to be projective; this assumption can be removed using [**Mon15**] or [**AV15**, Theorem 1.17 and 1.19]). The Kum_m-case is due to Yoshioka [**Yos16**] (see also [**Mon16**]).

Theorem 3.4.3 (Bayer–Macrì, Bayer–Hassett–Tschinkel, Mongardi; Yoshioka). *Let* X *be a hyperkähler manifold of* $\mathrm{K3}^{[m]}$*-type or* Kum_m*-type. Under the embedding*

$$\eta\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} \Lambda \hookrightarrow \widetilde{\Lambda},$$

we denote by $\widetilde{\Lambda}_{\mathrm{alg}}$ *the saturation of* $\eta(\mathrm{NS}(X)) \oplus \mathbf{Z}v$*. Consider the set*

$$S := \begin{cases} \left\{ s \in \widetilde{\Lambda} \;\middle|\; s^2 \geq -2,\ |(s,v)| \leq \widetilde{m} = m-1 \right\} \smallsetminus \{0\} & \text{if } X \text{ is of } \mathrm{K3}^{[m]}\text{-type;} \\ \left\{ s \in \widetilde{\Lambda} \;\middle|\; s^2 \geq 0,\ 0 < |(s,v)| \leq \widetilde{m} = m+1 \right\} & \text{if } X \text{ is of } \mathrm{Kum}_m\text{-type.} \end{cases}$$

Then the Kähler cone $\mathcal{K}_X$ *is one of the connected components of the positive cone* $\mathcal{C}_X$ *cut out by the hyperplanes* $s^\perp$ *in* $\mathrm{NS}(X)_{\mathbf{R}}$*, for all* $s \in S \cap \widetilde{\Lambda}_{\mathrm{alg}}$*.*

Note that the particular choice of the embedding η does not matter here: because η is unique up to the action of $\mathrm{O}(\widetilde{\Lambda})$, and the set S is clearly $\mathrm{O}(\widetilde{\Lambda})$-invariant.

This description depends on the larger lattice $\widetilde{\Lambda}$, which is inconvenient to work with. Note that each $s \in S$ together with v span a rank-2 sublattice of $\widetilde{\Lambda}$, so we may consider its intersection with Λ, which is of rank 1, and pick a generator $\kappa \in \Lambda$. The hyperplane $s^\perp$ can then also be expressed as $\kappa^\perp$. Since the class κ lies in $\mathrm{NS}(X)$ if and only if s lies in $\widetilde{\Lambda}_{\mathrm{alg}}$, we may conclude that all wall classes arise this way from some $s \in S$. We now give a lattice theoretical result, which will yield a numerical criterion for wall classes $\kappa \in \Lambda$ that is intrinsic to the smaller lattice Λ.

Proposition 3.4.4. *Let* $\widetilde{\Lambda}$ *be a lattice of the form* $\Lambda_0 \oplus U$*, where* Λ_0 *is an even unimodular lattice and* U *is the hyperbolic plane with basis* u_1, u_2*. Let* $v = u_1 + \widetilde{m}u_2$ *and* $\delta = u_1 - \widetilde{m}u_2$*, and let* Λ *be the sublattice* $v^\perp = \Lambda_0 \oplus \mathbf{Z}\delta$*. Let* $\kappa \in \Lambda$ *be a primitive vector and write* $\kappa^2 = 2l$ *and* $\kappa_* = k\delta_* \in D(\Lambda) \simeq \mathbf{Z}/2\widetilde{m}\mathbf{Z}$*, where* $|k| \leq \widetilde{m}$*. Set* $d := \gcd(2\widetilde{m}, k)$*.*

(i) There is a unique integer c *such that*

$$l = c\left(\frac{2\widetilde{m}}{d}\right)^2 - \widetilde{m}\left(\frac{k}{d}\right)^2.$$

(ii) Let $a \in \mathbf{Z}_{\geq 0}$ be a non-negative integer. There is a non-zero element $s \in \widetilde{\Lambda}$ contained in the saturation of the sublattice generated by κ and v, such that

$$s^2 \geq -2a, \quad |(s,v)| \leq \widetilde{m},$$

if and only if the integer c in (i) satisfies $c \geq -a$. When this is the case, there is one such element s with $s^2 = 2c$ and $(s,v) = -k$.

Proof. First we may assume that $k \geq 0$ by changing κ to $-\kappa$ if needed. Since $\kappa_* = [\kappa/\operatorname{div}(\kappa)]$ is equal to $k\delta_* = [k\delta/2\widetilde{m}]$ in $D(\Lambda)$, we may write

$$\frac{\kappa}{\operatorname{div}(\kappa)} = x + b\delta + \frac{k\delta}{2\widetilde{m}},$$

where $x \in \Lambda_0$ and $b \in \mathbf{Z}$. Since κ is integral and primitive, we see that $\operatorname{div}(\kappa) = \frac{2\widetilde{m}}{d}$. Now we let

$$s := \frac{d\kappa - kv}{2\widetilde{m}} = x + b\delta - ku_2,$$

which is an integral class in $\widetilde{\Lambda} \setminus \{0\}$, with $|(s,v)| = |-k| \leq \widetilde{m}$. Let $s^2 = 2c$. We can easily verify that c is the integer satisfying the equality in (i). Moreover, if $c \geq -a$, the vector s provides the element we need in (ii).

Conversely, suppose that there is some other vector s' satisfying the condition in (ii), then we will show that $c \geq -a$ so the vector s itself satisfies the condition. We let ${s'}^2 = 2c'$ with $c' \geq -a$, and $(s', v) = -k'$ with $|k'| \leq \widetilde{m}$. Since $2\widetilde{m}s'$ lies in the direct sum $\mathbf{Z}\kappa \oplus \mathbf{Z}v$, there exists a unique integer d' such that

$$2\widetilde{m}s' = d'\kappa - k'v \quad \text{or equivalently} \quad s' = \frac{d'\kappa - k'v}{2\widetilde{m}}.$$

As κ is of divisibility $\frac{2\widetilde{m}}{d}$ in Λ, there is some $y \in \Lambda$ such that $(\kappa, y) = \frac{2\widetilde{m}}{d}$. We then have $(s', y) = \frac{d'}{d}$, so d divides d'. Set $d' = \lambda d$. We must have $\lambda \neq 0$: otherwise, s' is equal to $-\frac{k'}{2\widetilde{m}}v$; but $|k'| \leq \widetilde{m}$, so s' can only be 0, which contradicts our hypothesis. By changing s' to $-s'$ if needed, we may suppose that $\lambda \geq 1$. Then by looking at the integral class $s' - \lambda s$, we have $k' \equiv \lambda k \pmod{2\widetilde{m}}$. Write $k' = \lambda k - \mu \cdot 2\widetilde{m}$. Since $k' \leq \widetilde{m}$, we must have $\mu \geq 0$. Then we have $s' = \lambda s + \mu v$ and thus

$$\begin{aligned} {s'}^2 &= \lambda^2 s^2 + 2\lambda\mu(s,v) + \mu^2 v^2 \\ &= \lambda^2 s^2 - \mu(2\lambda k - \mu \cdot 2\widetilde{m})) \\ &= \lambda^2 s^2 - \mu(2k' + \mu \cdot 2\widetilde{m}) \leq \lambda^2 s^2, \end{aligned}$$

where the last inequality is due to $k' \geq -\widetilde{m}$ and $\mu \geq 0$. So we get $-a \leq c' \leq \lambda^2 c$ for some $\lambda \geq 1$, and we may conclude that $c \geq -a$. □

Proposition 3.4.5. *Let X be a hyperkähler manifold of $\mathrm{K3}^{[m]}$-type or Kum_m-type. Let g be one of the canonical generators of $D(H^2(X,\mathbf{Z}))$. The Kähler cone $\mathcal{K}_X$ is one of the components of the positive cone cut out by the hyperplanes $\kappa^\perp$, for all classes $\kappa \in \mathrm{NS}(X)$*

satisfying the following numerical condition: writing $\kappa^2 = 2l$, $\kappa_* = kg$ *with* $0 \le k \le \widetilde{m}$, *and* $d = \gcd(2\widetilde{m}, k)$, *then*

(3.4)
$$\begin{cases} l = c\left(\frac{2m-2}{d}\right)^2 - (m-1)\left(\frac{k}{d}\right)^2 & \text{for an integer } -1 \le c < \frac{k^2}{4(m-1)} \quad \text{if } X \text{ is of } \mathrm{K3}^{[m]}\text{-type;} \\ l = c\left(\frac{2m+2}{d}\right)^2 - (m+1)\left(\frac{k}{d}\right)^2 & \text{for an integer } 0 \le c < \frac{k^2}{4(m+1)} \quad \text{if } X \text{ is of } \mathrm{Kum}_m\text{-type.} \end{cases}$$

Proof. The $\mathrm{K3}^{[m]}$-case is obtained by combining Theorem 3.4.3 and Proposition 3.4.4 for $a = 1$, and the upper bound for c comes from $\kappa^2 = 2l < 0$. For the Kum_m-case, we use $a = 0$ and we note that k cannot take the value 0 because l needs to be negative. So we will only consider κ with $1 \le k \le \widetilde{m} = m + 1$, and for such κ we indeed get an element s with $s^2 \ge 0$ and $0 < |(s, v)| = |-k| \le \widetilde{m} = m + 1$. □

Remark 3.4.6.

- To enumerate all the wall divisors, we let k run from 0 to $\widetilde{m}$ and for each k, we let c run from -1 or 0 to $\left\lceil \frac{k^2}{4\widetilde{m}} \right\rceil - 1$ to get the corresponding l.
- As an example, for $\mathrm{K3}^{[2]}$-type, the pair (k, l) has three possibilities: $(0, -1)$, $(1, -5)$, and $(1, -1)$. Thus we get $\kappa^2 = -2$ and $\mathrm{div}(\kappa) = 1, 2$, or $\kappa^2 = -10$ and $\mathrm{div}(\kappa) = 2$. This was first conjectured in [**HT09b**]. See also [**Mon15**], where the cases of $\mathrm{K3}^{[m]}$-type for $m \le 4$ are worked out; and [**HT09a**], where some examples for Kum_m-type are given.
- Analogous results for OG_6 and OG_{10} are also established: wall divisors on a hyperkähler manifold X of OG_6-type are given by elements $\kappa \in \mathrm{NS}(X)$ with $\kappa^2 = -2$, or $\kappa^2 = -4$ and $\mathrm{div}(\kappa) = 2$ [**MR21**]; wall divisors on a hyperkähler manifold X of OG_{10}-type are given by elements $\kappa \in \mathrm{NS}(X)$ with $0 > \kappa^2 \ge -4$, or $0 > \kappa^2 \ge -24$ and $\mathrm{div}(\kappa) = 3$ [**MO22**].
- In particular, the Kawamata–Morrison conjecture holds for all known deformation types of hyperkähler manifolds by a result of Amerik–Verbitsky [**AV15**, Theorem 1.21]: for a given deformation type, since the square of a wall class is bounded below, the automorphism group $\mathrm{Aut}(X)$ acts on the set of faces of $\mathcal{K}_X$ with finitely many orbits.

We will now describe the image of the period map. Let τ be the deformation type of a polarization, and take an element $h \in \tau$. For a vector $u \in \Lambda$ with negative square and linearly independent of h, the hyperplane $u^\perp \subset \mathbf{P}(\Lambda_\mathbf{C})$ cuts a hyperplane section in the subset Ω_h and induces a divisor $\mathcal{H}_u$ in the period domain $\mathcal{P}_\tau = \Omega_h / \mathrm{Mon}(\Lambda, h)$ which is called a *Heegner divisor.* By abuse of notation, its image in $\mathcal{P}_T$ will also be denoted as $\mathcal{H}_u$.

Proposition 3.4.7. *Take a deformation type of hyperkähler manifolds for which the Kawamata–Morrison conjecture holds. Let* τ *be the deformation type of a polarization and take* $h \in \tau$. *The complement of the image of the period map* $\wp_\tau$ *in* $\mathcal{P}_\tau$ *is the union of the Heegner divisors* $\mathcal{H}_\kappa$ *induced by wall classes* $\kappa \in \Lambda$ *that are orthogonal to* h.

PROOF. For $x \in \kappa^\perp$, if there is a polarized pair (X, H) of deformation type τ such that $\wp(X) = [x] \in \Omega_{\mathrm{marked}}$, take a marking η such that $\eta(H) = h$. Then $\eta^{-1}(\kappa)$ is of type $(1,1)$ and thus algebraic. The class H is contained in the wall $\eta^{-1}(\kappa)^\perp$ and thus not ample by the description of the Kähler cone, a contradiction.

Conversely, we consider a point $[x] \in \Omega_h$ not belonging to any Heegner divisor $\mathcal{H}_\kappa$. By Proposition 3.2.4, we know that $\mathcal{M}_h^{\mathrm{amp}}$ can be identified as a dense open subset of Ω_h by the marked period map $\wp$. If $[x]$ lies in this subset, then we know that $[x]$ is the period for some marked pair $(X, \eta) \in \mathcal{M}_h^{\mathrm{amp}}$ for which $\eta^{-1}(h)$ is ample; otherwise, since nefness is a closed condition, we can choose (X, η) so that $\eta^{-1}(h)$ is strictly nef, that is, it lies on the boundary of the Kähler cone $\mathcal{K}_X$. Since Kawamata–Morrison conjecture holds for X, we may conclude that $\eta^{-1}(h)$ lies on a hyperplane $D^\perp$ for some wall divisor $D \coloneqq \eta^{-1}(\kappa)$. But this means that the period $[x]$ is contained in the Heegner divisor $\mathcal{H}_\kappa$, where the wall class κ is orthogonal to h, and this is not the case by assumption. □

Finally we give a criterion for the existence of a wall class κ in $h^\perp$ for $\mathrm{K3}^{[m]}$-type and Kum_m-type.

PROPOSITION 3.4.8. *For* $\mathrm{K3}^{[m]}$*-type or* Kum_m*-type, let* $h \in \Lambda$ *be an element of divisibility* γ*. Let* k *and* l *be integers satisfying the condition* (3.4)*. Then there is a wall divisor* $\kappa \in h^\perp$ *with* $\kappa^2 = 2l$ *and* $\kappa_* = k\delta_*$ *if and only if we have* $\gamma \mid k$*. Equivalently, this is the condition* $\operatorname{div} h \cdot \operatorname{div} \kappa \mid 2\widetilde{m}$*.*

PROOF. Recall that $\Lambda = \Lambda_0 \oplus \mathbf{Z}\delta$. Write $h = \gamma a x + b\delta$ with $x \in \Lambda_0$ primitive, $\gcd(\gamma a, 2\widetilde{m}) = \gamma$, and $\gcd(\gamma a, b) = 1$. Write

$$\kappa = \frac{2\widetilde{m}}{d}(y + e\delta) + \frac{k}{d}\delta,$$

with $y \in \Lambda_0$. Thus κ being orthogonal to h is equivalent to

$$\gamma a (x, y) = b(2\widetilde{m}e + k).$$

Since $\gcd(\gamma, b) = 1$ and $\gamma \mid 2\widetilde{m}$, the condition $\gamma \mid k$ is clearly necessary. Conversely, if this condition is met, we show that there exist a suitable vector y and an integer e that give the desired κ. Since $\gcd(\gamma a, 2\widetilde{m}) = \gamma$, we may choose e such that

$$a \left| \frac{2\widetilde{m}}{\gamma} e + \frac{k}{\gamma} \right. .$$

Thus we only need to find $y \in \Lambda_0$ with required y^2 and (x, y). By Eichler's criterion, this can be done by taking $\phi \in \mathrm{O}(\Lambda_0)$ such that $\phi(x) = u_1' + \frac{x^2}{2} u_2'$ and then choosing y such that $\phi(y) = (x, y)u_2' + u_1'' + \frac{y^2}{2} u_2''$, where $\langle u_1', u_2' \rangle$ and $\langle u_1'', u_2'' \rangle$ are two copies of hyperbolic plane U in Λ_0. □

In the proof, since we have explicitly described the classes h and κ, if we look at the sublattice $\langle h, \kappa, v \rangle$ in $\widetilde{\Lambda}$, its saturation is generated by the three classes $\frac{h - bv}{\gamma}$, $s = \frac{d\kappa - kv}{2\widetilde{m}}$, and v. So for this particular choice of κ, the discriminant of the saturation is $\left| \frac{2d^2 nl}{\gamma^2 \widetilde{m}} \right|$, while in general the discriminant would be this number divided by some square. Since the Mukai lattice $\widetilde{\Lambda}$ is unimodular, this is also the discriminant of the orthogonal $\langle h, \kappa, v\rangle^{\perp}$, which can be identified with the orthogonal $\langle h, \kappa\rangle^{\perp}$ in Λ. The latter is called the *transcendental lattice* of the Heegner divisor $\mathcal{H}_\kappa$. Its discriminant is also referred to as the discriminant of the Heegner divisor $\mathcal{H}_\kappa$. Therefore we have the following corollary.

COROLLARY 3.4.9. *Let T be a polarization type of square $2n$ and divisibility γ on hyperkähler manifolds of $\mathrm{K3}^{[m]}$-type or Kum_m-type. Let k and l be integers satisfying the condition (3.4) (which only depends on m) such that $\gamma \mid k$. For each connected component $\mathcal{M}_\tau$ of $\mathcal{M}_T$, the period map $\wp_\tau$ avoids at least one irreducible Heegner divisor $\mathcal{H}_\kappa$ of discriminant $\left|\frac{2d^2nl}{\gamma^2\widetilde{m}}\right|$ in $\mathcal{P}_\tau$, where $d = \gcd(2\widetilde{m}, k)$.*

For example, for $\mathrm{K3}^{[2]}$-type, we have already seen that (k,l) can be $(0,-1)$, $(1,-5)$, and $(1,-1)$. For a polarization type T of square $2n$, if the divisibility γ is equal to 2, the only possible case is $(0,-1)$ and we get a Heegner divisor of discriminant $2n$; if the divisibility γ is equal to 1, the three cases are all present and we get Heegner divisors of discriminant $8n$, $10n$, and $2n$. This result is however not exhaustive, since the sublattice we used above to compute the discriminant might still not be primitive in general, and the discriminant will be divided by some square. For example, when $\gamma = 1$, by [**DM19**, Theorem 6.1] it is also possible to have a Heegner divisor of discriminant $2n/5$ in the complement. Note also that there might be several irreducible Heegner divisors with the same discriminant while we have only obtained one of them.

Another simple example works for almost every polarization type T:

- If $\gamma \le \widetilde{m}$ we may take (k,l) to be $(\gamma, -\widetilde{m})$ (and $c = 0$), so the discriminant is equal to $2n$. In other words, for such a polarization type T, the restriction of the period map to every connected component of $\mathcal{M}_T$ will avoid an irreducible Heegner divisor of discriminant $2n$ in the period domain
- For a polarization type T not satisfying $\gamma \le \widetilde{m}$, γ is necessarily equal to the maximal value $2\widetilde{m}$. For $\mathrm{K3}^{[m]}$-type, we may take (k,l) to be $(0,-1)$ (so $c = -1$), and the discriminant is then equal to $\frac{2n}{m-1}$, so we get a similar conclusion.

3.5. Two examples

Using the numerical condition (3.4), we can now compare the images by the period map of various components. Recall the picture of the polarized period map from (3.3). We prove the following result for $\mathrm{K3}^{[m]}$-type. Clearly the same idea can be adapted to Kum_m-type.

PROPOSITION 3.5.1. *Let a be a positive integer.*

(i) For hyperkähler manifolds of $\mathrm{K3}^{[144^a+1]}$-type, there is a unique polarization type T of square 288 and divisibility 12, for which the polarized moduli space $\mathcal{M}_T$ has exactly two components, with different images in $\mathcal{P}_T$ under the period map.

(ii) For hyperkähler manifolds of $\mathrm{K3}^{[6^a+1]}$*-type, there is a unique polarization type* T *of square* 2 *and divisibility* 1*, for which the polarized moduli space* $\mathcal{M}_T$ *is connected. The group* G *is isomorphic to* $\mathbf{Z}/2\mathbf{Z}$*, and the image of the period map in* $\mathcal{P}_\mathcal{T}$ *is not* G*-invariant above* $\mathcal{P}_T$*.*

Proof. For (i), we may check by Proposition 3.3.5 that such polarization type is unique and the polarized moduli space $\mathcal{M}_T$ has exactly two components. Note that by Proposition 3.3.4, $\gamma = 12$ is the smallest divisibility for the moduli space $\mathcal{M}_T$ to have more than one component.

As $D(\Lambda) = \mathbf{Z}/(2\cdot 144^a)\mathbf{Z}$ and $\rho(2\cdot 144^a) = 2$, by Lemma 3.3.2 we have $\mathrm{O}(D(\Lambda)) = \{\pm 1, \pm g\}$. For $h \in T$, the class h_* is of order 12 in $D(\Lambda)$. So for any $\phi \in \mathrm{O}(\Lambda, h)$, we have $\chi(\phi) = 1$ since 1 is the unique element in $\mathrm{O}(D(\Lambda))$ that is $\equiv 1 \pmod{12}$. This shows that $\mathrm{O}(\Lambda, h) \subset \widetilde{\mathrm{O}}(\Lambda)$ and consequently, the group $\mathrm{O}^+(\Lambda,h)/\operatorname{Mon}(\Lambda,h)$ is trivial. In this case, both period domains $\mathcal{P}_\mathcal{T}$ are canonically isomorphic to $\mathcal{P}_T$.

Since $\mathcal{M}_T$ has two components, we may choose $h, h' \in T$ belonging to different $\operatorname{Mon}(\Lambda)$-orbits or equivalently, $\widehat{\mathrm{O}}(\Lambda)$-orbits, as we have seen in the proof of Proposition 3.3.4 that they are the same. There exists $\psi \in \mathrm{O}(\Lambda) \smallsetminus \widehat{\mathrm{O}}(\Lambda)$ such that $\psi(h) = h'$. We may assume that $\chi(\psi) = g$. Consider the period domain $\mathcal{P}_T$ realized as the quotient $\Omega_h/\,\mathrm{O}^+(\Lambda,h)$ or $\Omega_{h'}/\,\mathrm{O}^+(\Lambda,h')$. The automorphism ψ induces an identification between the two, which maps each Heegner divisor $\mathcal{H}_\kappa$ to $\mathcal{H}_{\psi(\kappa)}$.

We consider a wall class $\kappa \in h^\perp$ with square $2l$ and $\kappa_* = k\delta_*$. The class $\kappa' = \psi(\kappa)$ has the same square $2l$ while $\kappa'_* = k'\delta_*$ with $k' \equiv gk \pmod{2\cdot 144^a}$. For κ' to also define a wall class, we need

$$c' = c + \frac{k'^2 - k^2}{4\cdot 144^a} \geq -1$$

to hold. So the idea is to choose some suitable k, l for which this condition fails. We let $k = 12g_0$ such that $k \equiv 12g \pmod{2\cdot 144^a}$ (so g_0 is the residue of g modulo $24\cdot 144^{a-1}$). Since $g \neq \pm 1$ in $\mathrm{O}(D(\Lambda))$, g_0 cannot be ± 1 hence we have $g_0^2 > 1$. Then we can let $c = -1$ and find the value for l using (3.4). By Proposition 3.4.8, there exists indeed such a wall class $\kappa \in h^\perp$. On the other hand, the choice of k means $k' = 12$, so $c' = -1 + \frac{12^2 - 12^2 g_0^2}{4\cdot 144^a} < -1$, and κ' is not a wall class. This shows that the same Heegner divisor inside $\mathcal{P}_T$ is avoided by the period map for one component but not for the other. Thus their images in $\mathcal{P}_T$ by the period map are not the same.

For (ii), once again we may verify by Proposition 3.3.5 that there is a unique such polarization type T with one connected component. And by Lemma 3.3.2, since $D(\Lambda) = \mathbf{Z}/(2\cdot 6^a)\mathbf{Z}$ and $\rho(2\cdot 6^a) = 2$, we have $\mathrm{O}(D(\Lambda)) = \{\pm 1, \pm g\}$.

Since this $\mathrm{O}(\Lambda)$-orbit is unique, we may take $h = u'_1 + u'_2$, where $\langle u'_1, u'_2\rangle$ is a copy of U. The group $\mathrm{O}(\Lambda, h)$ contains $\mathrm{O}(\Lambda, U) \coloneqq \{\phi \in \mathrm{O}(\Lambda) \mid \phi|_U = \mathrm{Id}\}$ as a subgroup, which is isomorphic to $\mathrm{O}(U^\perp)$ since U is a direct summand. Moreover, the inclusion $\mathrm{O}(U^\perp) \simeq \mathrm{O}(\Lambda, U) \hookrightarrow \mathrm{O}(\Lambda)$ induces an isometry between the two discriminant groups. We use

Proposition 3.2.12 on $\mathrm{O}(U^\perp)$ to deduce that the homomorphism $\chi\colon \mathrm{O}(\Lambda) \to \mathrm{O}(D(\Lambda))$ when restricted to $\mathrm{O}(\Lambda, U)$, is still surjective. In particular, there is $\phi \in \mathrm{O}(\Lambda, h)$ such that $\chi(\phi) = g$. On the other hand, following the proof of Proposition 3.2.15, there is an element $R \in \mathrm{O}(\Lambda, h)$ such that $\sigma(R) = -1$ and $\chi(R) = 1$. Let ψ be ϕ if $\sigma(\phi)$, and $R \circ \phi$ otherwise. Then ψ is in $\mathrm{O}^+(\Lambda, h)$ with $\chi(\psi) = g$. Consequently, the group $\mathrm{O}^+(\Lambda, h)/\operatorname{Mon}(\Lambda, h)$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$.

As in the previous case, we consider a wall class $\kappa \in h^\perp$ with square $2l$ and $\kappa_* = k\delta_*$, for $k = g$ and $c = -1$. Such a class exists by Proposition 3.4.8. However, the class $\kappa' = \psi(\kappa)$ will have $k' = 1$, so $c' = -1 + \frac{1^2 - g^2}{4 \cdot 6^a} < -1$ and κ' is not a wall class. This shows that there are two Heegner divisors in $\mathcal{P}_\tau$ that can be mapped to each other under the action of $\mathrm{O}^+(\Lambda, h)/\operatorname{Mon}(\Lambda, h)$, but one is avoided by the period map and the other is not. Thus we see in particular that the group G is non-trivial and therefore also isomorphic to $\mathbf{Z}/2\mathbf{Z}$, and the image of the period map is not G-invariant. $\square$

Part II

Geometry of Debarre–Voisin varieties

CHAPTER 4

Debarre–Voisin varieties

In this chapter, we study the general properties of Debarre–Voisin fourfolds. Such manifolds are defined from a *trivector*, and there are two Fano varieties of K3-type that can also be associated with it. We first deduce the smoothness criteria for these three varieties, and provide a picture of the moduli space and the period map. Then we relate the integral Hodge structures of the three varieties, and show that the two Fano varieties satisfy the integral Hodge conjecture. This is obtained as a detailed analysis of the geometry of these varieties along three special divisors in the moduli space.

This is a joint work with Vladimiro Benedetti, and has appeared in [**BS21**]*. Section 4.8 appeared separately in* [**Obe21**, Appendix C].

Some computations are carried out using the computer algebra system `Macaulay2` [**GS**], and particularly, the package `Schubert2`. We provide the code used in these cases. The computer algebra system `Singular` [**DGPS**] is extensively used in complement, which we also acknowledge.

4.1. Introduction

Let V_{10} be a 10-dimensional complex vector space and let $\sigma \in \bigwedge^3 V_{10}^\vee$ be a trivector, that is, an alternating 3-form on V_{10}. Consider the Grassmannian $\mathrm{Gr}(6, V_{10})$ and denote by $\mathcal{U}_6$ the tautological subbundle. By viewing σ as a section of the vector bundle $\bigwedge^3 \mathcal{U}_6^\vee$, Debarre–Voisin showed in [**DV10**] that the zero locus $X_6^\sigma \subset \mathrm{Gr}(6, V_{10})$ of σ is a smooth hyperkähler fourfold for a general σ. Moreover, by varying σ, these fourfolds form a locally complete family of projective hyperkähler varieties of $\mathrm{K3}^{[2]}$-type.

Along with X_6^σ, there are several other degeneracy loci determined by the trivector σ that have interesting Hodge theoretical and categorical properties. To get a slightly unified notation, we will denote by X_k^σ a subvariety defined in the Grassmannian $\mathrm{Gr}(k, V_{10})$ as follows.

- The variety X_3^σ is the hyperplane section of $\mathrm{Gr}(3, V_{10})$ defined by σ

$$X_3^\sigma \coloneqq \{[V_3] \in \mathrm{Gr}(3, V_{10}) \mid \sigma|_{V_3} = 0\}.$$

- There is the variety X_1^σ, a 6-dimensional degeneracy locus in $\mathbf{P}(V_{10})$, also known as the Peskine variety. It is defined as

$$X_1^\sigma \coloneqq \{[V_1] \in \mathbf{P}(V_{10}) \mid \operatorname{rank} \sigma(V_1, -, -) \leq 6\}.$$

First, we will prove the following criteria for the smoothness of X_1^σ, X_3^σ, and X_6^σ.

PROPOSITION 4.1.1. *Consider the following two conditions on a trivector* σ

(1) for all $[V_3] \in \mathrm{Gr}(3, V_{10})$, $\sigma(V_3, V_3, -) \neq 0$*;*
(2) for all $[V_1] \in \mathbf{P}(V_{10})$, $\operatorname{rank} \sigma(V_1, -, -) \geq 6$.

Then X_3^σ *and* X_6^σ *are smooth of expected dimension if and only if condition (1) holds, and* X_1^σ *is smooth of expected dimension 6 if and only if conditions (1) and (2) both hold.*

All these varieties associated with σ are expected to have some common Hodge structures. One particularly interesting one is the second integral cohomology group of the hyperkähler fourfold X_6^σ, which carries the Beauville–Bogomolov–Fujiki quadratic form q. This provides a polarized Hodge structure on the primitive part $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}}$. On the two Fano varieties X_3^σ and X_1^σ, we can consider the middle degree *vanishing cohomologies*, which are generated by cohomology classes not coming from the ambient space. Both Hodge structures are polarized with the cup product as the polarization. Our main result is the following, which relates these three pieces of Hodge structures.

THEOREM 4.1.2 (see Theorem 4.5.3 and Theorem 4.5.11). *We have Hodge isometries*

$$\big(H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}}, \cdot\big) \simeq \big(H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}}, -q\big) \simeq \big(H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}, \cdot\big)$$

given by algebraic correspondences between X_3^σ *and* X_6^σ*, and between* X_6^σ *and* X_1^σ*, whenever they are smooth of expected dimension.*

The isometry between the integral Hodge structures of X_3^σ and X_1^σ was already established using a different method by Bernardara–Fatighenti–Manivel [**BFM21**]. Our method focuses on the geometry of these varieties along some special divisors in the moduli space for σ: we use the extra algebraic classes they admit to perform computations in order to show the isometries.

The two correspondences here closely resemble the correspondence between a cubic fourfold and its variety of lines. In [**MO20**], Mongardi–Ottem proved the integral Hodge conjecture for 1-cycles on hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type, and used this to deduce the integral Hodge conjecture for 2-cycles on cubic fourfolds. Following the same idea, we can prove the integral Hodge conjecture for X_3^σ and X_1^σ.

COROLLARY 4.1.3 (see Theorem 4.6.11 and Theorem 4.6.19). *The integral Hodge conjecture holds for both* X_3^σ *and* X_1^σ *in all degrees, whenever they are smooth of expected dimension.*

We briefly review known results on the moduli space of Debarre–Voisin varieties. On the one hand, there is a 20-dimensional irreducible quasi-projective GIT moduli space

$$\mathcal{M} := \mathbf{P}\big(\textstyle\bigwedge^3 V_{10}^\vee\big) /\!\!/ \mathrm{SL}(V_{10})$$

for the trivectors σ. On the other hand, there is a 20-dimensional irreducible quasi-projective moduli space $\mathcal{M}_{22}^{(2)}$ for polarized hyperkähler varieties of $\mathrm{K3}^{[2]}$-type with square 22 and divisibility 2. It parametrizes pairs (X, H) of hyperkähler varieties X equipped with a primitive ample class H such that $q(H) = 22$ and $q\big(H, H^2(X_6^\sigma, \mathbf{Z})\big) = 2\mathbf{Z}$. The Torelli theorem for polarized hyperkähler manifolds tells us that the period map

$$\mathfrak{p}\colon \mathcal{M}_{22}^{(2)} \hookrightarrow \mathcal{P}, \quad [X] \longmapsto [H^{2,0}(X)]$$

is an open immersion into the quasi-projective period domain $\mathcal{P}$ that parametrizes the corresponding Hodge structures. The Debarre–Voisin construction gives a rational map

$$\mathfrak{m}\colon \mathcal{M} \dashrightarrow \mathcal{M}_{22}^{(2)},$$

which was proved by O'Grady to be birational [**O'G19**, Theorem 1.9]. Therefore, divisors in the moduli space can be studied from different points of view. Namely, we will be mostly interested in the divisors in $\mathcal{M}$ coming from $\mathrm{SL}(V_{10})$-invariant hypersurfaces in $\mathbf{P}\big(\bigwedge^3 V_{10}^\vee\big)$ and Heegner divisors in $\mathcal{P}$ defined in terms of Hodge structures.

We will study three divisors $\mathcal{D}^{3,3,10}$, $\mathcal{D}^{1,6,10}$, and $\mathcal{D}^{4,7,7}$ in $\mathcal{M}$ coming from $\mathrm{SL}(V_{10})$-invariant hypersurfaces in $\mathbf{P}(\bigwedge^3 V_{10}^\vee)$, which are mapped by the period map to Heegner divisors $\mathcal{D}_{22}$, $\mathcal{D}_{24}$, and $\mathcal{D}_{28}$ respectively in the period domain. Here the $\mathrm{SL}(V_{10})$-invariant divisors are labelled using the degeneracy condition on σ, while the Heegner divisors are labelled using their discriminant.

The first divisor $\mathcal{D}_{22}$ was studied in the original article [**DV10**]. Although the variety X_6^σ is not smooth in this case, its singular locus contains (and we will show that it coincides with) a K3 surface S_{22} of degree 22. It was proved in [**DV10**] that X_6^σ is birational to the Hilbert square $S_{22}^{[2]}$. In particular, this means that S_{22} shares the same transcendental Hodge structure with X_6^σ.

For a very general member of the second divisor $\mathcal{D}_{24}$, we will give a geometric construction of a Brauer-twisted K3 surface (S_6, β) and we show that the Hodge structure of (S_6, β) embeds in that of X_6^σ. Moreover, we can recover X_6^σ as a moduli space of β-twisted sheaves on S_6. This case bears a lot of similarities with the case of cubic fourfolds containing a plane. In fact, both cases provide examples of Brill–Noether contractions with non-trivial Brauer class on hyperkähler fourfolds, and their general theory has been thoroughly studied in the recent [**KvG21**].

The third divisor $\mathcal{D}_{28}$ is related to the existence of Lagrangian planes on X_6^σ and is important for the study of the Hodge structures. There are however no associated K3 surfaces for very general members of this family.

We give a brief overview of the results. In Section 4.2, we introduce the three $\mathrm{SL}(V_{10})$-invariant hypersurfaces in $\mathbf{P}(\bigwedge^3 V_{10}^\vee)$ as well as the divisors that they induce in $\mathcal{M}$. They are all defined using some degeneracy condition on the trivector σ with respect to some special flag. In Section 4.3, we study the smoothness of the three varieties X_1^σ, X_3^σ, and X_6^σ, and we prove Proposition 4.1.1, which identifies the divisors $\mathcal{D}^{3,3,10}$ and $\mathcal{D}^{1,6,10}$ as the loci where these varieties turn singular. Then we provide a general picture for the moduli space and the period map in Section 4.4.

In Section 4.5, we study the algebraic correspondences providing the Hodge isometries in Theorem 4.1.2, but we postpone the proof to Section 4.6, where we study in detail the geometry of the three varieties when one specializes to the divisor $\mathcal{D}_{28}$.

In Section 4.7, we study the divisor $\mathcal{D}_{24}$. For a general member in $\mathcal{D}_{24}$, we define a twisted associated K3 surface (S_6, β) of degree 6 (see Proposition 4.7.3), and we prove that the hyperkähler variety X_6^σ is isomorphic to a moduli space of twisted sheaves on (S_6, β) (see Theorem 4.7.17).

In Section 4.8, we give a description of the singularity of a general member in $\mathcal{D}_{22}$.

In the following table, we sum up the results obtained concerning these divisors.

$\mathrm{SL}(V_{10})$-invariant hypersurface	$\Delta^{3,3,10}$	$\Delta^{1,6,10}$	$\Delta^{4,7,7}$
degeneracy condition on σ	$\sigma(V_3, V_3, V_{10}) = 0$	$\sigma(V_1, V_6, V_{10}) = 0$	$\sigma(V_4, V_7, V_7) = 0$
degree in $\mathbf{P}(\bigwedge^3 V_{10}^\vee)$	640	990	5500
induced divisor in $\mathcal{M}$	$\mathcal{D}^{3,3,10}$	$\mathcal{D}^{1,6,10}$	$\mathcal{D}^{4,7,7}$
Heegner divisor	$\mathcal{D}_{22}$	$\mathcal{D}_{24}$	$\mathcal{D}_{28}$
singular locus of X_1^σ	$\mathbf{P}(V_3)$	$\{[V_1]\}$	$\varnothing$
singular locus of X_3^σ	$\{[V_3]\}$	$\varnothing$	$\varnothing$
singular locus of X_6^σ	S_{22}	$\varnothing$	$\varnothing$
degree of associated K3	22	6 with a Brauer class of order 2	none
birational model of X_6^σ	$S_{22}^{[2]}$	$M(S_6, v, \beta)$	-

TABLE 3. Divisors in the moduli spaces

Finally, we mention the recent results of Oberdieck in [**Obe21**] regarding invariants of a generic pencil of Debarre–Voisin varieties, obtained using Gromov–Witten techniques and modular forms. The three Heegner divisors $\mathcal{D}_{22}, \mathcal{D}_{24}$, and $\mathcal{D}_{28}$ that we study are precisely the first three non-HLS divisors with lowest discriminants (see Section 4.4 for the definition of an HLS divisor). Moreover, the corresponding Noether–Lefschetz numbers (see Theorem 2 of *loc. cit.*) indeed coincide with the degrees of the $\mathrm{SL}(V_{10})$-invariant hypersurfaces that we have computed.

Notation. *Grassmannians.* We will denote by U_n, V_n, or W_n an n-dimensional complex vector space. We denote by $\mathrm{Flag}(k_1, \dots, k_r, V_n)$ the flag variety parametrizing nested subspaces of V_n of dimensions $k_1, \dots, k_r$. We will denote by $\mathcal{U}_{k_i}$ the tautological vector subbundle of $V_n \otimes \mathcal{O}_{\mathrm{Flag}(k_1,\dots,k_r,V_n)}$ of rank k_i. When $r = 1$, we recover the ordinary Grassmannian, which we denote by $\mathrm{Gr}(k, V_n)$ (or $\mathbf{P}(V_n)$ if $k = 1$); when no confusion arises, $\mathcal{U}, \mathcal{Q}$ will denote respectively the tautological and the quotient vector bundles on $\mathrm{Gr}(k, V_n)$.

For a trivector $\sigma \in \bigwedge^3 V_n$, its *rank* is defined as the dimension of the smallest subspace $V \subset V_n$ such that $\sigma \in \bigwedge^3 V$. The rank is a $\mathrm{GL}(V_n)$-invariant. If $\sigma \in \bigwedge^3 V_n^\vee$ and $V_i \subset V_n$, we will denote by $\sigma(V_i, V_i, V_i)$ the restriction $\sigma|_{V_i} \in \bigwedge^3 V_i^\vee$. Similarly, if $V_i \subset V_j \subset V_k \subset V_n$, we use $\sigma(V_i, V_j, V_k)$ to denote the image of σ in $(V_i \wedge V_j \wedge V_k)^\vee$ (seen as a quotient of $\bigwedge^3 V_k^\vee$).

The notation for Schubert varieties inside a Grassmannian $\mathrm{Gr}(k, V_n)$ is as follows. Let us fix a complete flag $0 = V_0 \subset V_1 \subset \cdots \subset V_n$. For any sequence of integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ with $\lambda_1 \leq n - k$ and $\lambda_k \geq 0$, we define the Schubert variety

$$\Sigma_\lambda = \left\{ W \in \mathrm{Gr}(k, V_n) \;\middle|\; \dim(W \cap V_{n-k-\lambda_j+j}) \geq j \text{ for } 1 \leq j \leq k \right\},$$

which is of codimension $\sum_i \lambda_i$ inside $\mathrm{Gr}(k, V_n)$. We let σ_λ be the Schubert class representing Σ_λ in cohomology.

Lattices. By a lattice we shall mean a finitely generated free $\mathbf{Z}$-module L endowed with an integral quadratic form q. The following basic properties can be found in [**BHPVdV04**, Chapter I.2].

The *discriminant group* $D(L)$ of L is defined as $L^\vee/L$, where $L^\vee := \mathrm{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$ is the dual. If M denotes the Gram matrix of q in an integral basis of L, its determinant $\det(M)$ is independent of the choice of the basis and is called the *discriminant* of L and denoted by $\mathrm{disc}(L)$. We have $|D(L)| = |\mathrm{disc}(L)|$. A lattice L is called *even* if $q(x) \in 2\mathbf{Z}$ for all $x \in L$, and *unimodular* if $D(L)$ is trivial. Beware that the discriminant of a unimodular lattice can either be 1 or -1. For a sublattice $A \subset L$ with $\mathrm{rank}(L) = \mathrm{rank}(A)$, its index $[L : A]$ is finite and satisfies $[L : A]^2 = |\mathrm{disc}(A)/\mathrm{disc}(L)|$.

A sublattice $A \subset L$ is called *saturated* if L/A is torsion-free. For any sublattice $A \subset L$, one can define the orthogonal sublattice $A^\perp \subset L$ with respect to q. If L is unimodular and A is a saturated sublattice, A and $A^\perp$ have isomorphic discriminant groups, hence the same discriminant up to sign $|\mathrm{disc}(A)| = |\mathrm{disc}(A^\perp)|$. In this case, the direct sum $A \oplus A^\perp$ has index $|\mathrm{disc}(A)|$ in L.

4.2. GIT quotient

In this section, we first study the moduli space of trivectors in $\bigwedge^3 V_{10}^\vee$. We consider the 119-dimensional projective space $\mathbf{P}(\bigwedge^3 V_{10}^\vee)$ and define the following GIT quotient

$$\mathcal{M} := \mathbf{P}\big(\textstyle\bigwedge^3 V_{10}^\vee\big) /\!\!/ \mathrm{SL}(V_{10}),$$

which is a projective variety of dimension 20 by a parameter count.

We introduce three $\mathrm{SL}(V_{10})$-invariant hypersurfaces in $\mathbf{P}(\bigwedge^3 V_{10}^\vee)$, each inducing an irreducible divisor in $\mathcal{M}$.

4.2.1. Condition $(3,3,10)$**.** The projective dual of the Grassmannian $\mathrm{Gr}(3,V_{10})$ in its Plücker embedding defines a hypersurface in $\mathbf{P}(\bigwedge^3 V_{10}^\vee)$ known as the *discriminant* hypersurface. It parametrizes singular hyperplane sections of $\mathrm{Gr}(3,V_{10})$ and can be characterized as the set

$$\Delta^{3,3,10} := \big\{[\sigma] \in \mathbf{P}(\textstyle\bigwedge^3 V_{10}^\vee) \;\big|\; \exists V_3 \quad \sigma(V_3,V_3,V_{10}) = 0\big\}. \tag{4.1}$$

The superscript indicates the vanishing condition that defines this hypersurface. Its degree is equal to 640, which was first calculated by Lascaux in [**Las81**, Section 3]. To compute this degree, we may consider the incidence variety

$$\begin{array}{ccc} \Sigma = \{([\sigma],[V_3]) \mid \sigma(V_3,V_3,V_{10}) = 0\} & \xrightarrow{\mathrm{pr}_2} & \mathrm{Gr}(3,V_{10}) \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \\ \mathbf{P}(\bigwedge^3 V_{10}^\vee) & & \end{array}$$

where the map pr_1 is a birational morphism onto the discriminant hypersurface $\Delta^{3,3,10}$, since for a general $[\sigma] \in \Delta^{3,3,10}$, the hyperplane section it defines admits exactly one ordinary double point. For a fixed point $[V_3]$, the condition on σ is linear. Therefore the second projection pr_2 identifies Σ with the projective bundle $\mathbf{P}(\mathcal{E})$ where $\mathcal{E}$ is the vector bundle

$$\mathcal{E} := (\textstyle\bigwedge^2 \mathcal{U}_3 \wedge V_{10})^\perp = \bigwedge^2 \mathcal{Q}_7^\vee \wedge V_{10}^\vee.$$

We can verify that Σ is indeed of dimension 118. To compute the degree of its image under pr_1, we carry out some standard Schubert calculus on $\mathrm{Gr}(3,V_{10})$ and obtain the number 640. This can be easily done in `Macaulay2`.

```
needsPackage "Schubert2";
G = flagBundle{3,7}; (U,Q) = bundles G;
print integral chern dual(exteriorPower_3 U+exteriorPower_2 U*Q); -- 640
```

By a parameter count, the hypersurface $\Delta^{3,3,10}$ induces an irreducible divisor in $\mathcal{M}$, which we denote by

$$\mathcal{D}^{3,3,10} := \{[\sigma] \in \mathcal{M} \mid \exists V_3 \quad \sigma(V_3,V_3,V_{10}) = 0\}. \tag{4.2}$$

This divisor is unirational since it is dominated by the projective bundle Σ.

4.2.2. Condition $(1,6,10)$**.** Consider the following subvariety

$$\Delta^{1,6,10} := \big\{[\sigma] \in \mathbf{P}(\textstyle\bigwedge^3 V_{10}^\vee) \;\big|\; \exists V_1 \subset V_6 \quad \sigma(V_1,V_6,V_{10}) = 0\big\}. \tag{4.3}$$

Similar to the case of the discriminant, we study the incidence variety

$$\begin{array}{ccc} \Sigma = \{([\sigma],[V_1 \subset V_6]) \mid \sigma(V_1,V_6,V_{10}) = 0\} & \xrightarrow{\mathrm{pr}_2} & \mathrm{Flag}(1,6,V_{10}) \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \\ \mathbf{P}(\bigwedge^3 V_{10}^\vee) & & \end{array}$$

where the first projection maps Σ onto $\Delta^{1,6,10}$. For a fixed flag $[V_1 \subset V_6]$, the condition on σ is linear. Therefore the second projection pr_2 identifies Σ with the projective bundle $\mathbf{P}(\mathcal{E})$ where $\mathcal{E}$ is the vector bundle

$$\mathcal{E} := (\mathcal{U}_1 \wedge \mathcal{U}_6 \wedge V_{10})^{\perp}$$

We may verify that Σ is of dimension 118. We again use Schubert calculus on $\mathrm{Flag}(1, 6, V_{10})$ to compute its degree with respect to the polarization on $\mathbf{P}(\bigwedge^3 V_{10}^{\vee})$ via pr_1.

```
needsPackage "Schubert2";
G = flagBundle{1,5,4}; (U1,U61,Q) = bundles G;
print integral chern dual(U1*exteriorPower_2 U61+U1*U61*Q); -- 990
```

Since the degree is non-zero, we see that $\Delta^{1,6,10}$ is a hypersurface in $\mathbf{P}(\bigwedge^3 V_{10}^{\vee})$. We claim that the projection map pr_1 is birational onto its image, so the degree of the hypersurface $\Delta^{1,6,10}$ is exactly 990.

PROPOSITION 4.2.1. *The projection map* $\mathrm{pr}_1 \colon \Sigma \to \Delta^{1,6,10}$ *is birational onto its image. In other words, for a general* $[\sigma] \in \Delta^{1,6,10}$*, the flag* $[V_1 \subset V_6]$ *satisfying the vanishing condition is unique. Consequently,* $\Delta^{1,6,10}$ *is a hypersurface of degree* 990 *in* $\mathbf{P}(\bigwedge^3 V_{10}^{\vee})$.

For the proof, we exhibit an explicit trivector σ admitting a unique such flag $[V_1 \subset V_6]$ and then check that the fiber is reduced.

PROOF. Consider the following randomly generated trivector

[067]+[089]+[143]+[145]+[149]+[172]+[183]+[193]+[217]+[235]+[246]+[257]+[374]+[379]+[685]+[687]

where $[ijk]$ stands for the form $e_i^{\vee} \wedge e_j^{\vee} \wedge e_k^{\vee}$. Note that a flag $[V_1 \subset V_6]$ satisfying the vanishing condition means that the skew-symmetric form $\sigma(V_1, -, -)$ has rank ≤ 4, so we can determine the set of such $[V_1]$ by computing the 6×6 Pfaffians.

```
F = QQ; S = F[x_0..x_9];
delta = (x,y,v) -> table(10,10,(i,j) -> if i==x and j==y then v else 0);
skew = (i,j,k) -> sum(delta \ {(i,j,x_k),(j,k,x_i),(k,i,x_j),
                               (j,i,-x_k),(k,j,-x_i),(i,k,-x_j)});
sigma = {(0,6,7),(0,8,9),(1,4,3),(1,4,5),(1,4,9),(1,7,2),(1,8,3),(1,9,3),
         (2,1,7),(2,3,5),(2,4,6),(2,5,7),(3,7,4),(3,7,9),(6,8,5),(6,8,7)};
X = variety pfaffians_6 matrix sum(skew \ sigma);
print (dim X, degree X); -- (0, 1)
```

Therefore there exists a unique such V_1, and one may easily check that it is given by $\langle e_0 \rangle$. The subspace V_6, being the kernel of the form $\sigma(e_0, -, -)$, is given by $\langle e_0, \dots, e_5 \rangle$.[1]

Note that we cannot conclude yet, since this could be a ramification point for the projection map pr_1. To show that the schematic fiber $\mathrm{pr}^{-1}([\sigma])$ is reduced, we look inside the flag variety $\mathrm{Flag}(1, 6, V_{10})$. Since we know that the set-theoretical fiber is just a point

[1]With this explicit example, one can compute the projectivized tangent cone of the Peskine variety X_1^{σ} at the point $[V_1]$: it is isomorphic to a hyperplane section of the Grassmannian $\mathrm{Gr}(2, 5)$.

$[V_1 \subset V_6]$, we can take an affine chart of the flag variety at $[V_1 \subset V_6]$, then compute the ideal of $\mathrm{pr}^{-1}([\sigma])$ and check its smoothness at the origin using the Jacobian criterion.

More explicitly, we take an affine chart $\mathbf{A}^{29}$ with coordinates $x_0, \ldots, x_{28}$ such that in the basis $e_0, \ldots, e_9$, each flag $V_1 \subset V_6$ has the following form

$$\begin{pmatrix} 1 & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ 0 & 1 & 0 & 0 & 0 & 0 & x_9 & x_{14} & x_{19} & x_{24} \\ 0 & 0 & 1 & 0 & 0 & 0 & x_{10} & x_{15} & x_{20} & x_{25} \\ 0 & 0 & 0 & 1 & 0 & 0 & x_{11} & x_{16} & x_{21} & x_{26} \\ 0 & 0 & 0 & 0 & 1 & 0 & x_{12} & x_{17} & x_{22} & x_{27} \\ 0 & 0 & 0 & 0 & 0 & 1 & x_{13} & x_{18} & x_{23} & x_{28} \end{pmatrix}.$$

Then we ask for the vanishing of $\sigma(V_1, V_6, V_{10})$ which gives us the ideal defining $\mathrm{pr}_1^{-1}([\sigma])$.

```
R = QQ[x_0..x_28];
-- M is the affine chart of Flag(1,6,V_10) at [V_1,V_6]
M = id_(R^6)_{0}|(genericMatrix(R,1,9)||(1|genericMatrix(R,x_9,5,4)));
V = entries M;
eval = (trivector,u,v,w) -> (
  d := (i,j,k) -> u#i*v#j*w#k;
  skew := (i,j,k) -> d(i,j,k)+d(j,k,i)+d(k,i,j)-d(k,j,i)-d(j,i,k)-d(i,k,j);
  sum(skew \ trivector));
-- ideal of the fiber, given by sigma(V1,V6,V10) = 0
I = ideal flatten for i in 1..5 list for j in 0..9 list (
  eval(sigma, V#0, V#i, entries (id_(R^10))_j)); -- sigma is the trivector
print rank sub (jacobian I, for i in 0..28 list (x_i=>0)); -- 29
```

We see that the fiber $\mathrm{pr}_1^{-1}([\sigma])$ is indeed reduced, thus we may conclude that pr_1 is generically injective and birational onto its image, the latter being a hypersurface of degree 990. ☐

We also remark that each flag $[V_1 \subset V_6]$ will provide some extra algebraic classes on the Debarre–Voisin variety X_6^σ, so the uniqueness of $[V_1 \subset V_6]$ for a general $[\sigma] \in \Delta^{1,6,10}$ can also be obtained from Hodge theory.

Again, by a parameter count, the hypersurface $\Delta^{1,6,10}$ induces an irreducible divisor in $\mathcal{M}$, which we denote by

$$\mathcal{D}^{1,6,10} := \{[\sigma] \in \mathcal{M} \mid \exists V_1 \subset V_6 \quad \sigma(V_1, V_6, V_{10}) = 0\}. \tag{4.4}$$

It is unirational by the above analysis.

4.2.3. Condition $(4, 7, 7)$.

We consider

$$\Delta^{4,7,7} := \left\{[\sigma] \in \mathbf{P}(\textstyle\bigwedge^3 V_{10}^\vee) \;\middle|\; \exists V_4 \subset V_7 \quad \sigma(V_4, V_7, V_7) = 0\right\}. \tag{4.5}$$

We follow the exact same procedure as the above two cases. Schubert calculus on $\mathrm{Flag}(4, 7, V_{10})$ gives 5500 as the degree of the incidence variety Σ, so we may deduce that $\Delta^{4,7,7}$ is a unirational hypersurface in $\mathbf{P}(\bigwedge^3 V_{10}^\vee)$.

```
needsPackage "Schubert2";
G = flagBundle{4,3,3}; (U4,U74,Q) = bundles G;
E = exteriorPower_3 U4+exteriorPower_2 U4*U74 +U4*exteriorPower_2 U74;
print integral chern dual E; -- 5500
```

To conclude that this is also the degree of $\Delta^{4,7,7}$, we need to show that for a general $[\sigma] \in \Delta^{4,7,7}$ there is a unique flag $[V_4 \subset V_7]$ satisfying the vanishing condition. We will do this later by showing that each flag $[V_4 \subset V_7]$ provides some extra algebraic classes on the Debarre–Voisin variety X_6^σ and conclude using Hodge theory.

Again, by a parameter count, we get an induced irreducible divisor in $\mathcal{M}$

$$\mathcal{D}^{4,7,7} := \{[\sigma] \in \mathcal{M} \mid \exists V_4 \subset V_7 \quad \sigma(V_4, V_7, V_7) = 0\}, \tag{4.6}$$

which is unirational.

Note that by restricting to a subspace of dimension 7, we have the following result which produces some equivalent degeneracy conditions.

Lemma 4.2.2. *Let V_7 be a complex vector space of dimension 7. For a non-zero trivector $\sigma \in \bigwedge^3 V_7^\vee$, the following conditions are equivalent*

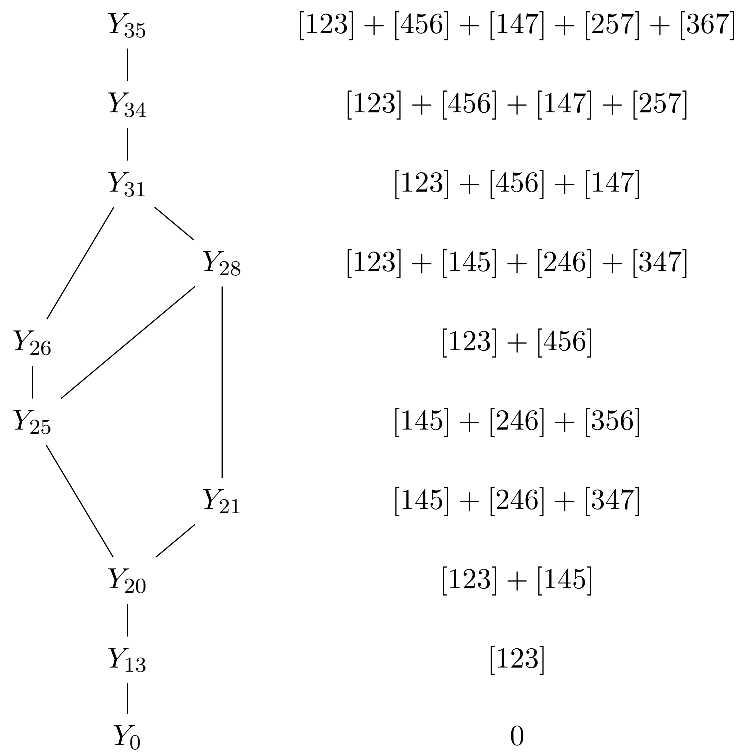
(1) σ is decomposable, that is, we can take a basis $\{e_i\}_{0\leq i\leq 6}$ such that $\sigma = e_4^\vee \wedge e_5^\vee \wedge e_6^\vee$;
(2) $\exists V_4 \subset V_7 \quad \sigma(V_4, V_7, V_7) = 0$, in other words, σ is of rank ≤ 3;
(3) $\exists V_3 \subset V_7 \quad \sigma(V_3, V_7, V_7) = 0$, in other words, σ is of rank ≤ 4;
(4) $\exists V_5 \subset V_7 \quad \sigma(V_5, V_5, V_7) = 0$.

Proof. The conditions (1) and (2) are clearly equivalent. The condition (3) says that σ is of rank ≤ 4, but there exists no trivector with rank 4, so we deduce that σ is of rank 3 and thus decomposable.

For the implication (1) $\Rightarrow$ (4), we simply take $V_5 = \langle e_0, \ldots, e_4\rangle$. Conversely, given V_5 as in (4), we pick two vectors e_5 and e_6 not in V_5 that generate V_7/V_5. Then we see that σ is uniquely determined by the form $\sigma(e_5, e_6, -)$, which show that σ is decomposable. □

This gives us some equivalent conditions to detect whether the ten dimensional trivector $\sigma \in \bigwedge^3 V_{10}^\vee$ is decomposable along a subspace V_7. Note however that in the above conditions, the subspaces V_3 and V_5 are not uniquely determined.

More generally, a classification of $\mathrm{GL}(V_7)$-orbits in $\bigwedge^3 V_7^\vee$ is known, which was first carried out by Schouten (see [**Gur64**, Section 35.3]). The following diagram is taken from [**KW13**, Section 3]. Here each Y_k is an orbit closure of affine dimension k, and $\Delta = Y_{34}$ is the discriminant hypersurface of degree 7.



We write as usual $[ijk]$ for the form $e_i^\vee \wedge e_j^\vee \wedge e_k^\vee$. The orbit closures $Y_0, Y_{13}, Y_{20}, Y_{25}, Y_{26}$ correspond to trivectors with rank ≤ 6.

Note that similar to the dimension 10 case we saw in Section 4.2.1, a trivector σ lies in the discriminant hypersurface $\Delta = Y_{34}$ if and only if there exists a V_3 such that $\sigma(V_3, V_7, V_7) = 0$.

For a general σ in the next orbit closure Y_{31}, we can write $\sigma = [123] + [456] + [147]$. We check that the subspaces $V_3 = \langle e_2, e_3, e_7 \rangle$ and $V_6 = \langle e_2, \dots, e_7 \rangle$ satisfy $\sigma(V_3, V_6, V_6) = 0$, in other words, σ is decomposable along V_6. A parameter count shows that the set of such σ has dimension 31, so we conclude that a trivector σ lies in Y_{31} if and only if there exists a flag $V_3 \subset V_6$ such that $\sigma(V_3, V_6, V_6) = 0$.

4.2.4. Further degeneracy. We give some explicit conditions for a trivector $\sigma \in \bigwedge^3 V_{10}^\vee$ to be unstable. For later studies, we will exclude such trivectors and only consider semi-stable ones.

LEMMA 4.2.3. *For a trivector $\sigma \in \bigwedge^3 V_{10}^\vee$, the point $[\sigma] \in \mathbf{P}(\bigwedge^3 V_{10}^\vee)$ is unstable in the following cases:*

(1) If there exists V_7 such that $\sigma(V_7, V_7, V_7) = 0$;
(2) If there exists V_1 such that $\sigma(V_1, V_{10}, V_{10}) = 0$;
(3) If there exists V_4 such that $\sigma(V_4, V_4, V_{10}) = 0$;

(4) If there exist $V_1 \subset V_6$ such that $\sigma(V_1, V_6, V_{10}) = 0$ and $\sigma(V_6, V_6, V_6) = 0$;
(5) If there exist $V_5 \subset V_8$ such that $\sigma(V_5, V_8, V_8) = 0$.

Proof. This can be easily verified using the Hilbert–Mumford criterion: in each case, we consider a 1-parameter subgroup of $\mathrm{SL}(V_{10})$ acting with the following weights

$$(3,3,3,3,3,3,3,-7,-7,-7), \quad (9,-1,-1,-1,-1,-1,-1,-1,-1,-1),$$
$$(3,3,3,3,-2,-2,-2,-2,-2,-2), \quad (13,3,3,3,3,3,-7,-7,-7,-7),$$
$$\text{and} \quad (5,5,5,5,5,-1,-1,-1,-11,-11).$$

The point $[\sigma]$ would admit only negative weights with respect to the 1-parameter subgroup, so it is indeed unstable. $\square$

We also consider the following situation, where a family of very degenerate trivectors are strictly semi-stable and are identified to a single point in the GIT quotient $\mathcal{M}$.

Lemma 4.2.4. *Consider the following special trivector σ_0: decompose V_{10} as a direct sum $V_{10} = V_7 \oplus V_3$, and let*

$$\sigma_0 \coloneqq \alpha + \beta, \quad \text{where } \alpha \in \textstyle\bigwedge^3 V_7^\vee \text{ is general and } \beta \text{ is a generator of } \bigwedge^3 V_3^\vee.$$

Consider a trivector σ satisfying the following vanishing condition

$$\exists V_1 \subset V_8 \quad \sigma(V_1, V_8, V_{10}) = 0. \tag{4.7}$$

Then $[\sigma]$ is semi-stable if and only if

$$\sigma|_{V_8/V_1} \in \textstyle\bigwedge^3 (V_8/V_1)^\vee \text{ is general and } \sigma(V_1, -, -)|_{V_{10}/V_8} \neq 0. \tag{4.8}$$

In this case, the class $[\sigma] \in \mathcal{M}$ is given by the class $[\sigma_0]$ of the special trivector σ_0.

Here $\alpha \in \bigwedge^3 V_7^\vee$ being general means that α does not lie in the discriminant hypersurface, or equivalently, there is no V_3 such that $\alpha(V_3, V_7, V_7) = 0$.

We also note that the special trivector σ_0 has been studied in [**DHOV20**, Section 5.2]. Namely, in Corollary 5.13 of *loc. cit.*, it was proved that the point $[\sigma_0] \in \mathbf{P}(\bigwedge^3 V_{10}^\vee)$ is polystable with respect to the $\mathrm{SL}(V_{10})$-action, and its stabilizer is isomorphic to $G_2 \times \mathrm{SL}(V_3)$, where G_2 is the stabilizer of α and $\mathrm{SL}(V_3)$ is the stabilizer of β. Moreover, it was shown that the point $[\sigma_0]$ is the preimage in $\mathcal{M}$ of the HLS Heegner divisor $\mathcal{D}_{18}$ by the period map.

Proof. Consider a trivector σ satisfying (4.7). We take a basis $\{e_i\}_{0 \le i \le 9}$ of V_{10} such that $V_1 = \langle e_0 \rangle$ and $V_8 = \langle e_0, \ldots, e_7 \rangle$. Let $V_7 \coloneqq \langle e_1, \ldots, e_7 \rangle$. Then the trivector σ can be written as

$$\omega + u \wedge e_8^\vee + v \wedge e_9^\vee + f \wedge e_8^\vee \wedge e_9^\vee + b \cdot e_0^\vee \wedge e_8^\vee \wedge e_9^\vee,$$

where $\omega \in \bigwedge^3 V_7^\vee$, $u, v \in \bigwedge^2 V_7^\vee$, $f \in V_7^\vee$, and $b \in \mathbf{C}$. We note that the condition (4.8) is equivalent to ω being general in $\bigwedge^3 V_7^\vee$ and $b \neq 0$.

We consider the 1-parameter subgroup of $\mathrm{SL}(V_{10})$ acting with the following weights

$$(2, 0, 0, 0, 0, 0, 0, 0, -1, -1).$$

In other words, for each $t \in \mathbf{C}^*$ we get a diagonal action on V_{10} given by

$$(x_0, \dots, x_9) \longmapsto (t^2 x_0, x_1, \dots, x_7, t^{-1} x_8, t^{-1} x_9).$$

The induced action on $\bigwedge^3 V_{10}^\vee$ maps σ to

$$\omega + t \cdot u \wedge e_8^\vee + t \cdot v \wedge e_9^\vee + t^2 \cdot f \wedge e_8^\vee \wedge e_9^\vee + b \cdot e_0^\vee \wedge e_8^\vee \wedge e_9^\vee.$$

Letting t go to 0, we see that the trivector

$$\sigma' \coloneqq \omega + b \cdot e_0^\vee \wedge e_8^\vee \wedge e_9^\vee$$

lies in the closure of the $\mathrm{SL}(V_{10})$-orbit of σ. Therefore $[\sigma]$ is semi-stable if and only if $[\sigma']$ is semi-stable.

We show that if the condition (4.8) is not satisfied, then $[\sigma']$ is unstable. First, we suppose that ω is not general. Then ω lies in the discriminant hypersurface, so there exists a $V_3 \subset V_7$ such that

$$\omega(V_3, V_3, V_7) = 0.$$

In particular, by letting $V_4 \coloneqq V_1 \oplus V_3$, we may verify that

$$\sigma'(V_4, V_4, V_{10}) = 0.$$

So by Lemma 4.2.3, the point $[\sigma']$ is indeed unstable. Similarly, if we suppose that $b = 0$, then we will have

$$\sigma'(V_1, V_{10}, V_{10}) = 0,$$

so again $[\sigma']$ is unstable by Lemma 4.2.3.

Conversely, if the condition (4.8) is satisfied, that is, we have ω general and $b \neq 0$, then σ' is $\mathrm{SL}(V_{10})$-equivalent to the special trivector σ_0, which is indeed semi-stable by [**DHOV20**, Corollary 5.13]. So we may conclude that $[\sigma]$ is semi-stable, and its class in $\mathcal{M}$ is given by the class $[\sigma_0]$. □

4.3. Smoothness criteria

In this section, we prove the criteria for the smoothness of the varieties X_1^σ, X_3^σ, and X_6^σ as stated in Proposition 4.1.1. The method is purely local and does not involve the global geometry of these varieties.

Notably, these smoothness criteria recover two of the divisors in the GIT quotient $\mathcal{M}$ that we defined in the last section. We will discuss the explicit geometry of the three varieties in these cases in later sections.

4.3.1. Smoothness of X_3^σ and X_6^σ. The smoothness of the hyperplane section X_3^σ is a well-known result, which we have already seen in Section 4.2.1.

PROPOSITION 4.3.1. *Set-theoretically, the singular locus* $\mathrm{Sing}(X_3^\sigma)$ *is given by*

$$\{[V_3] \in X_3^\sigma \mid \sigma(V_3, V_3, V_{10}) = 0\}.$$

PROOF. The hyperplane section X_3^σ is not smooth of dimension 20 at $[V_3]$ if and only if the differential

$$d\sigma\colon \mathcal{T}_{\mathrm{Gr}(3,V_{10}),[V_3]} \simeq \mathrm{Hom}(V_3, V_{10}/V_3) \longrightarrow \bigwedge^3 V_3^\vee$$

vanishes. Here $d\sigma$ maps an element $f \in \mathrm{Hom}(V_3, V_{10}/V_3)$ to the 3-form

$$d\sigma(f)\colon (v_1, v_2, v_3) \longmapsto \sigma(f(v_1), v_2, v_3) + \sigma(v_1, f(v_2), v_3) + \sigma(v_1, v_2, f(v_3)).$$

By varying $v_1, v_2, v_3 \in V_3$ and f, we get the desired vanishing condition. □

It turns out that the smoothness of X_6^σ can be given by the same criterion.

LEMMA 4.3.2. *Let* $[V_6]$ *be a point in* X_6^σ. *The Debarre–Voisin variety* X_6^σ *is not smooth of dimension* 4 *at* $[V_6]$ *if and only if there exists* $V_3 \subset V_6$ *such that* $\sigma(V_3, V_3, V_{10}) = 0$.

PROOF. The Zariski tangent space $\mathcal{T}_{X_6^\sigma,[V_6]}$ of the Debarre–Voisin variety X_6^σ at $[V_6]$ is given as the kernel of the differential

$$d\sigma\colon \mathcal{T}_{\mathrm{Gr}(6,V_{10}),[V_6]} \simeq \mathrm{Hom}(V_6, V_{10}/V_6) \longrightarrow \bigwedge^3 V_6^\vee,$$

which maps $f \in \mathrm{Hom}(V_6, V_{10}/V_6)$ to the 3-form

$$d\sigma(f)\colon (v_1, v_2, v_3) \longmapsto \sigma(f(v_1), v_2, v_3) + \sigma(v_1, f(v_2), v_3) + \sigma(v_1, v_2, f(v_3)).$$

Therefore X_6^σ is not smooth of dimension 4 if and only if the differential is not surjective, or equivalently, if there exists some non-zero $\omega \in (\bigwedge^3 V_6^\vee)^\vee \simeq \bigwedge^3 V_6$ such that $\omega|_{\mathrm{Im}(d\sigma)} = 0$, that is, for any $f \in \mathrm{Hom}(V_6, V_{10}/V_6)$ we have $d\sigma(f)(\omega) = 0$.

Suppose that $V_3 \subset V_6$ is a subspace satisfying the vanishing condition $\sigma(V_3, V_3, V_{10}) = 0$. Then a non-zero $\omega \in \bigwedge^3 V_3$ satisfies the above property, so X_6^σ is not smooth of dimension 4 at $[V_6]$.

Conversely, the orbit closures for the $\mathrm{GL}(V_6)$-action on $\bigwedge^3 V_6$ have long been classified (attributed to Reichel, see [**Gur64**, Section 35.2]): there are five of them, including $\{0\}$. So we study the four non-zero orbits case by case.

- If ω is completely decomposable, that is when $\omega = e_1 \wedge e_2 \wedge e_3$, consider a map f with $f(e_1) = f(e_2) = 0$: the property of ω shows that $\sigma(e_1, e_2, f(e_3)) = 0$, so by varying f we get $\sigma(e_1, e_2, V_{10}) = 0$. Similarly we have $\sigma(e_1, e_3, V_{10}) = \sigma(e_2, e_3, V_{10}) = 0$. So the subspace $V_3 = \langle e_1, e_2, e_3 \rangle$ satisfies the vanishing condition $\sigma(V_3, V_3, V_{10}) = 0$.

- If ω is of rank 5, it can be written as $e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_4 \wedge e_5$. Let $V_1 = \langle e_1 \rangle$ and $V_4 = \langle e_2, e_3, e_4, e_5 \rangle$. We get $\sigma(V_1, V_1 + V_4, V_{10}) = 0$. Consider the map
$$\varphi_\sigma \colon \bigwedge^2 V_4 \longrightarrow (V_{10}/V_6)^\vee$$
induced by σ (note that $\sigma|_{V_6} = 0$). The kernel of φ_σ is a subspace of dimension at least 2. Note also that the subset in $\bigwedge^2 V_4$ of decomposable elements is the affine cone over the Grassmannian $\mathrm{Gr}(2, V_4)$, which is a quadric hypersurface. This shows that there is some decomposable element $u \wedge v$ in the kernel of φ_σ. The subspace $\langle e_1, u, v \rangle$ thus provides the V_3 we want. Moreover, without loss of generality, we may suppose that $u \wedge v = e_2 \wedge e_3$; then $\langle e_1, e_4, e_5 \rangle$ gives another V_3 satisfying the vanishing condition.
- If ω is of type $e_1 \wedge e_2 \wedge e_4 + e_2 \wedge e_3 \wedge e_5 + e_1 \wedge e_3 \wedge e_6$, by considering a map f with $f(e_1) = f(e_2) = f(e_3) = 0$, we can see that $\langle e_1, e_2, e_3 \rangle$ gives a V_3 such that $\sigma(V_3, V_3, V_{10}) = 0$.
- If ω is general, so of type $e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6$, both $\langle e_1, e_2, e_3 \rangle$ and $\langle e_4, e_5, e_6 \rangle$ give a V_3 such that $\sigma(V_3, V_3, V_{10}) = 0$.

Therefore, the Zariski tangent space is not of dimension 4 if and only if there exists a V_3 satisfying the vanishing condition. □

We summarize the above results as follows.

Proposition 4.3.3. *The divisor $\mathcal{D}^{3,3,10}$ is the locus in $\mathcal{M}$ of $\mathrm{SL}(V_{10})$-classes of trivectors $[\sigma]$ for which X_3^σ and X_6^σ become singular. Moreover, for a general element $[\sigma] \in \mathcal{D}^{3,3,10}$ admitting a unique V_3 with $\sigma(V_3, V_3, V_{10}) = 0$, we have*
$$\mathrm{Sing}(X_3^\sigma) = \{[V_3]\},$$
$$\mathrm{Sing}(X_6^\sigma) = S_{22} = \{[V_6] \in X_6^\sigma \mid V_6 \supset V_3\},$$
where S_{22} is a K3 surface of degree 22. For a very general $[\sigma] \in \mathcal{D}^{3,3,10}$, the K3 surface S_{22} is of Picard rank 1.

The claim on the Picard rank follows from the projective model for K3 surfaces of degree 22 by Mukai (see [**Muk06**]). Note that in [**DV10**], it was only proved that the K3 surface S_{22} is contained in the singular locus, instead of an equality.

In fact, for a general $[\sigma] \in \mathcal{D}^{3,3,10}$, we can get a precise description of the type of singularity along S_{22}: similar to the nodal cubic case, by blowing up the singular locus, we obtain a smooth hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type, with the exceptional divisor being a $\mathbf{P}^1$-bundle over the K3 surface. See Section 4.8 for the proof.

4.3.2. Smoothness of X_1^σ. We study the smoothness of the Peskine variety X_1^σ. Recall that this is the locus in $\mathbf{P}(V_{10})$ where the rank of the skew-symmetric form $\sigma(V_1,-,-)$ drops to 6 or less.

We first remark that X_1^σ is smooth for σ general, which follows from the general theory of orbital degeneracy loci (ODL) from [**BFMT20**]. Consider the space of skew-symmetric bilinear forms on a vector space V_n. The $\mathrm{GL}(V_n)$-orbits are entirely determined by their ranks, so we have the following filtration

$$\{0\} = Y_0 \subset Y_2 \subset \cdots \subset Y_{2\lfloor \frac{n}{2} \rfloor} = \bigwedge^2 V_n^\vee,$$

where Y_{2r} consists of skew-symmetric bilinear forms of rank $\leq 2r$. Moreover, we have $\mathrm{Sing}(Y_{2r}) = Y_{2r-2}$ for $1 \leq r \leq \lfloor \frac{n}{2} \rfloor - 1$. In our case, by viewing σ as a section of the globally generated vector bundle $\bigwedge^2 \mathcal{Q}(1)$, the Peskine variety X_1^σ can be defined as the orbital degeneracy locus $D_{Y_6}(\sigma)$, where $Y_6 \subset \bigwedge^2 \mathbf{C}^9$ consists of skew-symmetric forms of rank ≤ 6 on a 9-dimensional vector space. By the Bertini theorem for orbital degeneracy loci [**BFMT20**, Proposition 2.3], for general σ we have

$$\mathrm{Sing}(D_{Y_6}(\sigma)) = D_{\mathrm{Sing}(Y_6)}(\sigma) = D_{Y_4}(\sigma) = \{[V_1] \in \mathbf{P}(V_{10}) \mid \operatorname{rank} \sigma(V_1,-,-) \leq 4\},$$

Since the codimension of Y_4 in $\bigwedge^2 \mathbf{C}^9$ is equal to 10, we may conclude that for a general σ, $D_{Y_4}(\sigma)$ is empty and therefore $X_1^\sigma = D_{Y_6}(\sigma)$ is smooth.

Now we remove the assumption of σ being general. We relate the Peskine variety to a certain zero locus in a flag variety. Namely, by viewing σ as a section of the vector bundle $(\mathcal{U}_1 \wedge \mathcal{U}_4 \wedge V_{10})^\vee$ on $\mathrm{Flag}(1,4,V_{10})$, we may consider the zero locus $Z(\sigma)$. The natural projection $\pi\colon \mathrm{Flag}(1,4,V_{10}) \to V_{10}$ restricts to a proper map $\pi\colon Z(\sigma) \to X_1^\sigma$. One can easily see that the fiber of this morphism at a point $[V_1]$ is the set of V_4 contained in the kernel of the skew-symmetric form $\sigma(V_1,-,-)$. In particular, we get an isomorphism by restricting π to the open locus where the rank of $\sigma(V_1,-,-)$ is exactly 6.

We first study the smoothness of $Z(\sigma)$.

LEMMA 4.3.4. *We view σ as a section of the vector bundle $(\mathcal{U}_1 \wedge \mathcal{U}_4 \wedge V_{10})^\vee$ on $\mathrm{Flag}(1,4,V_{10})$ and consider its zero-locus $Z(\sigma)$. A point $[V_1 \subset V_4] \in Z(\sigma)$ is not smooth of dimension 6 if and only if either $\sigma(V_1,-,-)$ is of rank ≤ 4 or there exists V_3 containing V_1 such that $\sigma(V_3,V_3,V_{10}) = 0$.*

PROOF. The variety $Z(\sigma)$ being not smooth of dimension 6 at $[V_1 \subset V_4]$ means that the differential

$$d\sigma\colon \mathcal{T}_{\mathrm{Flag}(1,4,V_{10}),[V_1 \subset V_4]} \simeq \mathrm{Hom}(V_1, V_{10}/V_1) \oplus \mathrm{Hom}(V_4/V_1, V_{10}/V_4) \longrightarrow (V_1 \wedge V_4 \wedge V_{10})^\vee,$$

which maps $f \in \mathrm{Hom}(V_1, V_{10}/V_1) \oplus \mathrm{Hom}(V_4/V_1, V_{10}/V_4)$ to the 3-form

$$d\sigma(f)\colon (v_1,v_2,v_3) \longmapsto \sigma(f(v_1),v_2,v_3) + \sigma(v_1,f(v_2),v_3) + \sigma(v_1,v_2,f(v_3)),$$

is not surjective (notice that f can be seen as a class of maps from V_4 to V_{10} modulo those maps that preserve V_1 and V_4). Equivalently, there exists a non-zero $\omega \in$

$((V_1 \wedge V_4 \wedge V_{10})^\vee)^\vee = V_1 \wedge V_4 \wedge V_{10}$ such that $\omega|_{\mathrm{Im}(d\sigma)} = 0$, that is, for any f as above, we have $d\sigma(f)(\omega) = 0$.

Modulo a change of coordinates, one can always take a suitable basis $\{e_i\}_{0\leq i\leq 9}$ such that $V_1 = \langle e_0\rangle$ and $V_4 = \langle e_0, \dots, e_3\rangle$, and that

$$\omega = e_0 \wedge \big(a(e_1 \wedge e_2) + b(e_1 \wedge e_4) + c(e_2 \wedge e_5) + d(e_3 \wedge e_6)\big)$$

for certain coefficients $a, b, c, d \in \mathbf{C}$. The proof is divided into three cases:

- If $d \neq 0$, consider a morphism f sending e_0, e_1, e_2 to 0. Then $d\sigma(f)(\omega) = 0$ shows that $\sigma(e_0, e_6, f(e_3)) = 0$. By varying f, one gets $\sigma(V_1, V_4 + \mathbf{C}e_6, V_{10}) = 0$, which implies that $\sigma(V_1, -, -)$ has rank at most 4.
- If $d = 0$ and $b \neq 0$, consider a morphism f sending e_0, e_2, e_3 to 0. Then $d\sigma(f)(\omega) = 0$ shows that $\sigma(e_0, e_4, f(e_1)) = 0$. By varying f, one gets $\sigma(V_1, V_4 + \mathbf{C}e_4, V_{10}) = 0$, again implying that $\sigma(V_1, -, -)$ has rank at most 4. Similarly one can treat the case when $d = 0$ and $c \neq 0$.
- If $b = c = d = 0$, consider a morphism f sending e_1, e_2, e_3 to 0. Then $d\sigma(f)(\omega) = 0$ shows that $\sigma(f(e_0), e_1, e_2) = 0$. By varying f and setting $V_3 = \langle e_0, e_1, e_2\rangle$, one gets $\sigma(V_3, V_3, V_{10}) = 0$.

Therefore, we may conclude that for a singular point $[V_1 \subset V_4]$ in $Z(\sigma)$, either $\sigma(V_1, -, -)$ is of rank ≤ 4, or there exists $V_3 \supset V_1$ with $\sigma(V_3, V_3, V_{10}) = 0$. □

PROPOSITION 4.3.5. *The locus of trivectors $[\sigma]$ for which $X_1^\sigma \subset \mathbf{P}(V_{10})$ becomes singular is the union of two divisors $\mathcal{D}^{1,6,10} \cup \mathcal{D}^{3,3,10}$ in $\mathcal{M}$. Moreover, we have the following set-theoretical descriptions of the singular locus.*[2]

- *If $[\sigma] \in \mathcal{D}^{1,6,10}$ is general such that $\sigma(V_1, V_6, V_{10}) = 0$, then* $\mathrm{Sing}(X_1^\sigma) = \{[V_1]\}$.
- *If $[\sigma] \in \mathcal{D}^{3,3,10}$ is general such that $\sigma(V_3, V_3, V_{10}) = 0$, then* $\mathrm{Sing}(X_1^\sigma) = \mathbf{P}(V_3)$.

PROOF. If $[\sigma] \notin \mathcal{D}^{1,6,10}$, then the projection $\pi\colon Z(\sigma) \to X_1^\sigma$ is an isomorphism. So by Lemma 4.3.4, we may conclude that X_1^σ is smooth if $[\sigma] \notin \mathcal{D}^{3,3,10} \cup \mathcal{D}^{1,6,10}$, while for a general $[\sigma] \in \mathcal{D}^{3,3,10} \smallsetminus \mathcal{D}^{1,6,10}$ such that there exists a unique V_3 with $\sigma(V_3, V_3, V_{10}) = 0$, we have $\mathrm{Sing}(X_1^\sigma) = \mathbf{P}(V_3)$.

For a general $[\sigma] \in \mathcal{D}^{1,6,10} \smallsetminus \mathcal{D}^{3,3,10}$, there exists a unique flag $[V_1 \subset V_6]$ with $\sigma(V_1, V_6, V_{10}) = 0$. In this case, the zero-locus $Z(\sigma)$ is reducible: it contains a component Z_0 that dominantes X_1^σ, as well as the Grassmannian $G \coloneqq \mathrm{Gr}(3, V_6/V_1)$ that is contracted to the point $[V_1]$ by π. Moreover, the projection π restricted to $Z_0 \smallsetminus G$ is an isomorphism onto the image $X_1^\sigma \smallsetminus [V_1]$. Again by Lemma 4.3.4, we see that X_1^σ is smooth away from $[V_1]$.

[2]It is in theory possible to analyze the type of the singularities in these two cases. We have chosen some random trivectors and computed the tangent cones at the singular points: in the first case, the tangent cone at the singular $[V_1]$ is isomorphic to a hyperplane section of $\mathrm{Gr}(2, 5)$, which is a Fano 5-fold of degree 5 (one explicit example was given in (4.2.2)); in the second case, the tangent cone at a general point in $\mathbf{P}(V_3)$ is a smooth 5-dimensional quadric. One would expect these descriptions to hold for a general σ in both cases.

It remains to show that X_1^σ is indeed singular at $[V_1]$. Since X_1^σ is a degeneracy locus defined by the vanishing of Pfaffians, we have the following resolution (see **[Han15**, Section 2.3])

$$0 \longrightarrow \mathcal{O}(-7) \longrightarrow \mathcal{Q}(-4) \xrightarrow{f} \mathcal{Q}^\vee(-3) \longrightarrow \mathcal{I}_{X_1^\sigma} \longrightarrow 0,$$

where at each point $[V_1] = [\mathbf{C}v]$, the map f is given by

$$f\colon V_{10}/V_1 \longrightarrow V_1^\perp \subset V_{10}^\vee, \quad v' \longmapsto \sigma(v, v', -).$$

Therefore, if $\sigma(V_1, -, -) \leq 4$, the cokernel of f at $[V_1]$ will have rank ≥ 5, so the conormal bundle $\mathcal{N}^\vee_{X_1^\sigma/\mathbf{P}(V_{10})} \simeq \mathcal{I}/\mathcal{I}^2$ would not be locally free of rank 3 at $[V_1]$. In other words, X_1^σ is indeed singular at $[V_1]$.

Finally, since smoothness is an open condition, one may conclude that X_1^σ is singular not just for general members of $\mathcal{D}^{3,3,10} \cup \mathcal{D}^{1,6,10}$, but for all of them. □

4.4. Moduli space and period map

4.4.1. Stable trivectors. Before we study the moduli space of Debarre–Voisin fourfolds, we first show that a trivector $[\sigma] \in \mathbf{P}(\bigwedge^3 V_{10}^\vee)$ defining a smooth Debarre–Voisin fourfold X_6^σ is stable with respect to the $\mathrm{SL}(V_{10})$-action. This is based on the following result.

PROPOSITION 4.4.1. *Write G for the Grassmannian $\mathrm{Gr}(6, V_{10})$ and $X \coloneqq X_6^\sigma$ for a smooth Debarre–Voisin fourfold. Let $\mathcal{U}$ and $\mathcal{Q}$ be the tautological subbundle and quotient bundle on the Grassmannian $\mathrm{Gr}(6, V_{10})$. The restrictions $\mathcal{U}|_X$ and $\mathcal{Q}|_X$ are both simple, that is, $\mathrm{End}(\mathcal{U}|_X) \simeq \mathrm{End}(\mathcal{Q}|_X) \simeq \mathbf{C}$.*

PROOF. We write $\mathcal{F}$ for the 20-dimensional vector bundle $\bigwedge^3 \mathcal{U}^\vee$ on G. Consider the Koszul complex

$$0 \longrightarrow \textstyle\bigwedge^{20} \mathcal{F}^\vee \longrightarrow \cdots \longrightarrow \bigwedge^2 \mathcal{F}^\vee \longrightarrow \mathcal{F}^\vee \longrightarrow \mathcal{O}_G \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{4.9}$$

which gives a free resolution of the structure sheaf of $\mathcal{O}_X$. For a vector bundle $\mathcal{E}$ on G, we can tensor the Koszul complex with $\mathcal{E}$ and obtain a spectral sequence

$$E_1^{-q,p} = H^p(G, \mathcal{E} \otimes \textstyle\bigwedge^q \mathcal{F}) \Longrightarrow H^{p-q}(X, \mathcal{E}|_X).$$

In our case, we consider $\mathcal{E} \coloneqq \mathcal{U}^\vee \otimes \mathcal{U}$. The cohomologies of the vector bundle $\mathcal{U}^\vee \otimes \mathcal{U} \otimes \bigwedge^q \mathcal{F}$ on the Grassmannian $\mathrm{Gr}(6, V_{10})$ can be computed using the Borel–Weil–Bott theorem (see **[BCP20**, Appendix A] for a detailed account). One may verify that there are only three terms that are non-zero

$$h^0(G, \mathcal{E}) = h^{24}(G, \mathcal{E} \otimes \det \mathcal{F}) = 1, \quad h^{12}(G, \mathcal{E} \otimes \textstyle\bigwedge^{10} \mathcal{F}) = 101.$$

In particular, the spectral sequence degenerates at the first page, so we can conclude that

$$h^0\big(X, \mathcal{E}|_X\big) = h^4\big(X, \mathcal{E}|_X\big) = 1, \quad h^2\big(X, \mathcal{E}|_X\big) = 101,$$

while $h^1 = h^3 = 0$. The computation for the bundle $\mathcal{Q}|_X$ is similar. □

Remark 4.4.2. In [**O'G19**, Corollary 8.5], O'Grady showed that for a very general X_6^σ, the bundles $\mathcal{U}|_{X_6^\sigma}$ and $\mathcal{Q}|_{X_6^\sigma}$ are slope-stable with respect to the Plücker polarization, so in particular they are simple. The above result shows that the simpleness holds whenever X_6^σ is smooth, without the hypothesis on X_6^σ being very general.

Corollary 4.4.3. *Consider a trivector* $[\sigma] \in \mathbf{P}(\bigwedge^3 V_{10}^\vee) \smallsetminus \Delta^{3,3,10}$. *Denote by* $\mathrm{Stab}(\sigma) \lhd \mathrm{SL}(V_{10})$ *the stabilizer of* σ *and consider the natural homomorphism*

$$\Phi\colon \mathrm{Stab}(\sigma) \longrightarrow \mathrm{Aut}(X_6^\sigma),$$

which maps each $\varphi \in \mathrm{SL}(V_{10})$ *to the induced automorphism* $\Phi(\varphi)$ *on* X_6^σ. *The kernel of the map is equal to* $\{\pm \mathrm{Id}\}$,

Proof. By Proposition 4.3.3, if $[\sigma]$ does not lie in the discriminant hypersurface $\Delta^{3,3,10}$, then it defines a smooth Debarre–Voisin fourfold X_6^σ. Suppose that $\varphi \in \mathrm{SL}(V_{10})$ induces the trivial automorphism on X_6^σ, then it will also induce an automorphism

$$f_\varphi \in \mathrm{End}(\mathcal{U}|_{X_6^\sigma}),$$

which acts fiberwise. But since the vector bundle $\mathcal{U}|_{X_6^\sigma}$ is simple by Proposition 4.4.1, up to multiplying by a non-zero scalar, f_φ must be the identity map. In other words, φ acts as the identity on each $\mathbf{P}(V_6)$ for $[V_6] \in X_6^\sigma$.

To conclude, we claim that all the six-dimensional vector spaces $[V_6] \in X_6^\sigma$ span the entire V_{10}. For example, this can be deduced from the fact that $h^0(X_6^\sigma, \mathcal{U}^\vee) = 10$ (see Proposition 4.4.6 below). So φ acts as the identity on V_{10}, and we may conclude that $\ker(\Phi) = \{\pm \mathrm{Id}\}$. □

Since X_6^σ is hyperkähler, the automorphism group $\mathrm{Aut}(X_6^\sigma)$ is always finite. So we may deduce that the stabilizer $\mathrm{Stab}(\sigma)$ is also finite, whenever $[\sigma]$ does not lie in the discriminant $\Delta^{3,3,10}$.

Corollary 4.4.4. *Let* $[\sigma] \in \mathbf{P}(\bigwedge^3 V_{10}^\vee)$ *be semi-stable with respect to the* $\mathrm{SL}(V_{10})$*-action. If* $[\sigma]$ *does not lie in the discriminant hypersurface* $\Delta^{3,3,10}$, *then* $[\sigma]$ *admits a finite stabilizer and is stable.*

Remark 4.4.5. To show that the $[V_6] \in X_6^\sigma$ span the entire V_{10}, we can also consider the following incidence variety

$$\begin{array}{ccc} \Sigma = \{([V_1],[V_6]) \mid V_1 \subset V_6, \sigma|_{V_6} = 0\} & \xrightarrow{\mathrm{pr}_2} & X_6^\sigma \\ \Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \\ \mathbf{P}(V_{10}) & & \end{array}$$

The second projection pr_2 realizes Σ as a projective bundle $\mathbf{P}(\mathcal{U})$ over X_6^σ, where $\mathcal{U}$ is the tautological subbundle. So Σ is smooth of dimension 9. Using Schubert calculus, one may compute that the degree of the first projection pr_1 is equal to 9 (which is the top Chern class of a suitable vector bundle on $\mathrm{Gr}(5,9)$). In other words, for a general point $[V_1] \in \mathbf{P}(V_{10})$, there are exactly 9 six-dimensional subspaces in X_6^σ containing V_1. In particular, the projection pr_1 is surjective and generically finite. However, it is not the case that the 9 six-dimensional subspaces intersect along V_1.

The method used in Proposition 4.4.1 is very standard, and can be adapted to the computation of the cohomologies for many other vector bundles on X_6^σ (for example, it was used in [**DV10**, Remark 2.6] to give another simple proof of the fact that X_6^σ is hyperkähler). We recollect some computations here.

PROPOSITION 4.4.6. *Let $\mathcal{U}$ and $\mathcal{Q}$ be the tautological subbundle of rank* 6 *and* 4 *on* $\mathrm{Gr}(6, V_{10})$ *respectively, and let $X = X_6^\sigma$ be a smooth Debarre–Voisin fourfold. We have the following descriptions of the cohomologies of vector bundles on X_6^σ (where blank means the corresponding cohomology vanishes).*

$\mathcal{E}$	$\mathcal{O}$	$\mathcal{U}^\vee$	$\mathrm{Sym}^2\mathcal{U}^\vee$	$\bigwedge^3\mathcal{U}^\vee$	$\mathcal{Q}$	$\bigwedge^2\mathcal{Q}$	$\mathrm{Sym}^2\mathcal{Q}$	$\mathrm{Sym}^3\mathcal{Q}$	$\mathcal{U}^\vee\otimes\mathcal{U}$	$\mathcal{Q}^\vee\otimes\mathcal{Q}$
$h^0(\mathcal{E}\vert_X)$	1	10	55	119	10	45	55	230	1	1
$h^1(\mathcal{E}\vert_X)$							9			
$h^2(\mathcal{E}\vert_X)$	1	10	55	20				1	101	1
$h^3(\mathcal{E}\vert_X)$										
$h^4(\mathcal{E}\vert_X)$	1								1	1

Note that since X has trivial canonical bundle, one can obtain the cohomologies of the dual vector bundles using the Serre duality.

PROOF. In each case, we tensor the Koszul complex (4.9) with the vector bundle $\mathcal{E}$ and obtain a spectral sequence. In most cases, the spectral sequence degenerates at the first page, and we obtain directly the cohomologies of $\mathcal{E}|_X$. This does not happen for the two vector bundles $\bigwedge^3\mathcal{U}^\vee$ and $\mathrm{Sym}^2\mathcal{Q}$, so we provide some further details in these two cases.

For the vector bundle $\mathcal{E} = \mathrm{Sym}^2\mathcal{Q}$, the only non-zero terms in the first page of the spectral sequence are given by

$$\dim E_1^{0,0} = h^0(G, \mathcal{E}) = 55, \quad \dim E_1^{-6,6} = h^6(G, \mathcal{E}\otimes \textstyle\bigwedge^6\mathcal{F}^\vee) = 1,$$
$$\dim E_1^{-9,10} = h^{10}(G, \mathcal{E}\otimes \textstyle\bigwedge^9\mathcal{F}^\vee) = 10.$$

This gives the vanishing of $h^k(X, \mathcal{E}|_X)$ for $k \geq 2$ as well as the holomorphic Euler characteristic $\chi(X, \mathcal{E}|_X) = 46$, so it suffices to show that $h^0(X, \mathrm{Sym}^2\mathcal{Q}|_X) = 55$. We use the resolution

$$0 \longrightarrow \textstyle\bigwedge^2\mathcal{U} \longrightarrow \mathcal{U}\otimes V_{10} \longrightarrow \mathrm{Sym}^2 V_{10}\otimes\mathcal{O}_G \longrightarrow \mathrm{Sym}^2\mathcal{Q} \longrightarrow 0.$$

Using again the same method, one may check that $h^1(X, \bigwedge^2\mathcal{U}|_X) = h^0(X, \mathcal{U}|_X) = 0$. So the spectral sequence of the above resolution shows that

$$\mathrm{Sym}^2 V_{10} \simeq H^0(X, \mathrm{Sym}^2 V_{10}\otimes\mathcal{O}_X) \longrightarrow H^0(X, \mathrm{Sym}^2\mathcal{Q}|_X)$$

is injective. This concludes the case of $\mathrm{Sym}^2\mathcal{Q}$.

Now we consider the case of $\bigwedge^3\mathcal{U}^\vee$. Note that this is the globally generated vector bundle $\mathcal{F}$ used in the definition of X_6^σ, and we have a short exact sequence

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_G|_X \longrightarrow \mathcal{F}|_X \longrightarrow 0. \tag{4.10}$$

The non-zero terms in the first page of the spectral sequence for $\mathcal{F}|_X$ are given by

$$\dim E_1^{0,0} = h^0(G,\mathcal{F}) = 120, \quad \dim E_1^{-1,0} = h^0(G,\mathcal{F}\otimes\mathcal{F}^\vee) = 1,$$
$$\dim E_1^{-11,12} = h^{11}(G,\mathcal{F}\otimes\textstyle\bigwedge^{11}\mathcal{F}^\vee) = 100, \quad \dim E_1^{-10,12} = h^{12}(G,\mathcal{F}\otimes\textstyle\bigwedge^{10}\mathcal{F}^\vee) = 120.$$

So we may deduce that $h^3(X,\mathcal{F}|_X) = h^4(X,\mathcal{F}|_X) = 0$. Similarly, we can tensor the Koszul complex with $\mathcal{T}_G$ to compute the cohomologies of $\mathcal{T}_G|_X$: the only non-zero terms in the first page of the spectral sequence are give by

$$\dim E_1^{0,0} = h^0(G,\mathcal{T}_G) = 99, \quad \dim E_1^{-10,11} = h^{11}(G,\mathcal{T}_G\otimes\textstyle\bigwedge^{10}\mathcal{F}^\vee) = 1,$$
$$\dim E_1^{-20,23} = h^{23}(G,\mathcal{T}_G\otimes\textstyle\bigwedge^{20}\mathcal{F}^\vee) = 1,$$

and we can deduce that $h^2(X,\mathcal{T}_G|_X) = h^4(X,\mathcal{T}_G|_X) = 0$, while $h^3(X,\mathcal{T}_G|_X) = 1$. Finally, since X is hyperkähler of $\mathrm{K3}^{[2]}$-type, using the knowledge of the Hodge diamond in this case, we deduce that

$$h^0(X,\mathcal{T}_X) = h^2(X,\mathcal{T}_X) = h^4(X,\mathcal{T}_X) = 0, \quad h^1(X,\mathcal{T}_X) = h^3(X,\mathcal{T}_X) = 21.$$

Using the long exact sequence for (4.10), we get the following dimensions

	$\mathcal{T}_X$	$\mathcal{T}_G\|_X$	$\mathcal{F}\|_X$
h^2	0	0	?
h^3	21	1	0
h^4	0	0	0

from which we may deduce that $h^2(X,\mathcal{F}|_X) = 20$. In particular, the differential map $d_1\colon E_1^{-11,12} \to E_1^{-10,12}$ must be injective, otherwise we would have $\dim E_2^{-10,12} > 20$. This means that the spectral sequence degenerates at the second page, so we may compute that $h^0(X,\mathcal{F}|_X) = 119$ and $h^1(X,\mathcal{F}|_X) = 0$, which concludes the proof. ☐

We have the following corollary.

Corollary 4.4.7. *Consider a smooth Debarre–Voisin fourfold $X \subset \mathrm{Gr}(6,V_{10})$ defined by some trivector σ. Then $[\sigma]$ is the unique class in $\mathbf{P}(\bigwedge^3 V_{10}^\vee)$ such that $X_6^\sigma = X$.*[3]

Proof. We consider the incidence variety

$$\begin{array}{ccc} I_{3,6}^\sigma = \{([V_3],[V_6]) \mid V_3 \subset V_6, \sigma|_{V_6} = 0\} & \xrightarrow{p_6} & X_6^\sigma \\ \downarrow{\scriptstyle p_3} & & \\ X_3^\sigma & & \end{array}$$

[3]Note that we are talking about an exact equality of zero sets in $\mathrm{Gr}(6,V_{10})$ instead of an isomorphism as polarized hyperkähler manifolds. An arbitrary automorphism of X_6^σ might not come from an element of $\mathrm{SL}(V_{10})$, hence the latter question is more difficult, and is solved by O'Grady (see Theorem 4.4.11).

The map p_6 is a Grassmannian fibration with fibers isomorphic to $\mathrm{Gr}(3,6)$, so $I^\sigma_{3,6}$ is of dimension 13, and the map p_3 is not surjective. However, we claim that the image $p_3(I^\sigma_{3,6})$ spans a hyperplane of dimension 118 in $\mathbf{P}(\bigwedge^3 V_{10})$, which would then allow us to uniquely determine σ up to a constant.

To show that the image spans a hyperplane, it suffices to show that the map

$$\varphi\colon \textstyle\bigwedge^3 V_{10}^\vee \longrightarrow H^0(I^\sigma_{3,6}, \mathcal{O}(1)) \simeq H^0(X_6^\sigma, \bigwedge^3 \mathcal{U}^\vee|_{X_6^\sigma}) \simeq \mathbf{C}^{119} \tag{4.11}$$

is surjective. Here $\mathcal{O}(1)$ is the relative ample bundle for the Grassmannian fibration p_6, so its direct image by p_6 is $\bigwedge^3\mathcal{U}^\vee|_{X_6^\sigma}$. The space of global sections is of dimension 119 by Proposition 4.4.6.

We consider the resolution

$$0 \longrightarrow \mathrm{Sym}^3 \mathcal{Q}^\vee \longrightarrow \mathrm{Sym}^2 \mathcal{Q}^\vee \otimes V_{10}^\vee \longrightarrow \mathcal{Q}^\vee \otimes \textstyle\bigwedge^2 V_{10}^\vee \longrightarrow \bigwedge^3 V_{10}^\vee \otimes \mathcal{O}_G \longrightarrow \bigwedge^3 \mathcal{U}^\vee \longrightarrow 0.$$

The cohomologies of these bundles restricted to X_6^σ have been computed in Proposition 4.4.6. We treat the first three terms as a resolution $\mathcal{F}^\bullet$ of the kernel $\mathcal{K}$ of $\bigwedge^3 V_{10}^\vee \otimes \mathcal{O}_G \to \bigwedge^3\mathcal{U}^\vee$. Using the spectral sequence

$$E_1^{q,p} = h^p(X, \mathcal{F}^q) \Longrightarrow H^{p+q}(X, \mathcal{K})$$

whose first page looks like

		h^0	h^1	h^2	h^3	h^4
$\mathcal{F}^0$	$\mathcal{Q}^\vee \otimes \bigwedge^2 V_{10}^\vee$	0	0	0	0	450
$\mathcal{F}^{-1}$	$\mathrm{Sym}^2 \mathcal{Q}^\vee \otimes V_{10}^\vee$	0	0	0	90	550
$\mathcal{F}^{-2}$	$\mathrm{Sym}^3 \mathcal{Q}^\vee$	0	0	1	0	230

we may compute that $h^0(X, \mathcal{K}) = 1$. So the kernel of the map φ in (4.11) is of dimension 1 and φ is indeed surjective. $\square$

4.4.2. Moduli space. For a general trivector σ, the variety X_6^σ was shown to be a smooth hyperkähler fourfold of $\mathrm{K3}^{[2]}$-deformation type in [**DV10**]. The Plücker polarization coming from the ambient Grassmannian provides an ample class H on X_6^σ of square 22 and divisibility 2. We recall some basic properties for such manifolds.

The second cohomology group $H^2(X_6^\sigma, \mathbf{Z})$ is equipped with the the Beauville–Bogomolov–Fujiki quadratic form q. The lattice $(H^2(X_6^\sigma, \mathbf{Z}), q)$ is isomorphic to the following

$$\Lambda \coloneqq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle,$$

where U is the hyperbolic plane, $E_8(-1)$ is the E_8 lattice with negative definite form, and $\langle -2 \rangle$ is the lattice generated by one element with square -2. The discriminant of Λ is equal to 2. The polarization H on X_6^σ is of square 22 and divisibility 2.

By the general discussion of the moduli spaces and period maps from Chapter 3, we have a moduli space $\mathcal{M}_{22}^{(2)}$ as well as a polarized period map

$$\mathfrak{p}\colon \mathcal{M}_{22}^{(2)} \longrightarrow \mathcal{P}$$

which is an open immersion of algebraic varieties.

For the reader's convenience, we briefly recall the construction: the two invariants $q(H) = 22$ and $\mathrm{div}(H) = 2$ together determine a unique $\mathrm{O}(\Lambda)$-orbit, so we may fix one element $h \in \Lambda$ in this orbit. By the property of the quadratic form q, the primitive cohomology $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}}$ can be identified with $H^\perp$, the sublattice orthogonal to H with respect to q. This is a lattice of signature $(2, 20)$ and discriminant 11. It carries a polarized integral Hodge structure of type $(1, 20, 1)$. The *period domain* that parametrizes such Hodge structures is the normal quasi-projective variety

$$\mathcal{P} \coloneqq \Omega_{h^\perp} / \mathrm{O}(\Lambda, h) \quad \text{where} \quad \Omega_{h^\perp} \coloneqq \{[x] \in \mathbf{P}(\Lambda_{\mathbf{C}}) \mid q(x, x) = q(x, h) = 0, q(x, \bar{x}) > 0\}.$$

In other words, we consider the domain of period points $\Omega_{h^\perp}$, and take its quotient by the group $\mathrm{O}(\Lambda, h) \coloneqq \{\varphi \in \mathrm{O}(\Lambda) \mid \varphi(h) = h\}$.[4] There exists also an irreducible coarse moduli space $\mathcal{M}_{22}^{(2)}$ for polarized hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type with a polarization of degree 22 and divisibility 2, and the polarized period map

$$\mathfrak{p} \colon \mathcal{M}_{22}^{(2)} \hookrightarrow \mathcal{P}, \quad [X] \longmapsto [H^{2,0}(X)]$$

is an open immersion by the polarized global Torelli theorem [**Mar11**, Theorem 1.10].

For each saturated sublattice $K \subset \Lambda$ of rank 2 and signature $(1, 1)$ containing h, its orthogonal complement $K^\perp$ defines a codimension 2 subspace $\mathbf{P}(K^\perp \otimes \mathbf{C}) \subset \mathbf{P}(\Lambda_{\mathbf{C}})$, whose image in $\mathcal{P}$ is an irreducible algebraic hypersurface $\mathcal{D}_K$ called a *Heegner divisor*. The discriminant of $K^\perp$ is a negative even integer, and we refer to $|\,\mathrm{disc}(K^\perp)|$ as the *discriminant* of the Heegner divisor. Following Hassett and Debarre–Macrì, we will label each Heegner divisor using its discriminant. In our case, [**DM19**, Proposition 4.1] shows that each Heegner divisor $\mathcal{D}_{2e}$ with given discriminant $2e$ is irreducible if non-empty (in *loc. cit.*, the divisor $\mathcal{D}_{2e}$ is denoted by $\mathcal{D}_{22,2e}^{(2)}$, referring to the fact that we are working with the moduli space $\mathcal{M}_{22}^{(2)}$ of hyperkähler fourfolds with a divisibility-2 polarization of square 22). Since the lattice Λ is of discriminant 2, *a priori* the discriminant $\mathrm{disc}(K)$ can either be $2\,\mathrm{disc}(K^\perp)$ or $\frac{1}{2}\,\mathrm{disc}(K^\perp)$. But by assuming that K contains the class h, we are always in the first case. This very useful fact is obtained in the proof of [**DM19**, Proposition 4.1]. We extracted it as a lemma.

Lemma 4.4.8 (Debarre–Macrì). *Let n be an positive integer with $n \equiv -1 \pmod 4$, and let $h \in \Lambda = \Lambda_{\mathrm{K3}^{[2]}}$ be a class of square $2n$ and divisibility 2. Let $K \subset \Lambda$ be a saturated sublattice of rank 2 and signature $(1, 1)$ containing h. Then we have $\mathrm{disc}(K) = 2\,\mathrm{disc}(K^\perp)$.*

As mentioned in the introduction, we also have the GIT quotient

$$\mathcal{M} \coloneqq \mathbf{P}(\textstyle\bigwedge^3 V_{10}^\vee) /\!\!/ \mathrm{SL}(V_{10})$$

[4]Note that in general, $\Omega_{h^\perp}$ admits two connected components Ω_h and Ω_{-h}, and to define the period domain $\mathcal{P}$, one need to pick one of the connected components and take its quotient by a smaller group $\mathrm{Mon}(\Lambda, h) \coloneqq \mathrm{Mon}(\Lambda) \cap \mathrm{O}(\Lambda, h)$, which consists of all monodromy operators fixing the class h. But for $\mathrm{K3}^{[2]}$-type, the two descriptions of $\mathcal{P}$ are equivalent. See Section 3.2 for more details.

for the trivectors σ. The class of a trivector σ is denoted by $[\sigma] \in \mathcal{M}$, and we denote by $\mathcal{M}^{\mathrm{smooth}} := \mathcal{M} \smallsetminus \mathcal{D}^{3,3,10}$ the open locus of classes $[\sigma]$ such that X_6^σ is smooth of dimension 4. Therefore we get the following diagram.

$$\begin{array}{ccccc} \mathbf{P}(\bigwedge^3 V_{10}^\vee) & & & & \\ & \overset{\pi}{\searrow} & & & \\ & & \mathcal{M} \overset{\mathfrak{m}}{\dashrightarrow} \mathcal{M}_{22}^{(2)} & \overset{\mathfrak{p}}{\hookrightarrow} & \mathcal{P} \\ & & [\sigma] \longmapsto [X_6^\sigma] & & [X] \longmapsto [H^{2,0}(X)] \end{array} \tag{4.12}$$

Here π is the GIT quotient map, and $\mathfrak{m}$ is the modular map given by the Debarre–Voisin construction. It is dominant by the following crucial lemma from [**DV10**, Lemma 4.6], which shows that the differential of $\mathfrak{m} \circ \pi$ is surjective everywhere.

Lemma 4.4.9 (Debarre–Voisin). *Write (X, L) for the pair $\big(X_6^\sigma, \mathcal{O}_{X_6^\sigma}(1)\big)$. Whenever X is of dimension 4, any first-order deformation of the pair (X, L) is given by a deformation of σ. More precisely, the Kodaira–Spencer map*

$$\mathrm{KS}\colon \textstyle\bigwedge^3 V_{10}^\vee/\langle\sigma\rangle \longrightarrow \mathrm{Def}_{(X,L)}(\mathbf{C}[\varepsilon]) \simeq \mathrm{Ext}^1\left(\mathcal{P}_{X,L}, \mathcal{O}_X\right)$$

is surjective, where the bundle $\mathcal{P}_{X,L}$ is the extension

$$0 \longrightarrow \Omega_X \longrightarrow \mathcal{P}_{X,L} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

given by $c_1(L) \in \mathrm{Ext}^1(\mathcal{O}_X, \Omega_X)$.

Corollary 4.4.10. *The map $\mathfrak{m}$ restricted to the open locus $\mathcal{M}^{\mathrm{smooth}}$ is quasi-finite and dominant.*

Proof. This essentially follows from the surjectiveness of the differential of the map $\mathfrak{m} \circ \pi$ and the fact that both $\mathcal{M}$ and $\mathcal{M}_{22}^{(2)}$ are of dimension 20.

More precisely, suppose that a curve C inside $\mathcal{M}^{\mathrm{smooth}}$ is contracted by the map $\mathfrak{m}$. Since all elements in $\mathcal{M}^{\mathrm{smooth}}$ are stable with respect to the $\mathrm{SL}(V_{10})$-action by Corollary 4.4.4, each orbit is of codimension 20 in $\mathbf{P}(\bigwedge^3 V_{10}^\vee)$. So the preimage of the curve C in $\mathbf{P}(\bigwedge^3 V_{10}^\vee)$ has codimension 19 and is contracted to a point in $\mathcal{M}_{22}^{(2)}$. This would contradict the surjectiveness of the differential of the map $\mathfrak{m} \circ \pi$ given in Lemma 4.4.9. ☐

The modular map $\mathfrak{m}$ is also birational, which was proved by O'Grady in [**O'G19**, Theorem 1.9].

We recollect the results concerning the moduli spaces and period maps in the following theorem.

Theorem 4.4.11. *Consider the diagram* (4.12).

(1) *The modular map $\mathfrak{m}$ is dominant and birational. Moreover, when restricted to the open locus $\mathcal{M}^{\mathrm{smooth}}$, the map $\mathfrak{m}$ is an open immersion.*

(2) The polarized period map $\mathfrak{p}$ is an open immersion. The complement of the image of $\mathfrak{p}$ is an irreducible Heegner divisor $\mathcal{D}_{22}$.

PROOF. For the second point, the period map $\mathfrak{p}$ is an open immersion by the polarized global Torelli theorem of Markman (see [**Mar11**, Theorem 1.10]), and the complement of the image of $\mathfrak{p}$ is always a union of Heegner divisors. In the $\mathrm{K3}^{[2]}$-case, the set of these Heegner divisors has been completely determined by Debarre–Macrì in [**DM19**, Theorem 6.1].

For the first point, we already know that the restriction of $\mathfrak{m}$ to the open locus $\mathcal{M}^{\text{smooth}}$ is quasi-finite and dominant by Corollary 4.4.10. The birationality of $\mathfrak{m}$ has been proved by O'Grady in [**O'G19**, Theorem 1.9]. Since the period domain $\mathcal{P}$ is normal, so is the moduli space $\mathcal{M}_{22}^{(2)}$, and we may conclude using the Zariski Main Theorem. □

To relate the divisors of $\mathcal{M}$ and $\mathcal{P}$, we state the following useful lemma.

LEMMA 4.4.12. *Let $f\colon X \to Y$ be a birational morphism of varieties. Assume that Y is regular in codimension 1. Let D be an irreducible divisor of X that is not contracted by f, that is, $f(D)$ is a divisor of Y. Then D is mapped birationally onto its image. Conversely, let D' be an irreducible divisor of Y such that the preimage $f^{-1}(D')$ is non-empty, then f restricts to a birationl morphism from $f^{-1}(D')$ to D'.*

PROOF. Since f is a birational morphism, it induces an isomorphism between the function fields

$$f^\sharp\colon k(Y) \xrightarrow{\sim} k(X).$$

If $f(D)$ is a divisor of Y, since Y is regular in codimension 1, the local ring at the generic point of $f(D)$ is a discrete valuation ring and therefore integrally closed in $k(Y)$. For dimension reasons, the morphism $f^\sharp\colon \mathcal{O}_{Y,\eta_{f(D)}} \to \mathcal{O}_{X,\eta_D}$ is finite. So this gives an isomorphism of local rings and induces also an isomorphism of function fields $k(f(D)) \xrightarrow{\sim} k(D)$. The converse follows from the same argument. □

In Section 4.3.1, we have seen that the complement in $\mathcal{M}$ of the locus $\mathcal{M}^{\text{smooth}}$ is given by an irreducible divisor $\mathcal{D}^{3,3,10}$, and this divisor is induced by the $\mathrm{SL}(V_{10})$-invariant discriminant hypersurface $\Delta^{3,3,10}$ in $\mathbf{P}\big(\bigwedge^3 V_{10}^\vee\big)$. We can extend the period map $\mathfrak{p}\circ\mathfrak{m}$ so that it is defined in codimension 1. We denote this extension by

$$\widetilde{\mathfrak{p}}\colon \mathcal{M} \dashrightarrow \mathcal{P}.$$

LEMMA 4.4.13. *The extended period map $\widetilde{\mathfrak{p}}$ maps the divisor $\mathcal{D}^{3,3,10}$ birationally onto the Heegner divisor $\mathcal{D}_{22}$.*

Proof. In Proposition 4.3.3, we have seen that for a very general $[\sigma] \in \mathcal{D}^{3,3,10}$, the singular locus of X_6^σ is a very general K3 surface S_{22} of degree 22 and Picard rank 1. In [**DV10**], it was shown that X_6^σ is birational to the Hilbert square $S_{22}^{[2]}$. Hence the extended period map $\widetilde{\mathfrak{p}}$ is defined over an open set of $\mathcal{D}^{3,3,10}$ by mapping $[\sigma]$ to the period of $S_{22}^{[2]}$. In this case, the fourfold $S_{22}^{[2]}$ has Picard rank 2, and the Picard group is generated by the classes H and δ, where H is induced by the polarization on S_{22}, and δ is half the class of the diagonal. Hence the Picard group has intersection matrix

$$\begin{pmatrix} 22 & 0 \\ 0 & -2 \end{pmatrix}$$

which is of discriminant -44. So we may use Lemma 4.4.8 to deduce that the extended period map $\widetilde{\mathfrak{p}}$ maps the divisor $\mathcal{D}^{3,3,10}$ onto the Heegner divisor $\mathcal{D}_{22}$. We conclude that $\widetilde{\mathfrak{p}}$ restricts to a birational map between $\mathcal{D}^{3,3,10}$ and $\mathcal{D}_{22}$ using Lemma 4.4.12. ☐

Combining these facts, we get the following picture of the moduli spaces

$$\begin{array}{ccccccc}
\mathbf{P}(\bigwedge^3 V_{10}^\vee) \smallsetminus \Delta^{3,3,10} & & \mathcal{M} & & & & \mathcal{P} \\
\sqcup & \overset{\pi}{\searrow} & \parallel & & & & \parallel \\
\Delta^{3,3,10} & & \mathcal{M}^{\mathrm{smooth}} & \overset{\mathfrak{m}}{\hookrightarrow} & \mathcal{M}_{22}^{(2)} & \underset{\sim}{\overset{\mathfrak{p}}{\longrightarrow}} & \mathrm{Im}(\mathfrak{p}) \\
 & \overset{\pi}{\dashrightarrow} & \sqcup & & & & \sqcup \\
 & & \mathcal{D}^{3,3,10} & & \underset{\text{bir.}}{\overset{\widetilde{\mathfrak{p}}}{\dashrightarrow}} & & \mathcal{D}_{22}.
\end{array}$$

In particular, each irreducible divisor of $\mathcal{M}$ is mapped birationally onto its image, and conversely, for each irreducible divisor of $\mathcal{P}$ with non-empty preimage, the preimage will be an irreducible divisor of $\mathcal{M}$, and the two divisors are birational via $\widetilde{\mathfrak{p}}$. Divisors of $\mathcal{P}$ with empty preimage are called *Hassett–Looijenga–Shah divisors* (HLS for short) and they are the main focus of the paper [**DHOV20**]. These divisors correspond to $\mathrm{SL}(V_{10})$-orbits of higher codimension in $\mathcal{M}$ that need to be blown up in order to resolve the indeterminacy of $\widetilde{\mathfrak{p}}$. Such loci are necessarily contained in the divisor $\mathcal{D}^{3,3,10}$.

We will study the other two divisors $\mathcal{D}^{1,6,10}$ and $\mathcal{D}^{4,7,7}$ in $\mathcal{M}$ coming from $\mathrm{SL}(V_{10})$-invariant hypersurfaces. Via the extended period map $\widetilde{\mathfrak{p}}$, they are birationally mapped onto some Heegner divisors $\mathcal{D}_{2e}$ in $\mathcal{P}$. We shall determine the discriminants by studying in detail the geometry of the three varieties X_1^σ, X_3^σ, and X_6^σ in these cases. In principle, for each Heegner divisor in $\mathcal{P}$ that is not HLS, we could try to describe it in terms of a divisor in $\mathcal{M}$. In the case of the three divisors that we study, this is done by imposing various degeneracy conditions on σ. Such descriptions also allow us to characterize these Heegner divisors as the loci where the varieties X_k^σ become singular.

Finally, we call the preimage $\mathfrak{p}^{-1}(\mathcal{D}_{2e})$ in $\mathcal{M}_{22}^{(2)}$ of each Heegner divisor $\mathcal{D}_{2e}$ a *Noether–Lefschetz divisor*, and we denote it by $\mathcal{C}_{2e}$. Thanks to Theorem 4.4.11, Noether–Lefschetz divisors and Heegner divisors give almost the same notion: $\mathcal{C}_{2e}$ can be identified with $\mathcal{D}_{2e} \smallsetminus \mathcal{D}_{22}$ via the period map, and in particular $\mathcal{C}_{22}$ is empty. A very general member X of each Noether–Lefschetz divisor has Picard rank 2, and the Picard group can be identified with the sublattice $K \subset \Lambda$ of rank 2 via the identification $H^2(X, \mathbf{Z}) \simeq \Lambda$. The transcendental sublattice $H^2(X, \mathbf{Z})_{\mathrm{trans}}$ is defined as the orthogonal complement of the Picard group inside $H^2(X, \mathbf{Z})$, and can be identified with $K^\perp \subset \Lambda$. By Lemma 4.4.8, the discriminant $2e$ of a Noether–Lefschetz divisor $\mathcal{C}_{2e}$/a Heegner divisor $\mathcal{D}_{2e}$ is equal to $|\frac{1}{2}\operatorname{disc}(\operatorname{Pic}(X))|$ for a very general member X. A Noether–Lefschetz $\mathcal{C}_{2e}$ is called HLS if the corresponding Heegner divisor $\mathcal{D}_{2e}$ is HLS. Such a divisor parametrizes polarized hyperkähler fourfolds in $\mathcal{M}_{22}^{(2)}$ that do not arise from the Debarre–Voisin construction.

4.5. Hodge structures

4.5.1. Hodge structures of X_3^σ. In this section, we suppose that $[\sigma]$ lies in the open locus $\mathcal{M}^{\mathrm{smooth}} = \mathcal{M} \smallsetminus \mathcal{D}^{3,3,10}$, so that X_3^σ and X_6^σ are both smooth of respective expected dimensions 20 and 4. We study the Hodge structures of X_3^σ. We first note that the integral cohomology ring $H^*(X_3^\sigma, \mathbf{Z})$ is torsion-free, due to the following lemma.

Lemma 4.5.1. *Let $Y \subset X$ be a smooth hyperplane section of a smooth projective variety X of dimension n. Suppose that the integral cohomology ring $H^*(X, \mathbf{Z})$ is torsion-free, then so is $H^*(Y, \mathbf{Z})$.*

Proof. This is an application of the Lefschetz hyperplane theorem, the Poincaré duality, and the universal coefficient theorem.

Write T^k and T_k for the torsion part of $H^k(Y, \mathbf{Z})$ and $H_k(Y, \mathbf{Z})$ respectively. Let $i\colon Y \hookrightarrow X$ be the inclusion. By Lefschetz hyperplane theorem, we get isomorphisms $i^*\colon H^k(X, \mathbf{Z}) \to H^k(Y, \mathbf{Z})$ for $0 \le k \le n-2$. So $T^0, \dots, T^{n-2}$ are all zero. Similarly, $i_*\colon H_k(Y, \mathbf{Z}) \to H_k(X, \mathbf{Z})$ are isomorphisms for $0 \le k \le n-2$, so $T_0, \dots, T_{n-2}$ are all zero. By Poincaré duality, this means that $T^n, \dots, T^{2n-2}$ are all zero. Finally, for the middle term T^{n-1}, we have $T^{n-1} \simeq T_{n-2}$ by the universal coefficient theorem, so we may conclude since we have already shown that T_{n-2} is zero. □

We introduce one interesting Hodge structure on X_3^σ. Denote by $j\colon X_3^\sigma \to \operatorname{Gr}(3, V_{10})$ the inclusion. For a given coefficient ring R, the *vanishing cohomology*, studied in the original work [**DV10**], is defined as

$$H^{20}(X_3^\sigma, R)_{\mathrm{van}} \coloneqq \ker\big(j_*\colon H^{20}(X_3^\sigma, R) \longrightarrow H^{22}(\operatorname{Gr}(3, V_{10}), R)\big).$$

When the coefficient ring is $\mathbf{Q}$, the vanishing cohomology can also be characterized as the orthogonal complement of $j^*H^{20}(\operatorname{Gr}(3, V_{10}), \mathbf{Q})$ with respect to the cup-product on $H^{20}(X_3^\sigma, \mathbf{Q})$; indeed, for $\beta \in H^{20}(\operatorname{Gr}(3, V_{10}), \mathbf{Q})$ and $\alpha \in H^{20}(X_3, \mathbf{Q})$, we have that $\alpha \cdot j^*\beta = j_*\alpha \cdot \beta$, and moreover j^* is injective in degree 20 by the Lefschetz hyperplane theorem. Hence there is an orthogonal decomposition

$$H^{20}(X_3^\sigma, \mathbf{Q}) = H^{20}(X_3^\sigma, \mathbf{Q})_{\mathrm{van}} \overset{\perp}{\oplus} j^*H^{20}(\operatorname{Gr}(3, V_{10}), \mathbf{Q}). \tag{4.13}$$

This decomposition does not work for $\mathbf{Z}$-coefficients, as the sum of the two sublattices is not saturated. In fact, $H^{20}(X_3^\sigma, \mathbf{Z})$ is a unimodular lattice and the lattice $j^*H^{20}(\mathrm{Gr}(3, V_{10}), \mathbf{Z})$ is generated by ten Schubert classes

$$j^*\sigma_{730}, j^*\sigma_{721}, j^*\sigma_{640}, j^*\sigma_{631}, j^*\sigma_{622}, j^*\sigma_{550}, j^*\sigma_{541}, j^*\sigma_{532}, j^*\sigma_{442}, j^*\sigma_{433},$$

with intersection product given by $j^*\alpha \cdot j^*\beta = \alpha \cdot \beta \cdot \sigma_{100}$. Thus we can explicitly write out the intersection matrix of $j^*H^{20}(\mathrm{Gr}(3, V_{10}), \mathbf{Z})$ as

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

which has determinant 11.[5] Therefore, $j^*H^{20}(\mathrm{Gr}(3, V_{10}), \mathbf{Z})$ is a saturated sublattice of discriminant 11, and so is its orthogonal $H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}}$. The whole lattice $H^{20}(X_3^\sigma, \mathbf{Z})$ is not even (for example $(j^*\sigma_{730})^2 = 1$), while the vanishing cohomology is, as it is generated by the vanishing cycles, whose self-intersection is always 2 (see [**Voi03**, Chapter 2.3.3, Lemma 2.26 and Remark 3.21]).

The Hodge structure on the vanishing cohomology $H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}}$ has been described in [**DV10**, Theorem 2.2]: it is of K3-type, that is, it has Hodge numbers $h^{9,11} = h^{11,9} = 1$, $h^{10,10} = 20$, and the other Hodge numbers are all zero; moreover, for very general σ (those outside the union of all Noether–Lefschetz divisors), there are no Hodge classes of type $(10, 10)$, so this Hodge structure is simple.

To relate the varieties X_3^σ and X_6^σ, we consider the diagram

$$\begin{array}{ccc} & I_{3,6}^\sigma := \{[V_3 \subset V_6] \in \mathrm{Flag}(3, 6, V_{10}) \mid \sigma|_{V_6} = 0\} & \\ & {\scriptstyle p_3} \swarrow \qquad\qquad \searrow {\scriptstyle p_6} & \\ X_3^\sigma & & X_6^\sigma, \end{array} \tag{4.14}$$

where p_6 is a fibration with fibers isomorphic to $\mathrm{Gr}(3, 6)$.[6] It is clear that the incidence variety $I_{3,6}^\sigma$ is smooth of expected dimension 13 whenever X_6^σ is smooth of dimension 4. Note that the projection p_3 is not surjective since X_3^σ has dimension 20, although we have seen in Corollary 4.4.7 that the image of p_3 does span the hyperplane in $\mathbf{P}(\bigwedge^3 V_{10})$ given by σ, which allows us to uniquely determine X_3^σ.

[5]See also the proof of Proposition 4.7.2, where the `Macaulay2` code for computing the intersection matrix is available.

[6]Usually the two projections are denoted by p and q. The notation of p_3 and p_6 is slightly cumbersome, but it allows us to easily distinguish the two. The same applies for the correspondence $I_{1,6}^\sigma$ below.

This correspondence induces a morphism

$$p_{6*}p_3^*\colon H^{20}(X_3^\sigma, R) \longrightarrow H^2(X_6^\sigma, R).$$

When $R = \mathbf{Q}$, it was proven in [**DV10**] that $p_{6*}p_3^*$ gives an isomorphism between the two $\mathbf{Q}$-Hodge structures $H^{20}(X_3^\sigma, \mathbf{Q})_{\mathrm{van}}$ and $H^2(X_6^\sigma, \mathbf{Q})_{\mathrm{prim}}$. We briefly recall the idea of the proof: the first step is to prove that the morphism is not identically 0. Then, since the Hodge structure on $H^2(X_6^\sigma, \mathbf{Q})_{\mathrm{prim}}$ is simple for σ very general, the map $p_{6*}p_3^*$ is an isomorphism for such σ. Finally, since the topology does not change when we deform σ, the isomorphism holds whenever X_3^σ and X_6^σ are smooth.

We will show that $p_{6*}p_3^*$ also gives a Hodge isometry with $\mathbf{Z}$-coefficients. This is analogous to the result of Beauville–Donagi for cubic fourfolds in [**BD85**]. Let us first state a lemma over $\mathbf{Q}$-coefficients.

Lemma 4.5.2. *The isomorphism of rational Hodge structures $p_{6*}p_3^*$ is a constant multiple of an isometry.*

Proof. We will compare the two quadratic forms on $H^2(X_6^\sigma, \mathbf{Q})_{\mathrm{prim}}$ and conclude using the uniqueness of q_X shown in Lemma 1.2.2.

Consider the intersection form on $H^{20}(X_3^\sigma, \mathbf{Q})_{\mathrm{van}}$. The subspace $H^{11,9}(X_3^\sigma)$ is orthogonal to $H^{11,9}(X_3^\sigma) \oplus H^{10,10}(X_3^\sigma)_{\mathrm{van}}$ for degree reasons. Since $p_{6*}p_3^*$ is a morphism of Hodge structures, the intersection form transports to a second quadratic form q' on $H^2(X_6^\sigma, \mathbf{Q})_{\mathrm{prim}}$ which satisfies the desired orthogonal condition:

$$H^{2,0}(X_6^\sigma) \text{ is orthogonal to } H^{2,0}(X_6^\sigma) \oplus H^{1,1}(X_6^\sigma)_{\mathrm{prim}} \text{ with respect to } q'.$$

Now by varying σ we get a locally complete family of polarized hyperkähler manifolds, so we may conclude that q' is a constant multiple of q using Lemma 1.2.2. ☐

Therefore we know that $p_{6*}p_3^*$ is a constant multiple of an isometry. If we can show that this constant is -1, then since the discriminants of the two lattices are the same, this isometry will also be onto, which proves the following theorem.

Theorem 4.5.3. *When σ is such that X_3^σ and X_6^σ are both smooth (that is, when $[\sigma] \notin \mathcal{D}^{3,3,10}$), the morphism $p_{6*}p_3^*$ gives an isomorphism of polarized integral Hodge structures*

$$p_{6*}p_3^*\colon H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}} \xrightarrow{\sim} H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}}(-1), \tag{4.15}$$

where the (-1) means that $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}}$ is endowed with the quadratic form $-q$.

To show that the constant is -1, we will use the argument of continuity: the constant is the same over the moduli space, so it suffices to compute its value over the Heegner divisor $\mathcal{D}_{28}$, where we have some explicit Hodge classes to work with. We postpone the proof of Theorem 4.5.3 to Section 4.6, where we study in detail the divisor $\mathcal{D}_{28}$. We will also prove the integral Hodge conjecture on $H^{20}(X_3^\sigma, \mathbf{Z})$ as a corollary (see Corollary 4.6.10).

4.5.2. Hodge structures of X_1^σ. In this section, we will suppose that the trivector σ does not lie in $\mathcal{D}^{3,3,10} \cup \mathcal{D}^{1,6,10}$, so all three varieties X_1^σ, X_3^σ, and X_6^σ are smooth of expected dimension. The Peskine variety $X_1^\sigma \subset \mathbf{P}(V_{10})$ has many interesting geometric aspects. First, it is a Fano sixfold of degree 15 and index 3. One way to compute these invariants is by using a resolution of the structure sheaf, for example see [**Han15**, Section 2.3] or [**BFM21**, Section 4.3]. An alternative way is to identify X_1^σ as the zero-locus $Z(\sigma) \subset \mathrm{Flag}(1, 4, V_{10})$ when σ is general (see Section 4.3.2), then use Schubert calculus on the flag variety.

```
needsPackage "Schubert2";
F = flagBundle{1,3,6}; (U1,U41,Q) = F.Bundles;
-- construct X1 as the zero-locus of a general section
X1 = sectionZeroLocus dual (U1*(exteriorPower_2 U41+U41*Q));
h = chern_1 (dual U1*OO_X1);
print integral h^6; -- degree of X1 is equal to 15
assert (chern_1 tangentBundle X1 == 3*h); -- index is equal to 3
```

Denote by h the natural polarization on X_1^σ. We have the following result from [**Han15**].

Proposition 4.5.4 (Han). *Let σ be a general trivector. For $[V_6]$ general in the Debarre–Voisin variety X_6^σ, the intersection of $\mathbf{P}(V_6)$ and X_1^σ inside $\mathbf{P}(V_{10})$ is a* Palatini threefold*, that is, a smooth threefold in $\mathbf{P}^5$ of degree 7, which is a scroll over a smooth cubic surface. Otherwise stated, there is a 7-dimensional incidence variety $I_{1,6}^\sigma$ called the* universal Palatini variety *and a diagram*

$$I_{1,6}^\sigma := \{[V_1 \subset V_6] \in \mathrm{Flag}(1, 6, V_{10}) \mid [V_1] \in X_1^\sigma, [V_6] \in X_6^\sigma\} \tag{4.16}$$
$$X_1^\sigma \xleftarrow{p_1} I_{1,6}^\sigma \xrightarrow{p_6} X_6^\sigma,$$

where the general fiber of p_6 is a Palatini threefold.

Remark 4.5.5. The general fiber of p_1 is a smooth curve of genus 13 and degree 33 (with respect to the Plücker embedding). This can be checked by either explicitly computing the fiber for some given trivector σ and $[V_1] \in X_1^\sigma$, or by realizing the fiber as a certain degeneracy locus inside the Grassmannian $\mathrm{Gr}(5, V_{10}/V_1)$ and using a Bertini-type argument. We omit the details since we will not need this fact.

We would like to use this correspondence to relate the Hodge structures of X_1^σ and X_6^σ. However, unlike the case of $I_{3,6}^\sigma$, there are some subtleties. The Proposition 4.5.4 only holds for σ general, in which case the fiber of p_6 is a Palatini threefold only for general $[V_6] \in X_6$. *A priori* the incidence variety $I_{1,6}^\sigma$ might not be smooth, or not have the expected dimension at all. It turns out that it is not smooth but has the expected dimension generically, and more precisely whenever X_1^σ and X_6^σ are smooth. Therefore we can still use the correspondence $p_{6*}p_1^*$ in families. More precisely, we view $I_{1,6}^\sigma$ as a subvariety of $X_1^\sigma \times X_6^\sigma$, so when it has the expected dimension, it gives a well-defined cohomology class $[I_{1,6}^\sigma] \in H^6(X_1^\sigma \times X_6^\sigma)$. Then by abuse of notation, we will write $p_{6*}p_1^*$ for $\mathrm{pr}_{2*}\big([I_{1,6}^\sigma] \cdot \mathrm{pr}_1^*(-)\big)$ and similarly $p_{1*}p_6^*$ for $\mathrm{pr}_{1*}\big([I_{1,6}^\sigma] \cdot \mathrm{pr}_2^*(-)\big)$.

First we show that $I^\sigma_{1,6}$ is not smooth for a general σ. We introduce a resolution of $I^\sigma_{1,6}$ that can be realized as the zero-locus of some section on a flag variety, whose smoothness can then be deduced for general σ. Consider the diagram

$$\begin{array}{ccccc} & & \widetilde{I}^\sigma_{1,6} := \left\{ \begin{array}{c} [V_1 \subset V_3 \subset V_6] \\ \in \mathrm{Flag}(1,3,6,V_{10}) \end{array} \,\middle|\, \begin{array}{c} \sigma(V_1,V_3,V_{10}) = 0, \\ \sigma|_{V_6} = 0 \end{array} \right\} & & \\ & \swarrow & \downarrow s & \searrow & \\ & & I^\sigma_{1,6} & & \\ X^\sigma_1 & \xleftarrow{p_1} & & \xrightarrow{p_6} & X^\sigma_6. \end{array} \tag{4.17}$$

In other words, apart from the pair $[V_1 \subset V_6]$, we introduce the extra information of a subspace V_3 contained in both V_6 and the kernel of the form $\sigma(V_1, -, -)$ (which immediately ensures that $\operatorname{rank} \sigma(V_1, -, -) \leq 6$).

Proposition 4.5.6. *Let σ be a general trivector with both X^σ_1 and X^σ_6 smooth. Then the variety $\widetilde{I}^\sigma_{1,6}$ defined in* (4.17) *is smooth of expected dimension* 7*. The projection $s\colon \widetilde{I}^\sigma_{1,6} \to I^\sigma_{1,6}$ obtained by forgetting the subspace V_3 contracts a* 5*-dimensional subvariety of $\widetilde{I}^\sigma_{1,6}$ to the following* 3*-dimensional subvariety*

$$Z := \bigl\{[V_1 \subset V_6] \in I^\sigma_{1,6} \bigm| \exists K_4 \subset V_6 \quad \sigma(V_1, K_4, V_{10}) = 0\bigr\}.$$

On the complement of $s^{-1}(Z)$, the projection s is an isomorphism. In particular, we have $\mathrm{Sing}(I^\sigma_{1,6}) = Z$*.*

Proof. The variety $\widetilde{I}^\sigma_{1,6}$ is defined inside $\mathrm{Flag}(1,3,6,V_{10})$ as the zero-locus of σ viewed as a section of the vector bundle $(\bigwedge^3 \mathcal{U}_6 + \mathcal{U}_1 \wedge \mathcal{U}_3 \wedge V_{10})^\vee$. This vector bundle is a quotient of $\bigwedge^3 V_{10}^\vee$ and is therefore globally generated. So $\widetilde{I}^\sigma_{1,6}$ is smooth of expected dimension 7 for a general σ.

The locus where the projection $\widetilde{I}^\sigma_{1,6} \to I^\sigma_{1,6}$ is not an isomorphism is precisely above those $[V_1 \subset V_6]$ where the kernel K_4 of $\sigma(V_1, -, -)$ is contained in V_6, in which case the fiber is a projective plane $\mathbf{P}^2 = \mathbf{P}\bigl((K_4/V_1)^\vee\bigr)$ parametrizing V_3 with $V_1 \subset V_3 \subset K_4$. We may look at the locus of such $[V_1 \subset K_4 \subset V_6]$ inside the flag variety $\mathrm{Flag}(1,4,6,V_{10})$: this is again the zero-locus of σ viewed as a section of a certain homogeneous bundle, so for general σ we get the 3-dimensional smooth subvariety

$$Z' := \{[V_1 \subset K_4 \subset V_6] \mid \sigma(V_1, K_4, V_{10}) = 0, [V_6] \in X^\sigma_6\} \subset \mathrm{Flag}(1,4,6,V_{10}).$$

Since the kernel K_4 is uniquely determined by V_1, the subvariety Z' projects injectively onto its image in $I^\sigma_{1,6}$, which is given by the subvariety Z. The preimage $s^{-1}(Z)$ in $\widetilde{I}^\sigma_{1,6}$ therefore has dimension 5. This means that the projection $\widetilde{I}^\sigma_{1,6} \to I^\sigma_{1,6}$ is a small contraction for general σ, so $I^\sigma_{1,6}$ cannot be smooth. □

Remark 4.5.7. For general σ, the 3-dimensional subvariety Z dominates a divisor in X^σ_6. This divisor has class $10H$, which can be shown by computing the degree of the pullback of the polarization H. Note that this is a canonically defined effective divisor in X_6^σ, which could be useful in constructing compactifications of the moduli space.

Next, we will show that $I_{1,6}^\sigma$ always has the expected dimension 7. We need some results on 4-dimensional families of skew-symmetric forms, studied by Fania–Mezzetti in [**FM02**].

Consider a six-dimensional complex vector space V_6. Inside $\mathbf{P}(\bigwedge^2 V_6^\vee)$, there are two $\mathrm{SL}(V_6)$-invariant orbits given by the discriminant hypersurface $\mathrm{Gr}(2,V_6)^*$—which is the Pfaffian cubic—and the Grassmannian $\mathrm{Gr}(2,V_6^\vee)\simeq \mathrm{Gr}(4,V_6)$, representing skew-symmetric forms of ranks ≤ 4 and ≤ 2 respectively. For a general 4-dimensional family $\Delta\subset\bigwedge^2 V_6^\vee$ (which gives a 3-dimensional linear system, or a *web*), the degeneracy locus defines a Palatini scroll in $\mathbf{P}(V_6)$, so one obtains a rational map

$$\rho\colon\ \mathrm{Gr}(4,\textstyle\bigwedge^2 V_6^\vee) \dashrightarrow \mathcal{H},$$

where $\mathcal{H}$ is the irreducible component of the Hilbert scheme containing the Palatini scroll. We have the following theorem (see [**FM02**, Theorem 1.1, 4.3 and 4.9], and also the erratum).

Theorem 4.5.8 (Fania–Mezzetti). *The map ρ is birational. Moreover, it is not regular at a point $[\Delta]\in\mathrm{Gr}(4,\bigwedge^2 V_6^\vee)$ if and only if Δ belongs to one of the following cases, where we view Δ as a 3-dimensional projective subspace in $\mathbf{P}(\bigwedge^2 V_6^\vee)$.*

- *Δ is entirely contained in the discriminant hypersurface $\mathrm{Gr}(2,V_6)^*$ and not contained in the tangent space to $\mathrm{Gr}(2,V_6^\vee)$ at a point;*
- *Δ is not contained in the discriminant hypersurface $\mathrm{Gr}(2,V_6)^*$, but its intersection with the Grassmannian $\mathrm{Gr}(2,V_6^\vee)$ contains a line or a conic.*

We prove the following lemma.

Lemma 4.5.9. *Let $[\sigma]\in\mathcal{M}$. If the fiber of p_6 above $[V_6]$ is not of dimension 3, there is a flag $V_4\subset V_6\subset V_7$ such that $\sigma(V_4,V_7,V_7)=0$. This gives a plane $\mathbf{P}\big((V_7/V_4)^\vee\big)$ contained in X_6^σ, necessarily Lagrangian.*

Proof. For each $[V_6]\in X_6^\sigma$, since σ vanishes on V_6, we get an induced linear map

$$\varphi_\sigma\colon V_{10}/V_6 \longrightarrow \textstyle\bigwedge^2 V_6^\vee.$$

This map is injective: if some V_7/V_6 is mapped to 0, the trivector σ would vanish on V_7, which implies that σ is unstable by Lemma 4.2.3 so not possible.

Therefore σ defines a 4-dimensional subspace $\Delta\subset\bigwedge^2 V_6^\vee$, and the fiber of p_6 above $[V_6]$ is precisely the union of degeneracy loci in $\mathbf{P}(V_6)$ for the family of skew-symmetric forms parametrized by Δ. If the fiber $p_6^{-1}([V_6])$ is not of dimension 3, the rational map ρ is not defined at $[\Delta]$, so Δ must satisfy one of the two conditions in Theorem 4.5.8.

Moreover, by a result of Manivel–Mezzetti [**MM05**, Corollary 11], there does not exist a $\mathbf{P}^3$-family of skew-symmetric forms on V_6 with constant rank 4. In other words, if the projective 3-space Δ is entirely contained in the discriminant hypersurface $\mathrm{Gr}(2, V_6)^*$, it will necessarily intersect the Grassmannian $\mathrm{Gr}(2, V_6^\vee)$ where the rank drops to 2. So we may conclude that in both cases, the projective 3-space Δ intersects the Grassmannian $\mathrm{Gr}(2, V_6^\vee)$, which means that there is a $V_7 \supset V_6$ whose image is decomposable: we have $\varphi_\sigma(V_7/V_6) = f_1 \wedge f_2$ where $f_1, f_2 \in V_6^\vee$ are linear forms. The common kernel $V_4 \subset V_6$ of f_1 and f_2 therefore satisfies the desired property $\sigma(V_4, V_7, V_7) = 0$. □

PROPOSITION 4.5.10. *Let $[\sigma] \in \mathcal{M}$ be a trivector such that X_1^σ and X_6^σ are both smooth (that is, $[\sigma] \notin \mathcal{D}^{3,3,10} \cup \mathcal{D}^{1,6,10}$). The variety $I_{1,6}^\sigma$ defined in* (4.16) *has only one irreducible component of expected dimension* 7*, and this component is reduced.*

PROOF. For a trivector $\sigma \notin \mathcal{D}^{3,3,10} \cup \mathcal{D}^{1,6,10}$, since X_6^σ is hyperkähler of dimension 4, it contains only finitely many planes of the form $\mathbf{P}\big((V_7/V_4)^\vee\big)$, because any such plane is necessarily Lagrangian hence rigid. We saw that for any $[V_6]$ away from these planes, the fiber of p_6 is of expected dimension 3 and is generically smooth, so the irreducible component of $I_{1,6}^\sigma$ that dominates X_6^σ is reduced of expected dimension 7.

On the other hand, for each Lagrangian plane $P = \mathbf{P}\big((V_7/V_4)^\vee\big)$, the preimage $p_6^{-1}(P)$ is a subvariety inside the projective bundle $\mathbf{P}(\mathcal{U}_6)$ over P, which is of dimension 7. We claim that $p_6^{-1}(P)$ must be of dimension ≤ 6: otherwise, for any $[V_6] \in \mathbf{P}\big((V_7/V_4)^\vee\big)$ and any $V_1 \subset V_6$, we have $[V_1] \in X_1^\sigma$. Then $\mathbf{P}(V_7) = \mathbf{P}^6$ would be entirely contained in X_1^σ, which is impossible because X_1^σ is assumed to be smooth of dimension 6. □

Therefore whenever X_1^σ and X_6^σ are both smooth, the variety $I_{1,6}^\sigma$ has one unique reduced component of expected dimension 7. It defines a class on the product $X_1^\sigma \times X_6^\sigma$ with correct codimension, and we can thus consider the morphisms between Hodge structures given by this correspondence.

Note that the degeneracy condition $\sigma(V_4, V_7, V_7) = 0$ from Lemma 4.5.9 gives precisely the divisor $\mathcal{D}^{4,7,7}$ from Section 4.2.3 and is related to Lagrangian planes contained in X_6^σ, which we will study in Section 4.6.

We now begin the study of the Hodge structures of X_1^σ. The Hodge numbers of X_1^σ were computed in [**BFM21**, Table 4.1] (there it is denoted by P). The integral cohomology $H^*(X_1^\sigma, \mathbf{Z})$ is also shown to be torsion-free. Since all the cohomologies in odd degree vanish, we list only the even degree ones:

(4.18)

h^0				1			
h^2			0	1	0		
h^4		0	1	22	1	0	
h^6	0	0	1	22	1	0	0
h^8		0	1	22	1	0	
h^{10}			0	1	0		
h^{12}				1			

We see that there are three Hodge structures of K3-type on different levels. They are related by the Lefschetz operator (see Lemma 4.6.18) and there is a polarization given by the cup product on $H^6(X_1^\sigma, \mathbf{Z})$. In [**BFM21**], the authors showed that the Hodge structure of $H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}}$ can be mapped into each of the three Hodge structures of X_1^σ by using certain geometric constructions called *jumps* between the Grassmannians $\mathbf{P}(V_{10})$, $\mathrm{Gr}(2, V_{10})$, and $\mathrm{Gr}(3, V_{10})$. Here we show that this can also be done by using the incidence variety $I_{1,6}^\sigma$.

As in the case of X_3^σ, we first determine the suitable Hodge structure to study: we define the vanishing cohomology $H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}$ to be the orthogonal of the sublattice generated by h^3 and the class π of a Palatini threefold in X_1^σ (see Proposition 4.5.4). To check that these two classes generate a sublattice of rank 2, one can compute their intersection matrix as follows:

- The self-intersection number $h^3 \cdot h^3$ is the degree of X_1^σ, which is 15;
- The intersection number $h^3 \cdot \pi$ is the degree of the Palatini threefold, which is 7;
- To compute the self-intersection number $\pi \cdot \pi$, we take two general points $[V_6]$ and $[V_6']$ from X_6^σ. Their intersection $V_6 \cap V_6'$ is a 2-dimensional subspace V_2, and the sum $V_6 + V_6'$ is the whole V_{10}. So one obtains $\sigma(V_2, V_2, V_{10}) = 0$. Such a V_2 defines a 4-secant line $\mathbf{P}(V_2)$ of the variety X_1^σ (see [**Han14**, Section 3.1]). As the class π of the Palatini threefolds can be represented by both $\mathbf{P}(V_6) \cap X_1$ and $\mathbf{P}(V_6') \cap X_1$, its self-intersection number is 4.

The intersection matrix for $\mathbf{Z}h^3 + \mathbf{Z}\pi$ is therefore

$$\begin{pmatrix} 15 & 7 \\ 7 & 4 \end{pmatrix}.$$

This is a saturated sublattice of rank 2 and discriminant 11. Its orthogonal complement, the vanishing cohomology $H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}$—a polarized integral Hodge structure of type $(1, 20, 1)$—therefore also has discriminant 11.

For cohomologies in degree $k = 4, 8$, we first use the Lefschetz operator L_h over $\mathbf{Q}$ to identify $H^k(X_1^\sigma, \mathbf{Q})_{\mathrm{van}}$, then define the corresponding intersection with the integral cohomology to be $H^k(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}$. *A priori*, the Lefschetz operators might not remain isomorphisms over integral coefficients. We will clarify this in Lemma 4.6.18.

The following is the analogue of Theorem 4.5.3.

THEOREM 4.5.11. *When σ is such that X_1^σ and X_6^σ are both smooth (that is, when $[\sigma] \notin \mathcal{D}^{3,3,10} \cup \mathcal{D}^{1,6,10}$), the morphism*

$$p_{6*}p_1^* L_h \colon H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}} \xrightarrow{\sim} H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}}(-1) \tag{4.19}$$

is an isomorphism of polarized integral Hodge structures. Here $p_{6*}p_1^*$ is the correspondence defined by $I_{1,6}^\sigma$ in the diagram (4.16), L_h is the Lefschetz operator given by cup product with h, and the (-1) means that $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}}$ is endowed with the quadratic form $-q$.

The proof is essentially the same as the proof of Theorem 4.5.3 and involves the study of $\mathcal{D}_{28}$, so again we postpone it to Section 4.6.

REMARK 4.5.12. One can also derive Theorem 4.5.11 directly from Theorem 4.5.3 and [**BFM21**, Theorem 19], where the Hodge isometries between X_1^σ and X_3^σ has already been established.

4.6. The Heegner divisor of degree 28

4.6.1. The discriminant. In Section 4.2.3, we defined the divisor $\mathcal{D}^{4,7,7}$ in $\mathcal{M}$ given by the degeneracy condition

$$\exists [V_4 \subset V_7] \quad \sigma(V_4, V_7, V_7) = 0.$$

Moreover, we have seen in Section 4.3.1 that if $[\sigma] \notin \mathcal{D}^{1,3,10}$ then σ defines a smooth fourfold X_6^σ. So we will consider the case when $[\sigma]$ lies in $\mathcal{D}^{4,7,7} \smallsetminus \mathcal{D}^{3,3,10}$, where the corresponding X_6^σ is a smooth hyperkähler fourfold. For the given flag $[V_4 \subset V_7]$, we see that every V_6 in $\mathrm{Gr}(2, V_7/V_4) = \mathbf{P}\big((V_7/V_4)^\vee\big)$ is in X_6^σ. So the hyperkähler fourfold X_6^σ contains a plane P, necessarily Lagrangian.

We will first determine the discriminant of the corresponding Noether–Lefschetz/Heegner divisor. In fact, we will show that a Debarre–Voisin fourfold X_6^σ containing a linearly embedded Lagrangian plane (with respect to the Plücker polarization) is always in the family $\mathcal{C}_{28}$. This shows in particular that the divisor $\mathcal{D}^{4,7,7}$ is mapped onto $\mathcal{C}_{28}$ via the modular map $\mathfrak{m}$, and that *any* Lagrangian plane contained in a Debarre–Voisin fourfold is of the above form.

Recall from Example 2.2.6 that we have the following general description of a Lagrangian plane P contained in a hyperkähler fourfold X of $\mathrm{K3}^{[2]}$-type, in terms of the dual class L of a line ℓ in P (first obtained by Hassett–Tschinkel in [**HT09b**, Section 5]). As already explained there, we work with $\lambda \coloneqq 2L \in H^2(X, \mathbf{Z})$ instead of L.

PROPOSITION 4.6.1. *Let X be a smooth hyperkähler fourfolds of* $\mathrm{K3}^{[2]}$*-type and let P be a Lagrangian plane contained in X. Let $\ell \in H^6(X, \mathbf{Z})$ be the class of a line contained in the plane P and let $\lambda \in H^2(X, \mathbf{Z})$ be the unique class satisfying the property*

$$\forall \alpha \in H^2(X, \mathbf{Z}) \quad q(\lambda, \alpha) = 2\ell \cdot \alpha, \tag{4.20}$$

where q is the Beauville–Bogomolov–Fujiki form. The class λ is of square $q(\lambda, \lambda) = -10$ and divisibility 2*, and we have the following relation*

$$[P] = \tfrac{1}{8}\lambda^2 + \tfrac{1}{20}\mathfrak{q},$$

where $\mathfrak{q} \in H^4(X, \mathbf{Q})$ is the dual of q.

By Example 1.1.7, we have the generalized Fujiki constants $C(\mathfrak{q}) = 25$ and $C(\mathfrak{q}^2) = \mathfrak{q}^2 = 575$, which means that $\mathfrak{q} \cdot \alpha_1 \cdot \alpha_2 = 25 \cdot q(\alpha_1, \alpha_2)$ for all $\alpha_1, \alpha_2 \in H^2(X, \mathbf{Z})$. By Corollary 2.1.11, we know that $\mathfrak{q} = \frac{5}{6} c_2(X)$.

We prove an extra lemma.

Lemma 4.6.2. *Let X be a smooth hyperkähler fourfolds of* $\mathrm{K3}^{[2]}$*-type and* $\lambda \in H^2(X,\mathbf{Z})$ *be a class of square* -10 *and divisibility* 2*. Moreover, let H be a polarization on X. Then there is at most one linearly embedded plane P (with respect to H) whose associated* (-10)*-class is equal to λ.*

Proof. For two distinct linearly embedded planes P and P', their intersection can be empty, a point, or a line L. We show that the last case is not possible in general: we have an exact sequence

$$0 \longrightarrow \mathcal{T}_P \longrightarrow \mathcal{T}_X|_P \longrightarrow \mathcal{N}_{P/X} \simeq \Omega_P \longrightarrow 0,$$

which, when restricted to L, gives

$$0 \longrightarrow \mathcal{T}_P|_L \simeq \mathcal{O}_L(1) \oplus \mathcal{O}_L(2) \longrightarrow \mathcal{T}_X|_L \longrightarrow \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-2) \longrightarrow 0.$$

If $P \cap P' = L$, the other normal bundle $\mathcal{N}_{P'/L} \simeq \mathcal{O}_L(1)$ should be a subbundle of the quotient, which is not possible. In conclusion, the intersection number $[P] \cdot [P]'$ is either 0 or 1.

If we now assume that the (-10)-classes associated with P and P' are both λ, then we may compute the intersection number using

$$[P] \cdot [P'] = (\tfrac{1}{8}\lambda^2 + \tfrac{1}{20}\mathfrak{q})^2 = [P]^2 = 3,$$

which leads to a contradiction. □

Theorem 4.6.3.

(1) *A smooth Debarre–Voisin fourfold X_6^σ containing a Lagrangian plane P is always in the family $\mathcal{C}_{28}$.*
(2) *Consequently, for $[\sigma]$ very general in the divisor $\mathcal{D}^{4,7,7}$, the corresponding transcendental sublattice $H^2(X_6^\sigma,\mathbf{Z})_{\mathrm{trans}}$ is of discriminant -28. The divisor $\mathcal{D}^{4,7,7}$ is mapped birationally onto the Noether–Lefschetz divisor $\mathcal{C}_{28}$ by the modular map $\mathfrak{m}$, and then onto the Heegner divisor $\mathcal{D}_{28} \subset \mathcal{P}$ by the period map $\mathfrak{p}$.*
(3) *A very general X_6^σ in the family $\mathcal{C}_{28}$ contains exactly one Lagrangian plane.*
(4) *Finally, any Lagrangian plane P contained in a smooth Debarre–Voisin fourfold is of the form $\mathbf{P}\big((V_7/V_4)^\vee\big)$, for a flag $[V_4 \subset V_7]$ satisfying the degeneracy condition $\sigma(V_4,V_7,V_7) = 0$.*

Proof. For statement (1), let σ be such that X_6^σ is smooth of dimension 4 and contains a Lagrangian plane P. Let H be the polarization on X_6^σ induced by the Plücker polarization, which is primitive with square $q(H) = 22$ and divisibility 2. Let ℓ be the class of a line contained in the plane P. Consider the class $\lambda \in H^2(X_6^\sigma,\mathbf{Z})$ given by Proposition 4.6.1 with $q(\lambda) = -10$. Since $H \cdot \ell = 1$, we have $q(H,\lambda) = 2$. Therefore the intersection matrix between H and λ is

$$\begin{pmatrix} 22 & 2 \\ 2 & -10 \end{pmatrix},$$

with determinant $-224 = -7 \cdot 2^5$.

We study the saturation of the sublattice $\mathbf{Z}H + \mathbf{Z}\lambda$. Since the discriminant of the lattice $H^2(X_6^\sigma, \mathbf{Z})$ is 2, and since both H and λ are primitive of divisibility 2, the images of $\frac{1}{2}H$ and $\frac{1}{2}\lambda$ in the discriminant group are equal. The class $\frac{1}{2}(H+\lambda)$ is therefore integral. We may consider the sublattice generated by $\frac{1}{2}(H+\lambda)$ and λ, which has intersection matrix

$$\begin{pmatrix} 4 & -4 \\ -4 & -10 \end{pmatrix}. \tag{4.21}$$

The discriminant of this lattice is equal to $-56 = -7 \cdot 2^3$. Suppose that it is not saturated, then we can find a class $a \cdot \frac{1}{2}(H+\lambda) + b\lambda = 2x$ with $\gcd(a,b) = 1$, where x is still integral. We may compute that $q(x) = a^2 - 2ab - \frac{5}{2}b^2$ which is an integer, so b is even, a is odd, and the square $q(x)$ is an odd number. This contradicts the fact that $H^2(X_6^\sigma, \mathbf{Z})$ is an even lattice, so we may conclude that $\mathbf{Z}\frac{1}{2}(H+\lambda) + \mathbf{Z}\lambda$ is the saturation of $\mathbf{Z}H + \mathbf{Z}\lambda$ and has discriminant -56. Since its discriminant is always twice that of its orthogonal by Lemma 4.4.8, we get a member of the family $\mathcal{C}_{28}$. Note that if such X_6^σ is of Picard rank 2, then its Picard lattice coincides with $\mathbf{Z}\frac{1}{2}(H+\lambda) + \mathbf{Z}\lambda$.

We argue that $\mathcal{D}^{4,7,7}$ is distinct from $\mathcal{D}^{3,3,10}$ by comparing the degree of the corresponding $\mathrm{SL}(V_{10})$-invariant hypersurface $\Delta^{4,7,7}$ and $\Delta^{3,3,10}$ in $\mathbf{P}(\bigwedge^3 V_{10}^\vee)$. Therefore, a very general $[\sigma]$ in $\mathcal{D}^{4,7,7} \smallsetminus \mathcal{D}^{3,3,10}$ defines a smooth X_6^σ. Since there exists a distinguished flag $[V_4 \subset V_7]$ such that $\sigma(V_4, V_7, V_7) = 0$, the corresponding Debarre–Voisin variety X_6^σ contains a Lagrangian plane $\mathbf{P}\big((V_7/V_4)^\vee\big)$ and therefore is a member of the family $\mathcal{C}_{28}$ by (1). Thus the divisor $\mathcal{D}^{4,7,7}$ is mapped to $\mathcal{C}_{28}$ in the moduli space. By Theorem 4.4.11, we conclude that $\mathfrak{m}$ restricts to a birational map from $\mathcal{D}^{4,7,7}$ to $\mathcal{C}_{28}$. This shows the statement (2).

The divisor $\mathcal{D}^{4,7,7}$ being mapped birationally onto $\mathcal{C}_{28}$ shows that a very general member X_6^σ of the family $\mathcal{C}_{28}$ indeed contains a plane. Moreover, for such X_6^σ, the Picard group is of rank 2 and has intersection matrix as in (4.21), so there is only one class λ satisfying $q(H,\lambda) = 2$ and $q(\lambda,\lambda) = -10$.[7] By Lemma 4.6.2, this shows that a very general X_6^σ in $\mathcal{C}_{28}$ contains exactly one Lagrangian plane, which must be of the form $\mathbf{P}\big((V_7/V_4)^\vee\big)$.

Finally, for each Lagrangian plane P contained in a smooth Debarre–Voisin fourfold, we may consider a generic deformation which preserves the Lagrangian plane, using the results of Voisin **[Voi92]** on deformations of Lagrangian subvarieties. In this case, a very general members of the deformation has Picard rank 2, and the Lagrangian plane it contains is indeed of the form $\mathbf{P}\big((V_7/V_4)^\vee\big)$ for a certain flag $[V_4 \subset V_7]$. As this is a deformation of the pair (X_6^σ, P), the original plane P in the central fiber is necessarily also of this form: this follows from the properness of the flag variety $\mathrm{Flag}(4,7,V_{10})$ and the fact that $\sigma(V_4, V_7, V_7) = 0$ is a closed condition. This concludes the proof. □

[7] Let $\lambda' = aH + b\lambda$ be a class satisfying $2 = \mathfrak{q}(H,\lambda') = 22a + 2b$ and $-10 = \mathfrak{q}(\lambda',\lambda') = 22a^2 + 4ab - 10b^2$, then we may solve that $(a,b) = (\frac{2}{11}, -1)$ or $(0,1)$. For λ' to be integral, only the second case is possible, so λ' is indeed equal to λ.

REMARK 4.6.4. Here are a few remarks regarding this result.

- The uniqueness of the Lagrangian plane for a very general member of the family $\mathcal{C}_{28}$ shows that a very general $[\sigma] \in \mathcal{D}^{4,7,7}$ admits a unique flag $[V_4 \subset V_7]$ satisfying the vanishing condition $\sigma(V_4, V_7, V_7) = 0$. This shows that the $\mathrm{SL}(V_{10})$-invariant hypersurface $\Delta^{4,7,7}$ in $\mathbf{P}(\bigwedge^3 V_{10}^\vee)$ is indeed of degree 5500 (see Section 4.2.3).
- Moreover, since any Lagrangian plane contained in a smooth X_6^σ is of the given form, all such Lagrangian planes are in the same *polarized* monodromy orbit. In other words, any two pairs (X_6^σ, P) and $(X_6^{\sigma'}, P')$ are deformation equivalent, that is, there is a family $\pi\colon \mathcal{X} \to B$ of Debarre–Voisin fourfolds and a subvariety Π flat over B such that $\Pi_t \subset \mathcal{X}_t$ is a Lagrangian plane, and the two pairs are isomorphic to some fibers $(\mathcal{X}_b, \Pi_b)$ and $(\mathcal{X}_{b'}, \Pi_{b'})$ for $b, b' \in B$. This also implies that the classes λ and λ' are in the same polarized monodromy orbit, as well as the classes $[P]$ and $[P']$.
- The last statement of Theorem 4.6.3 has an alternative proof: a plane contained in $\mathrm{Gr}(6, V_{10})$ is either of the form $\mathbf{P}\big((V_7/V_4)^\vee\big)$ for a flag $V_4 \subset V_7$, or $\mathbf{P}(V_8/V_5)$ for a flag $V_5 \subset V_8$ (this is a general fact for Grassmannians, see for example [**Har92**, Exercise 6.9]). So it suffices to show that a Debarre–Voisin variety X_6^σ containing a plane of the second type is not smooth. In this case, the trivector σ satisfies the degeneracy condition $\sigma(V_5, V_5, V_8) = 0$. To show that X_6^σ is singular, we look for a V_3 contained in V_5 satisfying the extra degeneracy condition $\sigma(V_3, V_3, V_{10}/V_8) = 0$. Equivalently, we take the Grassmannian $\mathrm{Gr}(3, V_5)$ and consider the section of the rank-6 bundle $\big(\bigwedge^2 \mathcal{U}_3^\vee\big)^{\oplus 2}$ induced by σ. Since $\mathrm{Gr}(3, V_5)$ is of dimension 6 and the vector bundle has top Chern class 1, any section has a non-empty zero-locus, so such V_3 indeed exists.
- Concerning the possibility of associated K3 surfaces, the lattice in (4.21) does not represent 28,[8] so there is no associated K3 surface of degree 28.

4.6.2. The correspondence $I_{3,6}^\sigma$. We proceed to the proof of the Hodge isometries in Theorem 4.5.3 and Theorem 4.5.11. In order to prove Theorem 4.5.3, we will use the correspondence $X_3^\sigma \xleftarrow{p_3} I_{3,6}^\sigma \xrightarrow{p_6} X_6^\sigma$ from (4.14). The key point is to show that $p_{6*}p_3^*$ sends the intersection product to $-q$, as explained by Lemma 4.5.2 and the remarks thereafter. For this, it is enough to prove

$$\exists x \in H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}} \smallsetminus \{0\} \quad x^2 = -q(p_{6*}p_3^* x),$$

where we write $q = q_{X_6^\sigma}$ for the Beauville–Bogomolov–Fujiki form. By a continuity argument, we may specialize to the case of a general $[\sigma]$ in the divisor $\mathcal{D}^{4,7,7}$, for which X_3^σ and X_6^σ remain smooth.

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[8] Suppose that there exist $a, b \in \mathbf{Z}$ satisfying $4a^2 - 8ab - 10b^2 = 28$, by modulo 7 we may verify that $a \equiv b \pmod 7$. So we can write $a = 7a' + r$ and $b = 7b' + r$. By modulo 49 we get $r^2 \equiv 5 \pmod 7$, which is impossible.

Let us begin with some preliminary results. For $[\sigma] \in \mathcal{D}^{4,7,7}$ with $\sigma(V_4, V_7, V_7) = 0$, denote by ℓ the class of a line contained in the plane $P = \mathbf{P}\big((V_7/V_4)\big)^\vee$. Such a line can be expressed as

$$\{[V_6] \in X_6^\sigma \mid V_5 \subset V_6 \subset V_7\},$$

where V_5 is a subspace such that $V_4 \subset V_5 \subset V_7$. The class $z \coloneqq p_{3*}p_6{}^*\ell \in H^{20}(X_3^\sigma, \mathbf{Z})$ is represented by the subvariety

$$Z \coloneqq \{[V_3] \in X_3^\sigma \mid V_3 \subset V_7,\ \dim(V_3 \cap V_5) \geq 2\}. \tag{4.22}$$

We may decompose the class z as the sum of its vanishing part $z_0 \in H^{20}(X_3^\sigma, \mathbf{Q})_{\mathrm{van}}$ and its Schubert part $z_1 \in j^*H^{20}(\mathrm{Gr}(3, V_{10}), \mathbf{Q})$ according to the decomposition (4.13).

LEMMA 4.6.5. *In the notation above, the Schubert part z_1 of the class $z \in H^{20}(X_3^\sigma, \mathbf{Z})$ has square $z_1^2 = \frac{5}{11}$.*

PROOF. The class j_*z is the Schubert class σ_{443} on $\mathrm{Gr}(3, V_{10})$ represented by Z. We can compute $z \cdot j^*\sigma_{433} = 1$ while $z \cdot j^*\sigma_{abc} = 0$ for the rest of the Schubert classes. The intersection numbers allow us to completely determine z_1 in terms of the basis $j^*\sigma_{abc}$: we get

$$\begin{aligned} z_1 = \tfrac{1}{11}(&j^*\sigma_{730} - 3j^*\sigma_{721} - j^*\sigma_{640} + 2j^*\sigma_{631} + 3j^*\sigma_{622} \\ &+ j^*\sigma_{550} - j^*\sigma_{541} - 5j^*\sigma_{532} + 6j^*\sigma_{442} + 5j^*\sigma_{433}). \end{aligned} \tag{4.23}$$

We may then compute its self-intersection number and find $\frac{5}{11}$. □

To compute z_0^2, we will specialize the trivector further so that X_6^σ, while still smooth, contains two disjoint planes P and P'. Denote by λ and λ' their corresponding (-10)-classes as defined in Proposition 4.6.1. We have the following result.

LEMMA 4.6.6. *If X_6^σ is smooth and contains two disjoint planes P, P', then either $q(\lambda, \lambda') = 2$ or $q(\lambda, \lambda') = -2$.*

PROOF. As the two planes P, P' are disjoint, we use Proposition 4.6.1 to obtain

$$\begin{aligned} 0 = [P] \cdot [P'] &= \left(\tfrac{1}{8}\lambda^2 + \tfrac{1}{20}\mathfrak{q}\right) \cdot \left(\tfrac{1}{8}\lambda'^2 + \tfrac{1}{20}\mathfrak{q}\right), \\ &= \tfrac{1}{64}\lambda^2 \cdot \lambda'^2 + \tfrac{1}{160}\mathfrak{q} \cdot \lambda^2 + \tfrac{1}{160}\mathfrak{q} \cdot \lambda'^2 + \tfrac{1}{400}\mathfrak{q}^2 \\ &= \tfrac{1}{64}\big(2q(\lambda, \lambda')^2 + q(\lambda)q(\lambda')\big) + \tfrac{1}{160}\mathfrak{q} \cdot \lambda^2 + \tfrac{1}{160}\mathfrak{q} \cdot \lambda'^2 + \tfrac{1}{400}\mathfrak{q}^2. \end{aligned}$$

Using $\mathfrak{q}^2 = 575$, $q(\lambda) = q(\lambda') = -10$, and $\mathfrak{q} \cdot \lambda^2 = \mathfrak{q} \cdot \lambda'^2 = 25 \cdot (-10) = -250$, we find $q(\lambda, \lambda')^2 = 4$, therefore $q(\lambda, \lambda') = \pm 2$. □

Following [**DV10**, Section 2], we will consider the following two situations for two disjoint planes P, P' contained in X_6^σ:

Case 1. We have $V_4 \subset V_7$ and $V_4' \subset V_7'$ with $\dim(V_7 \cap V_7') = 4$ and $V_4 \cap V_4' = \{0\}$. For a suitable choice of basis $(e_0, \ldots, e_9)$, we may set $V_7 = \langle e_0, \ldots, e_6 \rangle$, $V_7' = \langle e_3, \ldots, e_9 \rangle$, $V_4 = \langle e_1, e_2, e_3, e_4 \rangle$, and $V_4' = \langle e_5, e_6, e_7, e_8 \rangle$. Note that $\dim(V_4 \cap V_7') = \dim(V_4' \cap V_7) = 2$.

Case 2. We have $V_4 \subset V_7$ and $V_4' \subset V_7'$ with $\dim(V_7 \cap V_7') = 4$ but $V_4 \cap V_4'$ one-dimensional. In this case, we may set $V_7 = \langle e_0, \dots, e_6 \rangle$, $V_7' = \langle e_3, \dots, e_9 \rangle$, $V_4 = \langle e_0, e_1, e_2, e_3 \rangle$, and $V_4' = \langle e_3, e_7, e_8, e_9 \rangle$.

In both cases, the planes $P \coloneqq \mathbf{P}\big((V_7/V_4)^\vee\big)$ and $P' \coloneqq \mathbf{P}\big((V_7/V_4')^\vee\big)$ are disjoint.

REMARK 4.6.7. Note that the existence of such σ was not proved in [**DV10**], although it can be verified using a computer: we choose random trivectors σ that satisfy the vanishing conditions as above, and check the smoothness of the hyperplane section X_3^σ. For example, the following trivectors with coefficients in $\{0, \pm 1\}$ suffice in the two cases:

$$\begin{gathered}[056]+[037]-[237]+[047]+[157]+[257]+[267]-[018]-[128]-[148]\\ -[058]+[258]+[168]-[078]-[129]+[249]+[349]+[059]+[269]-[289]\end{gathered}$$

and

$$\begin{gathered}[456]+[017]+[027]+[147]-[057]+[067]+[167]-[267]+[018]\\ +[138]+[238]-[148]-[258]+[039]+[149]+[169]+[189],\end{gathered}$$

where $[ijk]$ stands for the form $e_i^\vee \wedge e_j^\vee \wedge e_k^\vee$.

We are now ready to prove Theorem 4.5.3.

PROOF OF THEOREM 4.5.3. For $[\sigma]$ very general in the divisor $\mathcal{D}^{4,7,7}$, the Debarre–Voisin fourfold X_6^σ has Picard rank 2. Therefore the space $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}} \cap H^{1,1}(X_6^\sigma)$ of primitive algebraic classes has rank 1. Using the intersection matrix (4.21), we see that it is generated by the integral class $\frac{1}{2}(H - 11\lambda)$. Recall that we consider the class $z \coloneqq p_{3*}p_6^*\ell \in H^{20}(X_3^\sigma, \mathbf{Z})$ which has a vanishing part z_0 and a Schubert part z_1. As proved in [**DV10**], the map

$$p_{6*}p_3^* \colon H^{20}(X_3^\sigma, \mathbf{Q})_{\mathrm{van}} \longrightarrow H^2(X_6^\sigma, \mathbf{Q})_{\mathrm{prim}}$$

is an isomorphism of rational Hodge structures. Since z_0 is a Hodge class of type $(10, 10)$, there is some non-zero rational number $c \in \mathbf{Q}$ such that $p_{6*}p_3^* z_0 = c(H - 11\lambda)$. We can then express the self-intersection number z_0^2 in terms of c: namely, we transport it to the side of X_6^σ using the correspondence

$$z_0^2 = z \cdot z_0 = p_{3*}p_6^*\ell \cdot z_0 = \ell \cdot c(H - 11\lambda) = c \cdot \tfrac{1}{2} q(\lambda, H - 11\lambda) = 56c.$$

We now specialize the trivector σ so that X_6^σ contains two disjoint planes P and P'. We have two (-10)-classes λ, λ' in $H^2(X_6^\sigma, \mathbf{Z})$, and two classes z, z' in $H^{20}(X_3^\sigma, \mathbf{Z})$ represented by the subvarieties Z and Z' defined in (4.22). Since both Z and Z' are Schubert varieties of type Σ_{443}, the two classes z and z' share the same Schubert part $z_1 = z_1'$, which can be determined explicitly as in (4.23) of Lemma 4.6.5 and has square $\frac{5}{11}$.

Let us suppose that we are in **Case 1** above. Recall that Z is defined as the set of $[V_3]$ with $\dim(V_3 \cap V_5) \geq 2$ for a given V_5 sitting between $[V_4 \subset V_7]$, and Z' is similarly defined by a V_5' sitting between $[V_4' \subset V_7']$. We may pick V_5 and V_5' so that $V_5 \cap V_5' = 0$, then no V_3

can satisfy $\dim(V_3 \cap V_5) \geq 2$ and $\dim(V_3 \cap V_5') \geq 2$. So the two subvarieties Z and Z' are disjoint, and we have $0 = z \cdot z' = z_0 \cdot z_0' + z_1^2$. Therefore we obtain

$$z_0 \cdot z_0' = -z_1^2 = -\tfrac{5}{11}.$$

On the other hand, we can also compute this intersection number using the same method that we used to compute z_0^2:

$$z_0 \cdot z_0' = z \cdot z_0' = p_{3*}p_6^*\ell \cdot z_0' = \ell \cdot c(H - 11\lambda') = c\big(1 - \tfrac{11}{2}q(\lambda, \lambda')\big).$$

By Lemma 4.6.6, $q(\lambda, \lambda')$ has two possible values ± 2. We use it to find the value of c: if $q(\lambda, \lambda') = 2$, we get $c = \frac{1}{22}$, while if $q(\lambda, \lambda') = -2$, we get $c = -\frac{5}{132}$. So we can compute that

$$z^2 = z_0^2 + \tfrac{5}{11} = 56c + \tfrac{5}{11} = 3 \quad \text{or} \quad -\tfrac{5}{3},$$

in the two cases respectively. Since z is the class of the subvariety Z, the intersection number z^2 should be an integer, so we may conclude that $q(\lambda, \lambda') = 2$, $c = \frac{1}{22}$, and $z^2 = 3$. Finally, we get

$$z_0^2 = \tfrac{28}{11}, \quad q(p_{6*}p_3^* z_0) = q\left(\tfrac{1}{22}(H - 11\lambda)\right) = -\tfrac{28}{11},$$

which proves what we need. □

REMARK 4.6.8. By Lemma 4.6.6, we know that $q(\lambda, \lambda') = \pm 2$. The proof of the theorem showed that we must have $q(\lambda, \lambda') = 2$ when we specialize to **Case 1**. We could also have specialized to **Case 2** to prove the theorem, in which case one obtains $q(\lambda, \lambda') = -2$ instead.

In the proof, we showed that $z^2 = 3$. Since we also know that $j_* z = \sigma_{443}$, this allows us to write out the full intersection matrix of the sublattice $\mathbf{Z}z + j^* H^{20}(\mathrm{Gr}(3, V_{10}), \mathbf{Z})$, whose discriminant group can then be computed to be $\mathbf{Z}/28\mathbf{Z}$.[9] Since the middle cohomology $H^{20}(X_3^\sigma, \mathbf{Z})$ is a unimodular lattice, the orthogonal complement $H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}}^{\perp z}$ has the same discriminant group. This last lattice is mapped via $p_{6*}p_3^*$ onto the transcendental sublattice $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{trans}}$, so we may again conclude that $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{trans}}$ is of discriminant 28.

Another consequence of the theorem is the integral Hodge conjecture for $H^{20}(X_3^\sigma, \mathbf{Z})$, following ideas of Mongardi–Ottem [**MO20**] for cubic fourfolds.

We first state a basic lemma on abelian groups.

LEMMA 4.6.9. *Let L and M be two abelian groups and let $\varphi\colon L \to M$ be a homomorphism. Let L_1 be a saturated subgroup of L (that is, L/L_1 is torsion-free). Moreover, we suppose that $\varphi|_{L_1}\colon L_1 \to M$ is injective with saturated image (that is,* $\operatorname{coker}\varphi|_{L_1}$ *is torsion-free). Then writing $\varphi^\vee$ for the dual homomorphism $\varphi^\vee\colon M^\vee \to L^\vee$, we have*

$$\varphi^\vee(M^\vee) + L_1^\perp = L^\vee,$$

where $L_1^\perp \coloneqq \{f \in L^\vee \mid f|_{L_1} = 0\}$ is the orthogonal of L_1 in $L^\vee$.

[9]See the proof of Proposition 4.7.2 where we perform the analogous computation in the case of $\mathcal{D}_{24}$. The `Macaulay2` code there can be modified accordingly to compute the case of $\mathcal{D}_{28}$.

Proof. Since L_1 is saturated in L, we have a surjective map $p\colon L^\vee \twoheadrightarrow L_1^\vee$ where the kernel is precisely $L_1^\perp$. So we get an induced isomorphism

$$p\colon L^\vee/L_1^\perp \xrightarrow{\sim} L_1^\vee.$$

On the other hand, since $\varphi|_{L_1}\colon L_1 \to M$ is injective with saturated image, we get similarly a surjective map $\varphi^\vee\colon M^\vee \twoheadrightarrow L_1^\vee$, which factorizes as the composition of $\varphi^\vee\colon M^\vee \to L^\vee$ and the natural map $p\colon L^\vee \twoheadrightarrow L_1^\vee$. Comparing with the isomorphism that we obtained above, we get a surjective map

$$\varphi^\vee\colon M^\vee \longrightarrow\!\!\!\!\!\rightarrow L^\vee/L_1^\perp.$$

We may thus conclude that $\varphi^\vee(M^\vee)$ and $L_1^\perp$ generate $L^\vee$. □

Corollary 4.6.10. *The integral Hodge conjecture holds for* $H^{20}(X_3^\sigma, \mathbf{Z})$.

Proof. The maps in diagram (4.14) define an injective morphism

$$p_{6*}p_3^*\colon H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}} \xrightarrow{\sim} H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}} \hookrightarrow H^2(X_6^\sigma, \mathbf{Z})$$

of abelian groups. By definition of the primitive cohomology, this has saturated image, therefore we may apply Lemma 4.6.9 by letting $L = H^{20}(X_3^\sigma, \mathbf{Z})$ and $L_1 = H^{20}(X_3^\sigma, \mathbf{Z})_{\mathrm{van}}$ and obtain that

$$(p_{6*}p_3^*)^\vee(H^2(X_6^\sigma, \mathbf{Z})^\vee) + L_1^\perp = L^\vee.$$

The group L equipped with the intersection product is a unimodular lattice, so we can identify $L^\vee$ with L itself. By the construction of the vanishing cohomology, we have $L_1^\perp = j^*H^{20}(\mathrm{Gr}(3, V_{10}), \mathbf{Z})$. Using the Poincaré duality, we may identify $H^2(X_6^\sigma, \mathbf{Z})^\vee$ with $H^6(X_6^\sigma, \mathbf{Z})$ and $(p_{6*}p_3^*)^\vee$ with $p_{3*}p_6^*$. So we obtain

$$p_{3*}p_6^*H^6(X_6^\sigma, \mathbf{Z}) + j^*H^{20}(\mathrm{Gr}(3, V_{10}), \mathbf{Z}) = H^{20}(X_3^\sigma, \mathbf{Z}).$$

Since the integral Hodge conjecture holds for $H^6(X_6^\sigma, \mathbf{Z})$ by [**MO20**, Theorem 0.1], and since the map $p_{3*}p_6{}^*$ is given by an integral correspondence, every integral $(10, 10)$-class on X_3^σ is therefore algebraic. □

Theorem 4.6.11. *When σ is such that X_3^σ is smooth (that is, when $[\sigma] \notin \mathcal{D}^{3,3,10}$), the integral Hodge conjecture holds for X_3^σ in all degrees.*

Proof. Since X_3^σ is a hyperplane section of the Grassmannian $\mathrm{Gr}(3, V_{10})$, the cohomology classes in degrees 2 to 18 all come from Schubert classes thanks to the Lefschetz hyperplane theorem.

The degree 20 case is settled in Corollary 4.6.10.

For degree 22 to degree 38, by using Poincaré duality, we can verify that the Schubert classes also produce all the cohomology classes, except in degree 22, where they only generate a subgroup of $H^{22}(X_3^\sigma, \mathbf{Z})$ of index 3.

More precisely, in each degree $2k$ for $11 \leq k \leq 19$, by the Lefschetz hyperplane theorem, we already know that the pullback of Schubert classes $\{j^*\sigma_{abc}\}_{a+b+c=20-k}$ generate $H^{40-2k}(X_3^\sigma, \mathbf{Z}) \simeq H^{2k}(X_3^\sigma, \mathbf{Z})^\vee$. We then consider the pullback of Schubert classes $\{j^*\sigma_{def}\}_{d+e+f=k}$ and pair them with $\{j^*\sigma_{abc}\}_{a+b+c=20-k}$. Using

$$j^*\alpha \cdot j^*\beta = [X_3^\sigma] \cdot \alpha \cdot \beta = \sigma_{100} \cdot \alpha \cdot \beta,$$

we can explicitly compute the intersection matrix and then reduce it to Smith normal form. This gives us the desired description of the quotient group $H^{2k}(X_3^\sigma, \mathbf{Z})/j^*H^{2k}(\mathrm{Gr}(3, V_{10}), \mathbf{Z})$.

To conclude for degree 22, we note that there is an extra algebraic class g represented by the Grassmannian $\mathrm{Gr}(3, U_6)$ for any $[U_6] \in X_6^\sigma$. It is easy to see that the class g only intersects the Schubert class $j^*\sigma_{333} \in H^{18}(X_3^\sigma, \mathbf{Z})$ with intersection number 1. This allows us to extend the intersection matrix computed above and verify that $j^*H^{22}(\mathrm{Gr}(3, V_{10}), \mathbf{Z}) + \mathbf{Z}g$ generates $H^{18}(X_3^\sigma, \mathbf{Z})^\vee \simeq H^{22}(X_3^\sigma, \mathbf{Z})$. ☐

REMARK 4.6.12. The extra algebraic class g can be seen as $p_{3*}p_6^*[*]$, where $[*] \in H^8(X_6^\sigma, \mathbf{Z})$ is the class of a point in X_6^σ. We see that Schubert classes only produce the class $3g$. This phenomenon reappears below for the variety X_1^σ: if $\pi = p_{1*}p_6^*[*]$ is the class of a Palatini threefold, Schubert classes—in particular the Lefschetz operator—only produce the class 3π.

4.6.3. The correspondence $I_{1,6}^\sigma$. We will now prove the second Hodge isometry stated in Theorem 4.5.11. We will use the correspondence $X_1^\sigma \xleftarrow{p_1} I_{1,6}^\sigma \xrightarrow{p_6} X_6^\sigma$ from (4.16). Recall that $h \in H^2(X_1^\sigma, \mathbf{Z})$ is the polarization on X_1^σ, the class $\pi \in H^6(X_1^\sigma, \mathbf{Z})$ is the class of a Palatini threefold, and $H^6(X_1^\sigma, R)_{\mathrm{van}}$ is defined as $\langle h^3, \pi \rangle^\perp$ for $R = \mathbf{Q}, \mathbf{Z}$. Note that the Lefschetz operators L_h from $H^4(X_1^\sigma, \mathbf{Q})$ to $H^6(X_1^\sigma, \mathbf{Q})$ and from $H^6(X_1^\sigma, \mathbf{Q})$ to $H^8(X_1^\sigma, \mathbf{Q})$ both define an isomorphism of $\mathbf{Q}$-vector spaces.

PROOF OF THEOREM 4.5.11. We will first show that

$$p_{6*}p_1^*L_h \colon H^6(X_1^\sigma, \mathbf{Q})_{\mathrm{van}} \longrightarrow H^2(X_6^\sigma, \mathbf{Q})_{\mathrm{prim}}$$

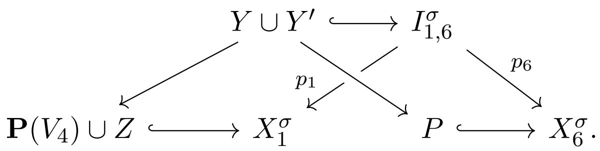
is an isomorphism of $\mathbf{Q}$-vector spaces. In fact, it suffices to show that the map is non-zero. This is because that for a very general σ, the Hodge structure $H^2(X_6^\sigma, \mathbf{Q})_{\mathrm{prim}}$ is simple, so the map must be an isomorphism by comparing the dimensions. But this is a topological property, so it will also suffice to show that the map is non-zero for some particular σ, not necessarily very general. Once we know that $p_{6*}p_1^*L_h$ is an isomorphism of $\mathbf{Q}$-vector spaces, we can use Lemma 1.2.2 to deduce that it is an isometry up to a scalar, and we will conclude by showing that the scalar is equal to -1.

As in the proof of Theorem 4.5.3, we consider a general $[\sigma]$ in the divisor $\mathcal{D}^{4,7,7}$, so that X_6^σ contains a unique plane $P = \mathbf{P}\big((V_7/V_4)^\vee\big)$. Denote by ℓ the class of a line contained in P, and consider the class $z \coloneqq p_{1*}p_6^*\ell \in H^4(X_1^\sigma, \mathbf{Z})$. We would like to study the intersection

$$\mathbf{P}(V_7) \cap X_1^\sigma \subset X_1^\sigma$$

and find a subvariety that represents the class z. It is easy to see that $\mathbf{P}(V_4)$ is always contained in X_1^σ, therefore this intersection is not irreducible, so we will use Z to denote the other component of $\mathbf{P}(V_7) \cap X_1^\sigma$ and show that when it is of expected dimension 4, it represents the class z.

We first describe the geometry of the incidence variety $I_{1,6}^\sigma$. Fibers of the map p_6 above $[V_6] \in P$ are degenerate Palatini threefolds having $\mathbf{P}(V_4)$ as one irreducible component. The preimage $p_6^{-1}(P)$ therefore consists of two components Y and Y': the map p_1 projects the first component Y onto $\mathbf{P}(V_4) \subset X_1^\sigma$, and the second component Y' onto Z. The fibers of $p_1\colon Y \to \mathbf{P}(V_4)$ are just copies of the plane P, while the fibers of $p_1\colon Y' \to Z$ away from $\mathbf{P}(V_4)$ can be described as follows: each $[V_1]$ not lying in $\mathbf{P}(V_4)$ spans a 5-dimensional subspace $V_1 \oplus V_4$, and since $\sigma(V_4, V_7, V_7) = 0$, each $[V_6]$ in the line $\mathbf{P}(V_7/(V_1 \oplus V_4))$ lies in X_6^σ. Therefore the generic fibers of $p_1\colon Y' \to Z$ are lines contained in P. For a fixed line $\ell \subset P$, the generic fibers of $p_1 : p_6^{-1}(\ell) \cap Y' \to Z$ are therefore intersections of two lines in P, so this is a birational map, and we may conclude that the class $p_{1*}p_6^*\ell \in H^4(X_1^\sigma, \mathbf{Z})$ is indeed represented by the class of Z with multiplicity 1. The geometry can be summarized in the following diagram



By intersecting z with h we get a class $z \cdot h \in H^6(X_1^\sigma, \mathbf{Z})$ which we can write as a sum $z \cdot h = x_0 + x_1$, where $x_0 \in H^6(X_1^\sigma, \mathbf{Q})_{\mathrm{van}}$ and $x_1 \in \mathbf{Q}h^3 + \mathbf{Q}\pi$. For σ very general in the divisor $\mathcal{D}^{4,7,7}$, X_6^σ has Picard rank 2 and $\frac{1}{2}(H - 11\lambda)$ generates the space of primitive algebraic classes $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}} \cap H^{1,1}(X_6^\sigma)$, so there is a rational number $c \in \mathbf{Q}$ such that $p_{6*}p_1^*(x_0 \cdot h) = c(H - 11\lambda)$. We may compute

$$x_0^2 = z \cdot h \cdot x_0 = p_{1*}p_6^*\ell \cdot h \cdot x_0 = \ell \cdot p_{6*}p_1^*(h \cdot x_0) = \ell \cdot c(H - 11\lambda) = 56c.$$

If c is non-zero, then the map $p_{6*}p_1^*L_h$ will also be non-zero.

To deduce the value of c, we consider again the two special cases where X_6^σ contains two planes P and P', and get two subvarieties Z and Z' and their classes z and z'. Recall from Remark 4.6.4 that the two planes P and P' are in the same polarized monodromy orbit, hence so are the classes ℓ and ℓ' for the two lines. The correspondence $p_{1*}p_6^*$ is clearly monodromy equivariant, therefore the two classes z and z' are also in the same monodromy orbit. In particular, this means that the components x_1 and x_1' are the same, since both classes $h^3, \pi \in H^6(X_1^\sigma, \mathbf{Z})$ are monodromy invariant. We can then conclude that $z \cdot z' \cdot h^2$ is equal to $x_0 \cdot x_0' + x_1^2$.

On the other hand, we may similarly compute

$$x_0 \cdot x_0' = z \cdot h \cdot x_0' = p_{1*}{p_6}^*\ell \cdot h \cdot x_0' = \ell \cdot p_{6*}{p_1}^*(h \cdot x_0') = \ell \cdot c(H - 11\lambda') = c - c \cdot \tfrac{11}{2}q(\lambda, \lambda').$$

By Remark 4.6.8, we know that $q(\lambda, \lambda')$ equals to 2 in **Case 1** and -2 in **Case 2**. So $x_0 \cdot x_0'$ is equal to $-10c$ and $12c$ in the two cases respectively.

Now we compute the intersection number $z \cdot z' \cdot h^2$ in an alternative way, using the geometry. It suffices to determine the intersection $\mathbf{P}(V_7) \cap \mathbf{P}(V_7') \cap X_1^\sigma$: we will show that in **Case 1**, it is a quadric surface and some lower-dimensional components, and in **Case 2**, it is a cubic surface and some lower-dimensional components.

More precisely, in the basis $(e_0, \dots, e_9)$ of **Case 1** described above, the intersection is defined inside $\mathbf{P}(V_7 \cap V_7') = \mathbf{P}(\langle e_3, e_4, e_5, e_6 \rangle)$ as the locus where the 10×10 skew-symmetric matrix

$$\left(\begin{array}{ccccc|ccccc}
 & & & & & a_{056}x_6 & -a_{056}x_5 & f_{07} & f_{08} & f_{09} \\
 & & & & & & & f_{17} & f_{18} & f_{19} \\
 & & & & & & & f_{27} & f_{28} & f_{29} \\
 & & & & & & & & & -a_{349}x_4 \\
 & & & & & & & & & a_{349}x_3 \\
\hline
-a_{056}x_6 & & & & & & & & & \\
a_{056}x_5 & & & & & & & & & \\
-f_{07} & -f_{17} & -f_{27} & & & & & & & \\
-f_{08} & -f_{18} & -f_{28} & & & & & & & \\
-f_{09} & -f_{19} & -f_{29} & a_{349}x_4 & -a_{349}x_3 & & & & &
\end{array}\right)$$

has rank ≤ 6. Here we only write down the non-zero entries: each $a_{ijk} := \sigma(e_i, e_j, e_k)$ is a constant, and each f_{ij} is the restriction of the linear form $\sigma(e_i, e_j, -)$ to $\langle e_3, e_4, e_5, e_6 \rangle$, a polynomial in x_3, x_4, x_5, x_6 of degree 1. Note that the constants a_{056} and a_{349} are both non-zero: otherwise σ would vanish entirely on $V_7 = \langle e_0, \dots, e_6 \rangle$ or $V_7' = \langle e_3, \dots, e_9 \rangle$, which is not possible by Lemma 4.2.3. Similarly, the quadratic form $f_{17}f_{28} - f_{27}f_{18}$ cannot be zero: otherwise one easily verifies that there exists a suitable 7-dimensional subspace of $\langle e_1, \dots, e_8 \rangle$ on which σ vanishes entirely. So the locus where the rank drops is the union of the quadric surface defined by $f_{17}f_{28} - f_{27}f_{18}$ and the two lines $x_3 = x_4 = 0$ and $x_5 = x_6 = 0$.

In **Case 2**, the matrix is instead the following

$$\left(\begin{array}{ccc|c|ccc|ccc}
 & & & & & & & f_{07} & f_{08} & f_{09} \\
 & & & & & & & f_{17} & f_{18} & f_{19} \\
 & & & & & & & f_{27} & f_{28} & f_{29} \\
\hline
 & & & 0 & & & & & & \\
\hline
 & & & & & ax_6 & -ax_5 & & & \\
 & & & & -ax_6 & & ax_4 & & & \\
 & & & & ax_5 & -ax_4 & & & & \\
\hline
-f_{07} & -f_{17} & -f_{27} & & & & & & & \\
-f_{08} & -f_{18} & -f_{28} & & & & & & & \\
-f_{09} & -f_{19} & -f_{29} & & & & & & &
\end{array}\right).$$

Again $a = a_{456} = \sigma(e_4, e_5, e_6)$ is a constant and each f_{ij} is the restriction of the linear form $\sigma(e_i, e_j, -)$ to $\langle e_3, e_4, e_5, e_6 \rangle$. Similarly, the constant a and the determinant $\det(f_{ij})$

are both non-zero, because otherwise σ would vanish entirely on a 7-dimensional subspace. Therefore the locus where the rank is ≤ 6 is the union of the cubic surface defined by $\det(f_{ij}) = 0$ and the point $x_4 = x_5 = x_6 = 0$.

Consequently, we see that $z \cdot z' \cdot h^2$ equals to 2 or 3 in the two cases respectively. Therefore we obtain the equations

$$-10c + x_1^2 = 2, \quad 12c + x_1^2 = 3,$$

from which we deduce that $c = \frac{1}{22}$ and $x_1^2 = \frac{27}{11}$. Since c is non-zero, the map $p_{6*}p_1^*L_h$ is indeed an isomorphism of $\mathbf{Q}$-vector spaces.

Once we know the isomorphism over $\mathbf{Q}$, we automatically get a Hodge isometry up to a scalar following the same argument as in Lemma 4.5.2: essentially we use the uniqueness of the Beauville–Bogomolov–Fujiki form q showed in Lemma 1.2.2. So it remains to show that the scalar is -1. We consider the class x_0 which satisfies

$$x_0^2 = 56c = \tfrac{28}{11} \quad \text{while} \quad q\left(p_{6*}p_1^*(x_0 \cdot h)\right) = q\left(\tfrac{1}{22}(H - 11\lambda)\right) = -\tfrac{28}{11}.$$

This allows us to conclude the proof. □

In the proof, we have seen that if we consider the class $z \coloneqq p_{1*}p_6^*\ell \in H^4(X_1^\sigma, \mathbf{Z})$, the class $z \cdot h$ admits components $x_0 \in H^6(X_1^\sigma, \mathbf{Q})_{\mathrm{van}}$ and $x_1 \in \mathbf{Q}h^3 + \mathbf{Q}\pi$, with $x_0^2 = \frac{28}{11}$ and $x_1^2 = \frac{27}{11}$. Hence $(z \cdot h)^2 = x_0^2 + x_1^2 = 5$, which is an integer as one would expect. This also allows us to determine the class x_1.

Lemma 4.6.13. *In the above notation, we have* $x_1 = \frac{3}{11}(h^3 + \pi)$.

Proof. The direct sum $H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}} \oplus (\mathbf{Z}h^3 + \mathbf{Z}\pi)$ is of index 11 in $H^6(X_1^\sigma, \mathbf{Z})$. Since the class $z \cdot h$ is integral, we see that $11x_1 \in \mathbf{Z}h^3 + \mathbf{Z}\pi$, so we may write $11x_1 = ah^3 + b\pi$. The value $x_1^2 = \frac{27}{11}$ leads to the equation

$$15a^2 + 14ab + 4b^2 = 297,$$

from which one may deduce that

$$b = -\tfrac{7}{4}a \pm \tfrac{1}{4}\sqrt{11(108 - a^2)}.$$

The only integer solutions are $(a, b) = (3, 3)$ or $(-3, -3)$, so $x_1 = \pm\frac{3}{11}(h^3 + \pi)$. Finally, since the class z is given by the subvariety Z and therefore is effective, we may conclude that $x_1 = \frac{3}{11}(h^3 + \pi)$. □

We can then compute the intersection matrix for h^3, π, and $z \cdot h$, and find

$$\begin{pmatrix} 15 & 7 & 6 \\ 7 & 4 & 3 \\ 6 & 3 & 5 \end{pmatrix}. \tag{4.24}$$

In particular, the discriminant group of the lattice generated by these three classes is $\mathbf{Z}/28\mathbf{Z}$. So this is also the discriminant group of the orthogonal $H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}^{\perp z \cdot h}$ and that of the transcendental lattice $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{trans}}$, again confirming that we are in the Heegner divisor $\mathcal{D}_{28}$.

Corollary 4.6.14. *When σ is such that X_1^σ is smooth (that is, when $[\sigma] \notin \mathcal{D}^{3,3,10} \cup \mathcal{D}^{1,6,10}$), we have*

$$p_{6*}p_1^*(h^4) = 6H, \quad p_{6*}p_1^*(\pi \cdot h) = 3H,$$

where $H \in H^2(X_6^\sigma, \mathbf{Z})$ is the Plücker polarization on X_6^σ.

If moreover, $[\sigma]$ lies in $\mathcal{D}^{4,7,7}$ so we have a Lagrangian plane $P = \mathbf{P}\big((V_7/V_4)^\vee\big)$ with (-10)-class $\lambda \in H^2(X_6^\sigma, \mathbf{Z})$. Write $\ell \in H^6(X_6^\sigma, \mathbf{Z})$ for the class of a line in P and consider the class $z := p_{1*}p_6^*\ell \in H^4(X_1^\sigma, \mathbf{Z})$. We have

$$p_{6*}p_1^*(z \cdot h^2) = \tfrac{1}{2}(5H - \lambda).$$

Proof. By the simplicity of the Hodge structure $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}}$ for a very general σ, we deduce that there exists constants a, b such that $p_{6*}p_1^*(h^4) = aH$ and $p_{6*}p_1^*(\pi \cdot h) = bH$. To determine their values, we specialize to the case $[\sigma] \in \mathcal{D}^{4,7,7}$ that we studied above. For ℓ the class of a line, we use the intersection numbers in (4.24) to compute

$$a = p_{6*}p_1^*(h^4) \cdot \ell = h^4 \cdot p_{1*}p_6^*\ell = h^4 \cdot z = 6,$$

and

$$b = p_{6*}p_1^*(\pi \cdot h) \cdot \ell = \pi \cdot h \cdot p_{1*}p_6^*\ell = \pi \cdot h \cdot z = 3.$$

For the class $z = p_{1*}p_6^*\ell$, we have the decomposition $z \cdot h = x_0 + x_1$, where we have shown that $p_{6*}p_1^*L_h(x_0) = \frac{1}{22}(H - 11\lambda)$ and $x_1 = \frac{3}{11}(h^3 + \pi)$. We may thus conclude that

$$p_{6*}p_1^*(z \cdot h^2) = \tfrac{1}{22}(H - 11\lambda) + \tfrac{3}{11}(6H + 3H) = \tfrac{1}{2}(5H - \lambda),$$

where we used the images for h^4 and $\pi \cdot h$ that we just obtained. □

Remark 4.6.15. As a side note, since the projective space $\mathbf{P}(V_4)$ is contained in X_1^σ, its class should be a linear combination of the three classes h^3, π, and $z \cdot h$. One may check that $[\mathbf{P}(V_4)] = \pi - z \cdot h$ using the intersection numbers.

As in the case of X_3^σ, we can obtain the integral Hodge conjecture for X_1^σ. From the Hodge diamond (4.18) and the fact that X_1^σ contains lines, we see that the only non-trivial cases are in degrees 4, 6, and 8. First we treat the case of the middle cohomology.

Corollary 4.6.16. *When σ is such that X_1^σ is smooth (that is, when $[\sigma] \notin \mathcal{D}^{3,3,10} \cup \mathcal{D}^{1,6,10}$), the integral Hodge conjecture holds for $H^6(X_1^\sigma, \mathbf{Z})$.*

Proof. The proof is similar to that of Corollary 4.6.10: namely, we apply Lemma 4.6.9 with $L = H^6(X_1^\sigma, \mathbf{Z})$ and $L_1 = H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}$ to get

$$(p_{6*}p_1^*L_h)^\vee(H^2(X_6^\sigma, \mathbf{Z})^\vee) + L_1^\perp = L^\vee,$$

which, since L has the structure of a unimodular lattice, translates into

$$L_h p_{1*}{p_6}^* H^6(X_6^\sigma, \mathbf{Z}) + (\mathbf{Z}h^3 + \mathbf{Z}\pi) = H^6(X_1^\sigma, \mathbf{Z}).$$

So we may again conclude from the integral Hodge conjecture for $H^6(X_6^\sigma, \mathbf{Z})$. □

The proof for degrees 4 and 8 is a bit more involved, due to the fact that the Lefschetz operator L_h is not an isomorphism over $\mathbf{Z}$-coefficients. We first construct some extra algebraic classes on X_1^σ.

Recall that when X_1^σ is smooth so $[\sigma]$ does not lie in the divisor $\mathcal{D}^{1,6,10}$, we can identify X_1^σ as the zero-locus $Z(\sigma)$ inside the 27-dimensional flag variety $F \coloneqq \mathrm{Flag}(1,4,V_{10})$ (see Section 4.3.2). Write $j\colon X_1^\sigma \hookrightarrow F = \mathrm{Flag}(1,4,V_{10})$ for the inclusion. For $k = 4,6,8$ and a coefficient ring $R = \mathbf{Z}$ or $\mathbf{Q}$, we can imitate the case of X_3^σ and consider the kernel

$$\ker(j_*)^k_R \coloneqq \ker\big(j_*\colon H^k(X_1^\sigma, R) \longrightarrow H^{k+42}(F,R)\big) = \big(j^*H^{12-k}(F,R)\big)^\perp. \tag{4.25}$$

The last equality holds due to the Poincaré duality and the projective formula $j_*\alpha\cdot\beta = j_*(\alpha\cdot j^*\beta)$. Namely, if $\alpha \in \ker(j_*)^k_R$, then indeed for any $\beta \in H^{12-k}(F,R)$ we have $\alpha\cdot j^*\beta = 0$; conversely, if $\alpha \in j^*H^{12-k}(F,R)^\perp$, then for any $\beta \in H^{12-k}(F,R)$, the class $j_*\alpha \in H^{42+k}(F,R)$ satisfies $j_*\alpha\cdot\beta = 0$, so it vanishes by the Poincaré duality.

We show that this construction gives us an alternative description of the vanishing cohomologies, in other words, $H^k(X_1^\sigma,R)_{\mathrm{van}}$ coincides with the kernel $\ker(j_*)^k_R$, which justifies the name of *vanishing cohomology*.

LEMMA 4.6.17. *We have*

$$j^*H^6(F,\mathbf{Z}) = \mathbf{Z}h^3 + 3\mathbf{Z}\pi.$$

Consider the Chern classes $c_2(\mathcal{Q})$ and $c_4(\mathcal{Q})$ of the tautological quotient bundle $\mathcal{Q}$ of rank 6. *We have*

$$j^*c_2(\mathcal{Q}) = 6h^2 - 3L_h^{-1}\pi, \quad j^*c_4(\mathcal{Q}) = (\tfrac{17}{3}h^3 - 7\pi)\cdot h.$$

In particular, the class $\frac{1}{3}h^4 \in H^8(X_1^\sigma,\mathbf{Z})$ is integral and algebraic.

Consequently, we can identify the vanishing cohomologies for $k = 4,6,8$ with the kernel of the map j_*

$$H^k(X_1^\sigma,R)_{\mathrm{van}} = \ker(j_*)^k_R,$$

defined as in (4.25).

PROOF. First, we may consider a very general σ in the moduli space. Theorem 4.5.11 shows that in this case, the Hodge structure $H^6(X_1^\sigma,\mathbf{Z})_{\mathrm{van}}$ is simple so there are no Hodge classes. Therefore, we may deduce that $j^*H^6(F,\mathbf{Z})$ lies in $\mathbf{Z}h^3 + \mathbf{Z}\pi$.

The cohomology ring $H^*(F,\mathbf{Z})$ of the flag variety $F = \mathrm{Flag}(1,4,10)$ is generated by the Chern classes of the three tautological vector bundles $\mathcal{U}_1$, $\mathcal{U}_{4/1} \simeq \mathcal{U}_4/\mathcal{U}_1$, and $\mathcal{Q}_6$. Since X_1^σ is realized as the zero-locus $Z(\sigma)$ with σ viewed as a section of the vector bundle $\mathcal{F} \coloneqq (\mathcal{U}_1 \wedge \mathcal{U}_4 \wedge V_{10})^\vee$ of rank 21, we have

$$[X_1^\sigma] = c_{21}(\mathcal{F}) \in H^{42}(F,\mathbf{Z}).$$

Therefore we can compute intersection products on X_1^σ using

$$j^*\alpha\cdot j^*\beta = [X_1^\sigma]\cdot\alpha\cdot\beta.$$

We then take an integral basis $\{\beta_i\}$ of $H^6(F,\mathbf{Z})$ and compute the intersection matrix $(j^*\beta_a \cdot j^*\beta_b)_{ab}$ and reduce it to Smith normal form. This shows that $j^*H^6(F,\mathbf{Z})$ is of discriminant 99, hence is a sublattice of $\mathbf{Z}h^3 + \mathbf{Z}\pi$ of index 3. The class h is clearly in the image since it is the pullback of $c_1(\mathcal{U}_1^\vee)$, so we may conclude that the image $j^*H^6(F,\mathbf{Z})$ is $\mathbf{Z}h^3 + 3\mathbf{Z}\pi$.

```
needsPackage "Schubert2";
F = flagBundle{1,3,6}; (U1,U41,Q) = F.Bundles;
HF = intersectionRing F; -- the ring H^*(F)
X1 = ctop dual (U1*(exteriorPower_2 U41+U41*Q)); -- the class [X_1]
beta = first entries basis_3 HF; -- an integral basis given by Chern classes
M = matrix (for b1 in beta list for b2 in beta list integral(X1*b1*b2));
print first smithNormalForm M; -- rank 2 with discriminant 99=11*3^2
h = chern_1 dual U1; c2 = chern_2 Q; c4 = chern_4 Q;
print (integral \ (X1*c2*h^4, X1*c2^2*h^2)); -- (69, 324)
print (integral \ (X1*c4*h^2, X1*c4*c2)); -- (36, 181)
```

We prove the statement on the pullback of the Chern classes. For a very general σ in the moduli space, the Hodge structure on $H^k(X_1^\sigma,\mathbf{Q})_{\mathrm{van}}$ is simple, so we have

$$j^*H^4(F,\mathbf{Q}) \subset \mathbf{Q}h^2 + \mathbf{Q}L_h^{-1}\pi, \quad j^*H^8(F,\mathbf{Q}) \subset \mathbf{Q}h^4 + \mathbf{Q}\pi \cdot h.$$

To determine the pullback of the Chern classes, it suffices to compute the corresponding intersection numbers. We first consider the class $j^*c_2(\mathcal{Q})\cdot h$ which lies in $\mathbf{Z}h^3 + 3\mathbf{Z}\pi$: we have

$$j^*c_2(\mathcal{Q})\cdot h \cdot h^3 = 69, \quad (j^*c_2(\mathcal{Q})\cdot h)^2 = 324.$$

Writing $j^*c_2(\mathcal{Q})\cdot h = ah^3 - 3b\pi$ with $a,b\in\mathbf{Z}$, we get $(a,b) = (6,-1)$ or $(\frac{16}{5},1)$, so only the first solution is possible. For $j^*c_4(\mathcal{Q})$, we similarly compute that

$$j^*c_4(\mathcal{Q})\cdot h^2 = 36, \quad j^*c_4(\mathcal{Q})\cdot j^*c_2(\mathcal{Q}) = 181.$$

Writing $j^*c_4(\mathcal{Q}) = ah^4 + b\pi\cdot h$ with $a,b\in\mathbf{Q}$, we get $(a,b) = (\frac{17}{3},-7)$.

We note that the space $j^*H^k(F,\mathbf{Q})$ is therefore of rank exactly 2 for $k\in\{4,6,8\}$. All three are related by the Lefschetz operators L_h. This shows that the vanishing cohomologies $H^k(X_1^\sigma,R)_{\mathrm{van}}$ can indeed be identified with the kernel $\ker(j_*)_R^k = \big(j^*H^{12-k}(F,R)\big)^\perp$. □

Now we study the Lefschetz operators L_h over $\mathbf{Z}$-coefficients.

LEMMA 4.6.18. *Suppose that X_1^σ is smooth. The image of the Lefschetz operator*

$$L_h\colon H^6(X_1^\sigma,\mathbf{Z}) \longrightarrow H^8(X_1^\sigma,\mathbf{Z})$$

is a subgroup of index 3*. By duality, the same is true for*

$$L_h\colon H^4(X_1^\sigma,\mathbf{Z}) \longrightarrow H^6(X_1^\sigma,\mathbf{Z}).$$

When restricted to the vanishing parts, both Lefschetz operators become isomorphisms.

PROOF. The Lefschetz operator L_h is an isomorphism over $\mathbf{Q}$-coefficients. Using Lemma 4.6.17, we get

$$L_h H^6(X_1^\sigma, \mathbf{Z}) + \tfrac{1}{3}\mathbf{Z}h^4 \subset H^8(X_1^\sigma, \mathbf{Z}). \tag{4.26}$$

Since $\frac{1}{3}h^3 \notin H^6(X_1^\sigma, \mathbf{Z})$, the index of the image of L_h in $H^8(X_1^\sigma, \mathbf{Z})$ is at least 3. We prove that it is equal to 3 by showing that the inclusion in (4.26) is an equality.

We will do this by studying the image of $p_{6*}p_1^*$. Since this is a topological property, we can specialize to the case of a general σ in the divisor $\mathcal{D}^{4,7,7}$, so we retain the notation of $\ell \in H_2(X_6^\sigma, \mathbf{Z})$, $z \in H^4(X_1^\sigma, \mathbf{Z})$, and $z \cdot h = x_0 + x_1$, where we have already shown that the algebraic part x_1 is equal to $\frac{3}{11}(h^3 + \pi)$. Note that since $z \cdot h$ does not lie in the direct sum $H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}} \oplus (\mathbf{Z}h^3 + \mathbf{Z}\pi)$ which is a sublattice of $H^6(X_1^\sigma, \mathbf{Z})$ of index 11, by adding the class $z \cdot h$ we get the entire lattice so

$$H^6(X_1^\sigma, \mathbf{Z}) = H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}} + \mathbf{Z}z \cdot h + (\mathbf{Z}h^3 + \mathbf{Z}\pi),$$

and thus

$$L_h H^6(X_1^\sigma, \mathbf{Z}) + \tfrac{1}{3}\mathbf{Z}h^4 = L_h H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}} + \mathbf{Z}z \cdot h^2 + (\tfrac{1}{3}\mathbf{Z}h^4 + \mathbf{Z}\pi \cdot h).$$

By Theorem 4.5.11, we see that $H^6(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}$ is mapped isomorphically onto $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}}$ via $p_{6*}p_1^*L_h$. Also, in Corollary 4.6.14, we have obtained the images of the classes $z \cdot h^2, h^4$, and $\pi \cdot h$ under the map $p_{6*}p_1^*$. So we have a complete description of the image of $p_{6*}p_1^*$

$$p_{6*}p_1^*\big(L_h H^6(X_1^\sigma, \mathbf{Z}) + \tfrac{1}{3}\mathbf{Z}h^4\big) = H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}} + \mathbf{Z}\tfrac{1}{2}(H + \lambda) + \mathbf{Z}H.$$

On the other hand, since we have $q\big(\frac{1}{2}(H + \lambda), H\big) = 12$ from the intersection numbers (4.21), the class $\frac{1}{2}(H + \lambda)$ does not lie in the direct sum $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{prim}} \oplus \mathbf{Z}H$. The latter is a sublattice of index 11 in $H^2(X_6^\sigma, \mathbf{Z})$, so we may conclude that the image is the full lattice

$$p_{6*}p_1^*\big(L_h H^6(X_1^\sigma, \mathbf{Z}) + \tfrac{1}{3}\mathbf{Z}h^4\big) = H^2(X_6^\sigma, \mathbf{Z}).$$

On the other hand, we also have

$$p_{6*}p_1^*\big(H^8(X_1^\sigma, \mathbf{Z})\big) = H^2(X_6^\sigma, \mathbf{Z}).$$

So to check that $L_h H^6(X_1^\sigma, \mathbf{Z}) + \frac{1}{3}\mathbf{Z}h^4$ generates $H^8(X_1^\sigma, \mathbf{Z})$, it suffices to check this on the kernel of $p_{6*}p_1^*$. By comparing the dimensions, the kernel is of rank 1, and one generator (over $\mathbf{Q}$-coefficients) is given by $h^4 - 2\pi \cdot h = L_h(h^3 - 2\pi)$. This class is clearly an element of $L_h H^6(X_1^\sigma, \mathbf{Z}) + \frac{1}{3}\mathbf{Z}h^4$; but it is also primitive in $H^8(X_1^\sigma, \mathbf{Z})$ since $(h^4 - 2\pi \cdot h) \cdot h^2 = 1$. We may thus conclude that $L_h H^6(X_1^\sigma, \mathbf{Z}) + \frac{1}{3}\mathbf{Z}h^4$ indeed generates $H^8(X_1^\sigma, \mathbf{Z})$.

We note that for degree 4, the class $j^*c_2(\mathcal{Q})$ is primitive, since by Lemma 4.6.17 we have

$$j^*c_2(\mathcal{Q}) \cdot (-\tfrac{13}{3}h^4 + 10\pi \cdot h) = (6h^3 - 3\pi) \cdot (-\tfrac{13}{3}h^3 + 10\pi) = 1.$$

But its image $L_h j^*c_2(\mathcal{Q}) = 6h^3 - 3\pi$ is divisible by 3. Therefore $\mathbf{Z}h^2 + \mathbf{Z}j^*c_2(\mathcal{Q})$ is saturated in $H^4(X_1^\sigma, \mathbf{Z})$.

Finally, for both Lefschetz operators, the extra 3-divisible class lies in the algebraic part. So when we restrict to the vanishing parts, we get isomorphisms. □

THEOREM 4.6.19. *When σ is such that X_1^σ is smooth (that is, when $[\sigma] \notin \mathcal{D}^{3,3,10} \cup \mathcal{D}^{1,6,10}$), the integral Hodge conjecture holds for X_1^σ in all degrees.*

PROOF. It remains to show the cases of $k \in \{4, 8\}$.

For $H^4(X_1^\sigma, \mathbf{Z})$, we see that the subgroup $\mathbf{Z}h^2 + \mathbf{Z}j^*c_2(\mathcal{Q})$ is saturated in $H^4(X_1^\sigma, \mathbf{Z})$. As before, we apply Lemma 4.6.9 to $L = H^8(X_1^\sigma, \mathbf{Z})$, $L_1 = H^8(X_1^\sigma, \mathbf{Z})_{\mathrm{van}}$, and the map $p_{6*}p_1^*$ to get

$$p_{1*}p_6^* H^6(X_6^\sigma, \mathbf{Z}) + \big(\mathbf{Z}h^2 + \mathbf{Z}j^*c_2(\mathcal{Q})\big) = H^4(X_1^\sigma, \mathbf{Z}),$$

which allows us to conclude.

For $H^8(X_1^\sigma, \mathbf{Z})$, we could proceed similarly as above. But since we have already obtained the integral Hodge conjecture for $H^6(X_1^\sigma, \mathbf{Z})$, and we have also seen that $\frac{1}{3}h^4$ is algebraic, we can directly conclude that all integral Hodge classes in $H^8(X_1^\sigma, \mathbf{Z})$ are algebraic. □

4.7. The Heegner divisor of degree 24

In the GIT moduli space $\mathcal{M}$ of trivectors, we have defined the divisor $\mathcal{D}^{1,6,10}$ given by trivectors satisfying the degeneracy condition $\sigma(V_1, V_6, V_{10}) = 0$ as in (4.4), which is also the locus where the Peskine variety X_1^σ becomes singular and generically admits an isolated singularity at $[V_1]$. In this section, we will study the geometry along this divisor. Notably we will give the geometric construction of a K3 surface S of degree 6 and a divisor D in X_6^σ ruled over S.

4.7.1. The discriminant. First we show that the divisor $\mathcal{D}^{1,6,10}$ is mapped to the Noether–Lefschetz divisor $\mathcal{C}_{24}$ under the modular map $\mathfrak{m}$ and to the Heegner divisor $\mathcal{D}_{24}$ by the period map $\mathfrak{p}$. We state a lemma which gives an alternative description for $\mathcal{D}^{1,6,10}$.

LEMMA 4.7.1. *For a trivector σ, there is a flag $[V_1 \subset V_6]$ such that $\sigma(V_1, V_6, V_{10}) = 0$ if and only if there is a flag $[V_1 \subset V_8]$ such that $\sigma(V_1, V_8, V_8) = 0$. Moreover, in this case, the flags $[V_1 \subset V_8]$ are parametrized by a 3-dimensional quadric.*

PROOF. If we have a flag $V_1 \subset V_8$ as above, the skew-symmetric 2-form $\sigma(V_1, -, -)$ is of rank at most 4, so there is a 6-dimensional V_6 in the kernel.

Conversely, if we have a flag $V_1 \subset V_6$ as in the lemma, the set of V_8 in $\mathrm{Gr}(2, V_{10}/V_6)$ such that $\sigma(V_1, V_8, V_8) = 0$ is exactly the set of subspaces which are isotropic with respect to $\sigma(V_1, -, -)$, which is a linear section of the quadric $\mathrm{Gr}(2, V_{10}/V_6)$. □

PROPOSITION 4.7.2. *The divisor $\mathcal{D}^{1,6,10}$ is mapped birationally onto the Noether–Lefschetz divisor $\mathcal{C}_{24}$ via the moduli map $\mathfrak{m}$, and then to the Heegner divisor $\mathcal{D}_{24}$ via the period map $\mathfrak{p}$.*

Proof. As with the other two divisors, it suffices to compute the discriminant. We pick a very general $[\sigma]$ in the divisor $\mathcal{D}^{1,6,10}$. In particular, we assume that $[\sigma] \notin \mathcal{D}^{3,3,10}$ so X_3^σ is smooth.

The degeneracy condition $\sigma(V_1, V_8, V_8) = 0$ shows that there is a Grassmannian $\mathrm{Gr}(2,7) = \mathrm{Gr}(2, V_8/V_1)$ contained in X_3^σ. Notice that the choice of V_8 is not canonical, as generally these V_8 are parametrized by a 3-dimensional quadric for a fixed flag $V_1 \subset V_6$, and so are the Grassmannians $\mathrm{Gr}(2,7)$ contained in X_3^σ.

If we fix one $Z = \mathrm{Gr}(2,7)$ contained in X_3^σ and look at its class $z \in H^{20}(X_3^\sigma, \mathbf{Z})$, we may compute the self-intersection number

$$z^2 = c_{10}(\mathcal{N}_{Z/X_3^\sigma}) = 2.$$

Indeed, using the two normal sequences

$$0 \longrightarrow \mathcal{T}_Z \longrightarrow \mathcal{T}_{X_3^\sigma}|_Z \longrightarrow \mathcal{N}_{Z/X_3^\sigma} \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{T}_{X_3^\sigma} \longrightarrow \mathcal{T}_{\mathrm{Gr}(3,V_{10})}|_{X_3^\sigma} \longrightarrow \mathcal{O}_{X_3^\sigma}(1) \longrightarrow 0,$$

the normal bundle $\mathcal{N}_{Z/X_3^\sigma}$ can be expressed in terms of homogeneous vector bundles $\mathcal{U}_2$ and $\mathcal{Q}_5$ on $Z = \mathrm{Gr}(2,7)$, so we may calculate explicitly its Chern classes using the splitting principle and Schubert calculus.

```
needsPackage "Schubert2";
(U,Q) = bundles flagBundle{2,5}; N = dual(U+1)*(Q+2)-det Q-dual U*Q;
print integral chern_10 N; -- 2
```

Moreover, we see that $j_*z = \sigma_{722}$ is a Schubert class. So we can compute the full intersection matrix for the lattice $\mathbf{Z}z + j^*H^{20}(\mathrm{Gr}(3,V_{10}), \mathbf{Z})$ and find that its determinant is 24. We can also compute the Smith normal form of the intersection matrix to show that the discriminant group is $\mathbf{Z}/24\mathbf{Z}$.

```
needsPackage "Schubert2";
G = flagBundle {3,7};
-- enumerate all the Schubert classes in codimension k
classes = k -> for p in partitions(k,7) list (
    if #p <= 3 then (p = toList p; while #p < 3 do p = p|{0}; schubertCycle(p, G))
    else continue);
jz = schubertCycle({7,2,2}, G); -- the class j_*z
X3 = schubertCycle({1,0,0}, G); -- the class [X_3]
H20G = classes 10;              -- the classes in H^20(G)
-- V: the intersection numbers of z with j^*H^20(G)
V = matrix {for a in H20G list integral(a*jz)};
-- M: the intersection matrix of j^*H^20(G); we have det(M)=11
M = matrix for a in H20G list for b in H20G list integral(a*b*X3);
MM = (matrix{{2}} | V) || (transpose V | M); -- the full intersection matrix
print det MM; -- 24
print first smithNormalForm MM; -- the discriminant group is Z/24Z
print prune cokernel MM; -- an alternative way to obtain Z/24Z
```

This concludes the proof. □

We also remark that the above code can be modified accordingly to compute the $\mathcal{D}_{28}$ case, as mentioned in Remark 4.6.8 (namely, we change σ_{722} to σ_{443} and $z^2 = 2$ to $z^2 = 3$).

4.7.2. Review: cubic fourfolds containing a plane. Before studying the geometry of the Debarre–Voisin hyperkähler manifold X_6^σ along the divisor $\mathcal{D}^{1,6,10}$, we first briefly review results for cubic fourfolds containing a plane, originally considered by Voisin in her thesis [**Voi86**] and by Hassett in [**Has00**], with later studies on their derived aspects by Kuznetsov [**Kuz10**], and moduli aspects by Macrì–Stellari [**MS12**] and Ouchi [**Ouc17**]. We will see that analogous results hold in our case.

Let X be a cubic fourfold in $\mathbf{P}(V_6)$ that contains a plane $\mathbf{P}(V_3)$ for some $V_3 \subset V_6$. As shown in [**Voi86**], the blow up $\mathrm{Bl}_{\mathbf{P}(V_3)} X$ projects onto the plane $\mathbf{P}^2 = \mathbf{P}(V_6/V_3)$ and the fibers are quadric surfaces which are generically smooth. The discriminant locus, that is, the locus where the quadrics are singular, is a sextic curve in $\mathbf{P}^2$. Let S be the variety parametrizing rulings of lines in these fibers: as the fibers are quadric surfaces, the projection $S \to \mathbf{P}^2$ is a generically 2-to-1 morphism, ramified along the discriminant curve, and S is therefore a K3 surface of degree 2.

The variety $F \subset \mathrm{Gr}(2, V_6)$ of lines contained in X is a hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type by [**BD85**]. The lines in the fibers of $\mathrm{Bl}_{\mathbf{P}(V_3)} X \to \mathbf{P}^2$ form a uniruled divisor D in F. Alternatively, it can also be defined as the closure of the set of lines in X that intersect $\mathbf{P}(V_3)$ at one point. Clearly, D admits a $\mathbf{P}^1$-fibration over S. In [**Voi86**], it was shown that the transcendental part $H^2(F, \mathbf{Z})_{\mathrm{trans}}$ (of discriminant 8 and Hodge type $(1, 19, 1)$) embeds as a sublattice of index two into the primitive cohomology $H^2(S, \mathbf{Z})_{\mathrm{prim}}$ (of discriminant 2). This sublattice is closely related to the Brauer class β induced by the $\mathbf{P}^1$-fibration $D \to S$, so it should be considered as the "primitive cohomology" of the twisted K3 surface (S, β) (see [**vG05**]). For a general X containing a plane, the class β is non-trivial and is related to rationality questions (see [**Has00**]). Finally, it was proved in [**MS12**] that for a general X containing a plane, the hyperkähler variety F can be recovered (birationally) as a moduli space of β-twisted sheaves on S.

4.7.3. Associated K3 surface. From now on, we consider a general $[\sigma] \in \mathcal{D}^{1,6,10}$, so there is a unique distinguished flag $[V_1 \subset V_6]$ such that $\sigma(V_1, V_6, V_{10}) = 0$. We study the geometry of the Debarre–Voisin variety X_6^σ, which resembles a lot that of a cubic fourfold containing a plane. Notably, we will construct a K3 surface S of degree 6 and a uniruled divisor D in X_6^σ that admits a $\mathbf{P}^1$-fibration over S. In the next section, we will compare the Hodge structures of X_6^σ and of the K3 surface S. The $\mathbf{P}^1$-fibration defines a non-trivial Brauer class $\beta \in \mathrm{Br}(S)$, and we will show that X_6^σ can be recovered as a moduli space of β-twisted sheaves on S (which is proved in a purely Hodge theoretical way).

Let W_7 be a complex vector space of dimension 7. We begin by recalling some properties on $\mathrm{GL}(W_7)$-orbit closures inside $\bigwedge^3 W_7$. Let $Y \subset \bigwedge^3 W_7^\vee$ be the unique $\mathrm{GL}(W_7)$-invariant hypersurface. It can also be characterized as the affine cone over the projective dual variety $\mathrm{Gr}(3, W_7)^*$ embedded in $\mathbf{P}(\bigwedge^3 W_7^\vee)$, which is a hypersurface of degree 7. In other words, the polynomial f defining Y lives inside $\mathrm{Sym}^7(\bigwedge^3 W_7^\vee)^\vee \simeq \mathrm{Sym}^7 \bigwedge^3 W_7$, and is usually referred to as the *discriminant* or the *hyperdeterminant*. Equivalently, $\mathbf{C}f$ can be characterized as the unique one-dimensional $\mathrm{GL}(W_7)$-subrepresentation of $\mathrm{Sym}^7 \bigwedge^3 W_7$. Since all one-dimensional representations of $\mathrm{GL}(W_7)$ are of the form $\det(W_7)^{\otimes i}$ for $i \in \mathbf{Z}$, weight invariance with respect to the torus $\mathbf{C}^* \,\mathrm{Id} \subset \mathrm{GL}(W_7)$ implies that we have $\mathbf{C}f \simeq \det(W_7)^{\otimes 3}$. This also means that we can canonically define the discriminant $\operatorname{disc} y$ of each $y \in \bigwedge^3 W_7^\vee$ as an element of $\det(W_7^\vee)^{\otimes 3}$.

Let us now return to our trivector σ.

PROPOSITION 4.7.3. *Suppose the trivector $\sigma \in \bigwedge^3 V_{10}^\vee$ is general in the divisor $\mathcal{D}^{1,6,10}$, that is, we have $\sigma(V_1, V_6, V_{10}) = 0$ for a unique flag $V_1 \subset V_6 \subset V_{10}$. Then it defines a smooth K3 surface S of degree 6 inside $\mathrm{Gr}(2, V_{10}/V_6)$, where the polarization is given by the Plücker line bundle.*

PROOF. The Grassmannian $\mathrm{Gr}(2, V_{10}/V_6)$ is a 4-dimensional quadric. The K3 surface S will be the intersection of a linear section and a cubic section of this quadric, hence the Plücker line bundle will be of degree 6. For clarity, we denote by $\mathcal{U}_{8/6}$ the tautological subbundle and by $\mathcal{Q}_{10/8}$ the quotient bundle on $\mathrm{Gr}(2, V_{10}/V_6)$ respectively.

The linear section is given by the condition $\sigma(V_1, V_8, V_8) = 0$: since $\sigma(V_1, V_6, V_{10}) = 0$, this is equivalent to the condition $\sigma(V_1, V_8/V_6, V_8/V_6) = 0$, which can be seen as the vanishing of a general section of the line bundle $\bigwedge^2 \mathcal{U}_{8/6}^\vee \simeq \mathcal{O}(1)$. The zero-locus is therefore a 3-dimensional quadric S'.

Now for each $[V_8/V_6] \in S'$, since we have $\sigma(V_1, V_8, V_8) = 0$, the form σ induces an element of $\bigwedge^3 (V_8/V_1)^\vee$. In the relative setting, by letting $\mathcal{W}_7 := V_6/V_1 \oplus \mathcal{U}_{8/6}$ where V_6/V_1 is the trivial bundle $(V_6/V_1) \otimes \mathcal{O}_{S'}$, we get a global section σ' of the vector bundle $\bigwedge^3 \mathcal{W}_7^\vee$. So we may define the orbital degeneracy locus

$$S := D_Y(\sigma') = \left\{ [V_8/V_6] \in S' \;\middle|\; \sigma'|_{V_8/V_1} \in Y \subset \bigwedge^3 (V_8/V_1)^\vee \simeq \left(\bigwedge^3 \mathcal{W}_7^\vee\right)_{[V_8/V_6]} \right\}.$$

As we have already seen, the hypersurface Y is defined in $\bigwedge^3 W_7^\vee$ by the vanishing of the discriminant. Therefore S is the hypersurface in S' defined by the vanishing of $\operatorname{disc} \sigma'$, which is a section of $\det(\mathcal{W}_7^\vee)^{\otimes 3} \simeq \mathcal{O}_{S'}(3)$.

As σ is general, so is σ' among sections of $\bigwedge^3 \mathcal{W}_7^\vee$. Moreover, the hypersurface Y is smooth in codimension 2, so by a Bertini-type theorem for orbital degeneracy loci (see [**BFMT20**, Proposition 2.3]), the zero-locus S is also smooth. In other words, we obtain a smooth surface defined as the intersection of a quadric and a cubic, that is, a K3 surface of degree 6. □

REMARK 4.7.4. A general element $y \in Y$ admits a unique point $[W_3] \in \mathrm{Gr}(3, W_7)$ for which we have the vanishing condition $y(W_3, W_3, W_7) = 0$. In the relative setting, this implies that a general point $[V_8/V_6]$ of S defines a unique 3-dimensional subspace of V_8/V_1, in other words a 4-dimensional subspace V_4 with $V_1 \subset V_4 \subset V_8$ such that

$$\sigma'(V_4/V_1, V_4/V_1, V_8/V_1) = 0 \text{ or equivalently, } \sigma(V_4, V_4, V_8) = 0.$$

In conclusion, having fixed the flag $V_1 \subset V_6$ and the trivector $\sigma \in \mathcal{D}^{1,6,10}$, the K3 surface S can also be defined as the set

$$Z(\sigma) := \left\{ [V_4 \subset V_8] \;\middle|\; \begin{array}{c} V_1 \subset V_4,\ V_6 \subset V_8,\ \sigma(V_1, V_8, V_8) = 0, \\ \text{and } \sigma(V_4, V_4, V_8) = 0 \end{array} \right\} \subset \mathrm{Flag}(4, 8, V_{10}). \tag{4.27}$$

The advantage of this description is that K3 surface can now be characterized as the zero-locus of a section of some vector bundle on a flag variety.

There is a natural projection map from $Z(\sigma)$ to $S \subset \mathrm{Gr}(2, V_{10}/V_6)$ by forgetting $[V_4]$. We claim that this is an isomorphism if $[\sigma] \notin \mathcal{D}^{4,7,7}$. Suppose that the map is not an isomorphism at a point $[V_8] \in S$, this means that the form $\sigma' = \sigma|_{V_8/V_1} \in \bigwedge^3 (V_8/V_1)^\vee$ admits at least 2 three-dimensional subspaces V_4/V_1 and V_4'/V_1 such that $\sigma(V_4, V_4, V_8) = \sigma(V_4', V_4', V_8) = 0$. In particular, it cannot be a general point of the discriminant hypersurface Y and must further degenerate. Using the description of the $\mathrm{GL}(W_7)$-orbits in $\bigwedge^3 W_7^\vee$ in Section 4.2.3, we conclude that $\sigma' \in Y_{31}$, and there exists a flag $W_3 \subset W_6$ such that $\sigma'(W_3, W_6, W_6) = 0$. But W_3 and W_6 are subspaces of $W_7 = V_8/V_1$, hence we get a flag $V_4 \subset V_7$ such that $W_3 = V_4/V_1$, $W_6 = V_7/V_1$, and $\sigma(V_4, V_7, V_7) = 0$, so we conclude that $[\sigma] \in \mathcal{D}^{4,7,7}$.

Note that the Picard group of $\mathrm{Flag}(4, 8, V_{10})$ is of rank 2, generated by the first Chern classes of the two tautological bundles $\mathcal{U}_{4/1} := \mathcal{U}_4/V_1$ and $\mathcal{U}_{8/4} := \mathcal{U}_8/\mathcal{U}_4$. By construction, the first Chern class of the bundle $\mathcal{U}_{8/6}^\vee$ gives the polarization h of degree 6 on S. One may check that $c_1(\mathcal{U}_{4/1}|_S) = -3h$ and $c_1(\mathcal{U}_{8/4}|_S) = 2h$, so no new polarizations are produced this way.

```
needsPackage "Schubert2";
(U86,Q) = bundles flagBundle{2,2}; -- first choose V8/V6 in V10/V6
(U41,U84) = bundles flagBundle({3,4},U86+5); -- then choose V4/V1 in V8/V1
S = sectionZeroLocus dual(det U86+det U41+exteriorPower_2 U41*U84);
h = chern_1(dual U86*OO_S);
-- verify the Chern classes
assert(chern_1(U41*OO_S) == -3*h and chern_1(U84*OO_S) == 2*h);
```

In fact, we will see later that the family of polarized K3 surfaces of degree 6 parametrized by $\mathcal{D}_{24}$ is a locally complete family, as a consequence of the study of their Hodge structures. Hence a very general member S of the family has Picard rank 1.

Next, we construct a uniruled divisor D in X_6^σ.

PROPOSITION 4.7.5. *For a general* $[\sigma] \in \mathcal{D}^{1,6,10} \setminus \mathcal{D}^{3,3,10}$*, the set*

$$D \coloneqq \{[U_6] \in X_6^\sigma \mid \exists [V_4 \subset V_8] \in S \quad V_4 \subset U_6 \subset V_8\}$$

defines a divisor in X_6^σ *which has a smooth conic fibration* $\pi\colon D \to S$ *over the K3 surface* S*.*

PROOF. First we construct the morphism $\pi\colon D \to S$ by showing that for each $[U_6] \in D$, the corresponding $[V_4 \subset V_8] \in S$ is unique. Since U_6 and V_6 are both subspaces of V_8, we have $\dim(U_6 \cap V_6) \geq 4$. We claim that the equality always holds. Otherwise, suppose that there exists some U_6 with $\dim(U_6 \cap V_6) \geq 5$. For any V_3 with $V_1 \subset V_3 \subset U_6 \cap V_6$, we have the vanishing $\sigma(V_1, V_3, V_{10}) = 0$ as well as $\sigma(V_3, V_3, U_6) = 0$, and we claim that there exists a such V_3 with $\sigma(V_3, V_3, V_{10}) = 0$. This is equivalent to study inside $\mathrm{Gr}(2, (U_6 \cap V_6)/V_1)$ the zero locus of σ seen as a section of the vector bundle $\bigwedge^2 \mathcal{U}_2^\vee \otimes (V_{10}/V_6)^\vee$, which is a condition of codimension 4 so such a V_3 must exist. This would contradict the hypothesis on $[\sigma]$, so we may conclude that $\dim(U_6 \cap V_6) = 4$. We may then recover V_8 as the sum $U_6 + V_6$ and get a morphism $\pi\colon D \to S$ (recall that for σ general, we have two equivalent descriptions of S, as a degeneracy locus in $\mathrm{Gr}(2, V_{10}/V_6)$ or one in $\mathrm{Flag}(4, 8, V_{10})$; in particular, the subspace V_4 can be uniquely determined from V_8).

Now we show that this morphism $\pi\colon D \to S$ is a smooth conic fibration. We first study the fiber $\{[U_6] \in X_6^\sigma \mid V_4 \subset U_6 \subset V_8\}$ above each $[V_4 \subset V_8] \in S$. This fiber can be seen as the locus

$$\{[U_6] \in \mathrm{Gr}(2, V_8/V_4) \mid \sigma|_{U_6} = 0\}.$$

The trivector σ, when restricted to V_8, becomes a section of $(V_4/V_1)^\vee \otimes \bigwedge^2 \mathcal{U}_2^\vee = \mathcal{O}(1)^{\oplus 3}$. So we get three hyperplane sections, whose intersection in $\mathrm{Gr}(2, V_8/V_4)$ is generically a conic.

In the relative setting, the fibers are defined inside the projectivization $\mathbf{P}_S(\mathcal{E})$ of a vector bundle $\mathcal{E}$ of rank 3 over S, which is realized as the kernel

$$\mathcal{E} \hookrightarrow \textstyle\bigwedge^2 \mathcal{U}_{8/4} \xrightarrow{\sigma} \mathcal{U}_{4/1}^\vee \longrightarrow 0.$$

Here the last arrow is surjective for a general σ: otherwise we would get a subspace $U_2 \supset V_1$ such that $\sigma(U_2, V_8, V_8) = 0$; then all the U_1 contained in U_2 will have $\operatorname{rank} \sigma(U_1, -, -) \leq 4$ which does not happen for σ general. The quadratic form q on $\mathcal{E}$ is given by

$$\mathrm{Sym}^2 \mathcal{E} \hookrightarrow \mathrm{Sym}^2 \textstyle\bigwedge^2 \mathcal{U}_{8/4} \xrightarrow{q} \mathcal{L} \coloneqq \det \mathcal{U}_{8/4}$$

where it takes value in the line bundle $\mathcal{L}$. We have $\det \mathcal{E} \simeq \mathcal{O}_S(3)$ while $\mathcal{L} \simeq \mathcal{O}_S(2)$. The discriminant locus is defined by a section of the line bundle $(\det \mathcal{E}^\vee)^{\otimes 2} \otimes \mathcal{L}^{\otimes \operatorname{rank} \mathcal{E}} \simeq \mathcal{O}_S$ and is therefore empty. Thus the conic fibration is everywhere smooth. □

We also have an alternative description of D.

PROPOSITION 4.7.6. *For a general* σ *in the divisor* $\mathcal{D}^{1,6,10}$*, let* $[V_1 \subset V_6]$ *be the distinguished flag. A point* $[U_6] \in X_6^\sigma$ *is contained in* D *if and only if* U_6 *contains* V_1*. In other words, we have*

$$D = \{[U_6] \in X_6^\sigma \mid U_6 \supset V_1\}.$$

Proof. Since any $[U_6]$ in D contains a subspace $V_4 \supset V_1$, one direction is evident.

Suppose now that U_6 contains V_1 and $\sigma|_{U_6} = 0$. Then U_6 is isotropic with respect to $\sigma(V_1, -, -)$ and is contained inside a maximal isotropic subspace V_8 of dimension eight. Let us consider a point $[V_4] \in \mathrm{Gr}(3, U_6/V_1)$. As $[U_6] \in X_6^\sigma$, any such V_4 satisfies $\sigma(V_4, V_4, U_6) = 0$. Therefore, the condition $\sigma(V_4, V_4, V_8) = 0$ is a codimension-6 condition, and there exists exactly one point $[V_4] \in \mathrm{Gr}(3, U_6/V_1)$ satisfying it since the bundle $(\bigwedge^2 \mathcal{U}_3^\vee) \otimes (V_8/U_6)^\vee$ has top Chern class 1. This tells us that $[V_4 \subset V_8]$ is a point of S as in (4.27) and therefore $[U_6] \in D$. $\square$

4.7.4. Hodge structures. Denote by $i\colon D \hookrightarrow X_6^\sigma$ the embedding of the divisor D constructed above. By abuse of notation, we denote the class $[D] \in H^2(X_6^\sigma, \mathbf{Z})$ also by D. We first compute the intersection matrix for the classes H and D under the Beauville–Bogomolov–Fujiki form q. Note that for $[\sigma]$ very general in $\mathcal{D}^{1,6,10}$, the Debarre–Voisin fourfold X_6^σ is of Picard rank 2, so H and D generate a subgroup of $\mathrm{Pic}(X_6^\sigma)$ of finite index.

Lemma 4.7.7. *The intersection matrix between H and D with respect to the Beauville–Bogomolov–Fujiki form q is*

$$\begin{pmatrix} 22 & 2 \\ 2 & -2 \end{pmatrix},$$

which has determinant 48. *For $[\sigma] \in \mathcal{D}^{1,6,10}$ very general, the Picard group* $\mathrm{Pic}(X_6^\sigma)$ *is generated by H and D.*

Proof. By the adjunction formula and the fact that X_6^σ has trivial canonical bundle, the canonical class K_D of the divisor D is the restriction i^*D. One can then compute explicitly the intersection numbers using Schubert calculus in `Macaulay2` and obtain:

$$D^4 = 12,\ D^3 \cdot H = -12,\ D^2 \cdot H^2 = -36,\ D \cdot H^3 = 132,\ H^4 = 1452.$$

```
needsPackage "Schubert2";
(U1,Q1) = bundles flagBundle{3,2}; -- first choose U4/V1 in V6/V1
(U2,Q2) = bundles flagBundle({2,4},Q1+4); -- then choose U6/U4 in V10/U4
D = sectionZeroLocus dual((1+U1)*det U2+det U1+exteriorPower_2 U1*U2);
h = chern_1(dual(1+U1+U2)*OO_D);
d = chern_1 cotangentBundle D;
(U,Q) = bundles flagBundle{6,4};
X = sectionZeroLocus dual exteriorPower_3 U;
h' = chern_1 OO_X(1);
print (integral \ (d^3,d^2*h,d*h^2,h^3,h'^4)); -- (12, -12, -36, 132, 1452)
```

Then we use the property of the Beauville–Bogomolov–Fujiki form and the fact that $q(H) = 22$ to obtain the desired numbers

$$132 = D \cdot H^3 = 3q(D,H)q(H) \quad \Longrightarrow \quad q(D,H) = 2,$$
$$-12 = D^3 \cdot H = 3q(D)q(D,H) \quad \Longrightarrow \quad q(D) = -2.$$

Since the divisors $\mathcal{D}^{1,6,10}$ is mapped to the Heegner divisor $\mathcal{D}_{24}$ by the period map, we may use the fact that the discriminant of the algebraic lattice is twice that of its orthogonal from Lemma 4.4.8 to conclude that the Picard group is generated by H and D. □

We have the following useful result.

COROLLARY 4.7.8. *The class D has divisibility* 1*, that is, there exists $C \in H^2(X_6^\sigma, \mathbf{Z})$ such that $q(C, D) = 1$.*

Note that the class C is not algebraic for a very general σ in the family $\mathcal{D}_{24}$.

PROOF. We recall that the lattice $\Lambda = H^2(X_6^\sigma, \mathbf{Z})$ has discriminant group $\mathbf{Z}/2\mathbf{Z}$. Suppose that D has divisibility 2, then the class $[D/2] \in D(\Lambda)$ gives the non-trivial element. Since H is also of divisibility 2, the class $[\frac{1}{2}(H + D)]$ is then trivial in $D(\Lambda)$, so $\frac{1}{2}(H + D)$ is integral. This contradicts the fact that the sublattice $\mathbf{Z}H + \mathbf{Z}D$ is saturated in Λ. □

Now we would like to compare the Hodge structures on $H^2(X_6^\sigma, \mathbf{Z})$ and $H^2(S, \mathbf{Z})$. Consider the diagram

$$\begin{array}{ccc} H^2(X_6^\sigma, \mathbf{Z}) & \xrightarrow{i^*} & H^2(D, \mathbf{Z}) \\ & & \uparrow{\scriptstyle \pi^*} \\ & & H^2(S, \mathbf{Z}). \end{array}$$

The idea is to make the comparison inside $H^2(D, \mathbf{Z})$. As we saw in Proposition 4.7.5, the natural projection $\pi \colon D \to S$ is a smooth conic fibration over the K3 surface S. We denote by $\ell \in H_2(D, \mathbf{Z})$ the class of a fiber of π.

LEMMA 4.7.9. *For σ general in the divisor $\mathcal{D}^{1,6,10}$, there exists a class $\zeta \in H^2(D, \mathbf{Z})$ with $\zeta \cdot \ell = 1$ such that*

$$H^2(D, \mathbf{Z}) = \pi^* H^2(S, \mathbf{Z}) \oplus \mathbf{Z}\zeta.$$

Let $H^2(X_6^\sigma, \mathbf{Z})^{\perp D} \subset H^2(X_6^\sigma, \mathbf{Z})$ denote the orthogonal of D with respect to q. Then

$$i^*(H^2(X_6^\sigma, \mathbf{Z})^{\perp D}) \subset \pi^* H^2(S, \mathbf{Z}).$$

PROOF. Since π is a conic fibration and $\ell \in H_2(D, \mathbf{Z})$ is the class of a fiber of π, we have $i^*H \cdot \ell = 2$. We can also compute $i^*D \cdot \ell$: by the adjunction formula and the fact that X_6^σ has trivial canonical class, we see that $i^*D = K_D$ is the canonical divisor of D; on the other hand, $\pi \colon D \to S$ being a conic fibration together with the fact that S has trivial canonical class shows that K_D is also the relative canonical class of π, which restricts to the canonical class on each fiber. So $i^*D \cdot \ell = -2$.

Consider $i_*\ell \in H_2(X_6^\sigma, \mathbf{C})$ as the class of a rational curve on X_6^σ which is of type $(3, 3)$. There exists a unique element $y \in H^2(X_6^\sigma, \mathbf{Q})$ such that

$$\forall x \in H^2(X_6^\sigma, \mathbf{Z}) \quad q(x, y) = x \cdot i_*\ell.$$

Moreover y must be of type $(1,1)$ so it is a $\mathbf{Q}$-linear combination of H and D. Since

$$q(D,D) = -2 = i^*D\cdot\ell = D\cdot i_*\ell,$$
$$q(H,D) = 2 = i^*H\cdot\ell = H\cdot i_*\ell,$$

we see that $y = D$. By Corollary 4.7.8, there exists a class $C \in H^2(X_6^\sigma,\mathbf{Z})$ such that $q(C,D) = 1$. We have $i^*C\cdot\ell = C\cdot i_*\ell = q(C,D) = 1$, so the class i^*C restricts to $\mathcal{O}(1)$ on each fiber ℓ of π. By the Leray–Hirsch theorem, the classes 1 and i^*C generate $H^*(D,\mathbf{Z})$ as a $H^*(S,\mathbf{Z})$-module, hence we have

$$H^*(D,\mathbf{Z}) = \pi^*H^*(S,\mathbf{Z}) \oplus \pi^*H^*(S,\mathbf{Z})(i^*C),$$

and in particular

$$H^2(D,\mathbf{Z}) = \pi^*H^2(S,\mathbf{Z}) \oplus \mathbf{Z}i^*C.$$

We may therefore choose i^*C as the class ζ that we want. For each class in $H^2(D,\mathbf{Z})$, its coefficient before i^*C is simply its intersection number with the fiber ℓ.

Any class $x \in H^2(X_6^\sigma,\mathbf{Z})$ with $q(x,D) = 0$ must satisfy $i^*x\cdot\ell = x\cdot i_*\ell = 0$. This shows that $i^*(H^2(X_6^\sigma,\mathbf{Z})^{\perp D})$ is indeed contained in $\pi^*H^2(S,\mathbf{Z})$. □

The intersection product on S can be pulled back to $\pi^*H^2(S,\mathbf{Z})$ via π^*. By the previous lemma, this also induces a form on $H^2(X_6^\sigma,\mathbf{Z})^{\perp D}$ via i^*, which we denote by q_S. In other words, for each $x \in H^2(X_6^\sigma,\mathbf{Z})^{\perp D}$, there exists a unique $u \in H^2(S,\mathbf{Z})$ such that $i^*x = \pi^*u$, and we define

$$q_S(x) \coloneqq q_S(u) = \int_S u\cdot u.$$

We can compare this form with the Beauville–Bogomolov–Fujiki form q.

Proposition 4.7.10. *Let σ be general in the divisor $\mathcal{D}^{1,6,10}$. For any $x \in H^2(X_6^\sigma,\mathbf{Z})^{\perp D}$, we have*

$$q(x) = q_S(x),$$

where q is the Beauville–Bogomolov–Fujiki form. As a consequence, the morphism i^* is injective.

Proof. By Lemma 4.7.9, for each $x \in H^2(X_6^\sigma,\mathbf{Z})^{\perp D}$, there exists a unique $u \in H^2(S,\mathbf{Z})$ such that $i^*x = \pi^*u$. We have $i^*x\cdot i^*x = \pi^*u\cdot\pi^*u = q_S(u)\ell = q_S(x)\ell$.

Consider the class $i^*D \in H^2(D,\mathbf{Z})$ and the class ℓ of a fiber of $\pi\colon D\to S$. By Lemma 4.7.9, since $i^*D\cdot\ell = -2$, we can write $i^*D = -2\zeta + \pi^*v$ for some $v \in H^2(S,\mathbf{Z})$. We then compute the intersection number using the fact that $\zeta\cdot\ell = 1$

$$\int_D i^*D\cdot i^*x\cdot i^*x = \int_D(-2\zeta+\pi^*v)\cdot q_S(x)\cdot\ell = -2q_S(x).$$

On the other hand, we have

$$\int_D i^*D\cdot i^*x\cdot i^*x = \int_{X_6^\sigma} D^2x^2 = q(D,D)q(x,x) + 2q(x,D)^2.$$

Since $q(x, D) = 0$ and $q(D, D) = -2$, we get the desired equality $q(x) = q_S(x)$.

This shows that i^* is injective when restricted to $H^2(X_6^\sigma, \mathbf{Z})^{\perp D}$. But any x such that $i^*x = 0$ will satisfy $i^*x \cdot \ell = 0$ so we have $q(x, D) = 0$ and hence $x = 0$. Thus i^* itself is injective. ☐

We see that the lattice $\big(H^2(X_6^\sigma, \mathbf{Z})^{\perp D}, q\big)$ embeds isometrically inside $\big(H^2(S, \mathbf{Z}), \cdot\big)$. It remains to determine the index of the embedding.

THEOREM 4.7.11. *For σ very general in the divisor $\mathcal{D}^{1,6,10}$, there is an embedding of integral Hodge structures*

$$\iota\colon \big(H^2(X_6^\sigma, \mathbf{Z})^{\perp D}, q\big) \hookrightarrow \big(H^2(S, \mathbf{Z}), \cdot\big)$$

as a sublattice of index 2. We have

$$\iota(H + D) = 2h,$$

where h is the polarization on S of degree 6. Restricted to the transcendental part, we get

$$\iota\colon \big(H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{trans}}, q\big) \hookrightarrow \big(H^2(S, \mathbf{Z})_{\mathrm{prim}}, \cdot\big)$$

again of index 2.

PROOF. By Corollary 4.7.8, the class D is of divisibility 1 in $H^2(X_6^\sigma, \mathbf{Z})$. As its orthogonal, the sublattice $H^2(X_6^\sigma, \mathbf{Z})^{\perp D}$ is of discriminant 4. On the other hand, $H^2(S, \mathbf{Z})$ is unimodular. Hence by comparing discriminants, the first statement follows.

Since the $(1,1)$ part of $H^2(X_6^\sigma, \mathbf{Z})^{\perp D}$ is generated by the class $H + D$ with square $q(H + D) = 24$, while the negative generator $-(H + D)$ is not effective, it is clear that $i^*(H + D)$ must be equal to $2\pi^*h$, so $\iota(H + D) = 2h$.

Finally, the second embedding follows by looking at the respective orthogonals of these two classes, while the index 2 is again obtained by comparing discriminants. ☐

It is possible to get a more precise description of the sublattice of index 2. We first define a class A in $H^2(S, \mathbf{Z})$ as follows. Consider the class C as in Corollary 4.7.8. Since $q(H - 2C, D) = 0$, we define

$$A \coloneqq \iota(H - 2C) \in H^2(S, \mathbf{Z}). \tag{4.28}$$

Note that, since $q(H) = 22$ and $q(C)$ is even, we have

$$A \cdot A = q(H - 2C, H - 2C) \equiv 6 \pmod 8,$$

so A is not divisible by 2. Also, for a very general $[\sigma] \in \mathcal{D}^{1,6,10}$ such that X_1^σ has Picard rank 2, we saw that the Picard lattice is generated by H and D, both having an even intersection number with D, so the class C is not algebraic. Since ι is a map of Hodge structures, the class A is also not algebraic.

Proposition 4.7.12. *For a very general $[\sigma] \in \mathcal{D}^{1,6,10}$, there exists a class $A \in H^2(S, \mathbf{Z})$ not divisible by 2 and not algebraic, that is, not a multiple of the polarization h, such that the lattice $H^2(X_6^\sigma, \mathbf{Z})^{\perp D}$ can be identified via the embedding ι as the sublattice*

$$\Lambda_{\frac{1}{2}A} \coloneqq \{u \in H^2(S, \mathbf{Z}) \mid u \cdot A \in 2\mathbf{Z}\},$$

while the sublattice $H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{trans}}$ can be identified as the sublattice

$$\Lambda_{\frac{1}{2}A,\mathrm{prim}} \coloneqq \{u \in H^2(S, \mathbf{Z})_{\mathrm{prim}} \mid u \cdot A \in 2\mathbf{Z}\}.$$

Proof. For each class $x \in H^2(X_6^\sigma, \mathbf{Z})^{\perp D}$, the intersection number

$$\iota(x) \cdot A = q(x, H - 2C) = q(x, H) - 2q(x, C)$$

is always even, because $\operatorname{div}(H) = 2$. So we get the inclusion in one direction. For the other direction: since the index is 2, it suffices to show that $\Lambda_{\frac{1}{2}A,\mathrm{prim}}$ is a proper sublattice of $H^2(S, \mathbf{Z})_{\mathrm{prim}}$, and the sublattice $\Lambda_{\frac{1}{2}A}$ will then also be proper in $H^2(S, \mathbf{Z})$.

Thus we search for a class $v \in H^2(S, \mathbf{Z})_{\mathrm{prim}}$ with $v \cdot A$ odd. First we claim that $h \cdot A$ is odd: this is equivalent to

$$2h \cdot A = q(H + D, H - 2C) = 22 - 2q(C, H) \equiv 2 \pmod 4,$$

which follows from the fact that the divisibility of H is 2.

Now since $H^2(S, \mathbf{Z})$ is unimodular, all classe of square 6 are in the same $\mathrm{O}\left(H^2(S, \mathbf{Z})\right)$-orbit. Hence we may assume that $h = e_1 + 3f_1$ for (e_1, f_1) a standard basis for a copy of the hyperbolic plane U in $H^2(S, \mathbf{Z})$, and take $v \coloneqq e_1 - 3f_1 \in H^2(S, \mathbf{Z})_{\mathrm{prim}}$. Since $(h + v) \cdot A = 2e_1 \cdot A$ is even, the intersection number $v \cdot A$ is odd as desired. ☐

We explain in the next section the interpretation of the class A in terms of a B-field lifting of a Brauer class $\beta \in \mathrm{Br}(S)$.

4.7.5. Moduli space of twisted sheaves. We first recall the notions of Brauer group and B-field lifting. We will only be interested in the case of K3 surfaces. We follow [**Huy16**, Chapter 18] (see also [**vG05**]).

The *Brauer group* of a K3 surface S can be characterized as the cohomology groups

$$\mathrm{Br}(S) \simeq H^2_{\text{ét}}(S, \mathbf{G}_m) \simeq H^2(S, \mathcal{O}_S^*)_{\mathrm{tors}}$$

in the algebraic and analytic context respectively. Since $H^3(S, \mathbf{Z}) = 0$, the exponential sequence allows us to have another description

$$\mathrm{Br}(S) \simeq \left(H^2(S, \mathbf{Z}) / \operatorname{Pic}(S)\right) \otimes (\mathbf{Q}/\mathbf{Z}) \simeq \operatorname{Hom}(\operatorname{Pic}(S)^\perp, \mathbf{Q}/\mathbf{Z}).$$

For an element β of $\mathrm{Br}(S)$, a representative $B \in H^2(S, \mathbf{Q})$ is called a *B-field lifting* of β.

Each β of order n in the Brauer group gives a morphism from the transcendental part $\operatorname{Pic}(S)^\perp$ (when S is of Picard rank 1, this coincides with $H^2(S, \mathbf{Z})_{\mathrm{prim}}$) to $\mathbf{Q}/\mathbf{Z}$, and the kernel is a sublattice of index n, which in particular does not depend on the choice of the B-field lifting.

In a more geometric setting, each class β gives a *Brauer–Severi variety* $\pi\colon X \to S$, which is a $\mathbf{P}^{n-1}$-fibration that is locally trivial in the étale topology. Equivalently, it is the projectivization $\mathbf{P}(\mathcal{E})$ of some β-twisted vector bundle $\mathcal{E}$ on S. We refer to [**HS05**] for the definitions of β-twisted coherent sheaves as well as the *twisted Chern classes* $c_i^B(E)$ and the *twisted Chern character* $\mathrm{ch}^B(E)$. We only emphasize that the definition of twisted Chern classes depends not just on β but also on the choice of a B-field lifting B.

Back to the situation of Section 4.7.4. In Proposition 4.7.12, we showed the existence of a class $A \in H^2(S, \mathbf{Z})$ not divisible by 2 and not algebraic. Hence the class $\frac{1}{2}A$ gives a Brauer class β of order 2, and the lattice $\Lambda_{\frac{1}{2}A,\mathrm{prim}}$ gives the index-2 sublattice defined by β, as explained above. This is the reason why we adopted the notation $\Lambda_{\frac{1}{2}A}$ instead of Λ_A.

We first find another B-field lifting that is easier to work with.

LEMMA 4.7.13. *There exists another B-field lifting B of the same Brauer class β such that $B \cdot B = B \cdot h = \frac{1}{2}$.*

PROOF. Recall that $\frac{1}{2}A \cdot \frac{1}{2}A \equiv \frac{1}{2}A \cdot h \equiv \frac{1}{2} \in \mathbf{Q}/\mathbf{Z}$. We try to find B by adding integral classes to $\frac{1}{2}A$.

Take (e_1, f_1) and (e_2, f_2) to be the standard bases of two copies of U inside $H^2(S, \mathbf{Z})$. Since $H^2(S, \mathbf{Z})$ is unimodular, we may assume that $h = e_1 + 3f_1$ and $A = ae_1 + bf_1 + ce_2 + df_2$. By adding e_1 and f_1, we can reduce the coefficients of e_1 and f_1 in $\frac{1}{2}A$ to 0 or $\frac{1}{2}$. The condition $\frac{1}{2}A \cdot h \equiv \frac{1}{2} \in \mathbf{Q}/\mathbf{Z}$ shows that we have either $\frac{1}{2}e_1 + 0f_1$ or $0e_1 + \frac{1}{2}f_1$. We may do the same for e_2 and f_2, and the condition $\frac{1}{2}A \cdot \frac{1}{2}A \equiv \frac{1}{2} \in \mathbf{Q}/\mathbf{Z}$ shows that we always get $\frac{1}{2}e_2 + \frac{1}{2}f_2$. In the two possible situations, we may choose B to be equal to either $\frac{1}{2}e_1 - f_1 + \frac{1}{2}e_2 + \frac{3}{2}f_2$ or $0e_1 + \frac{1}{2}f_1 + \frac{1}{2}e_2 + \frac{1}{2}f_2$. □

So the identifications in Proposition 4.7.12 become

$$\iota\colon H^2(X_6^\sigma, \mathbf{Z})^{\perp D} \xrightarrow{\sim} \Lambda_B \coloneqq \{u \in H^2(S, \mathbf{Z}) \mid u \cdot B \in \mathbf{Z}\}$$

and

$$\iota\colon H^2(X_6^\sigma, \mathbf{Z})_{\mathrm{trans}} \xrightarrow{\sim} \Lambda_{B,\mathrm{prim}} \coloneqq \{u \in H^2(S, \mathbf{Z})_{\mathrm{prim}} \mid u \cdot B \in \mathbf{Z}\}.$$

We remark that when we write $B = \frac{1}{2}A + u$, with $u \in H^2(S, \mathbf{Z})$, we have

$$2 = 2B \cdot 2B = A \cdot A + 4A \cdot u + 4u \cdot u.$$

Since $A \cdot A \equiv 6 \pmod 8$ while $u \cdot u$ is always even, we see that $A \cdot u$ is odd. In particular $u \neq 0$, so $B \neq \frac{1}{2}A$. Also, the intersection number $A \cdot B$ is even. This gives the following lemma that we will need shortly.

LEMMA 4.7.14. *Since $2B \cdot B = 1 \in \mathbf{Z}$, or equivalently $2B \in \Lambda_B$, we may set $2B = \iota(x_0)$ for some $x_0 \in H^2(X_6^\sigma, \mathbf{Z})^{\perp D}$. The class $D - x_0$ is divisible by 2 in $H^2(X_6^\sigma, \mathbf{Z})$.*

PROOF. It is equivalent to show that $D - x_0 + 2C$ is divisible by 2. We have

$$\iota(D - x_0 + 2C) = \iota((D + H) - x_0 - (H - 2C)) = 2(h - B - \tfrac{1}{2}A).$$

Now the class $h - B - \frac{1}{2}A$ is integral and the intersection number $(h - B - \frac{1}{2}A) \cdot B = -\frac{1}{2}A \cdot B$ is also integral since $A \cdot B$ is even. So $h - B - \frac{1}{2}A$ lies in Λ_B and thus comes from a class in $H^2(X_6^\sigma, \mathbf{Z})^{\perp D}$, and $D - x_0 + 2C$ is indeed divisible by 2. □

We now show that, for σ very general in $\mathcal{D}^{1,6,10}$, the projective bundle $\pi\colon D \to S$ is precisely the Brauer–Severi variety for the Brauer class β, which means that the Brauer class that we obtained Hodge-theoretically actually comes from geometry. In particular, for σ very general, the bundle $\pi\colon D \to S$ has a non-trivial Brauer class.

PROPOSITION 4.7.15. *For σ very general in the divisor $\mathcal{D}^{1,6,10}$, the $\mathbf{P}^1$-fibration $\pi\colon D \to S$ is the Brauer–Severi variety for the Brauer class β.*

PROOF. Recall that $K_{D/S} = K_D \otimes \pi^* K_S^{-1} = i^* D$, since X_6^σ and S both have trivial canonical bundles.

Denote by β' the Brauer class defined by $\pi\colon D \to S$. We may suppose that $D = \mathbf{P}(\mathcal{E})$ with $\mathcal{E}$ a β'-twisted vector bundle on S of rank 2. The relative $\mathcal{O}(1)$ is a $(-\pi^*\beta')$-twisted line bundle on D, and its square $\mathcal{O}(2)$ is a non-twisted line bundle. Moreover, $\mathcal{O}(2) = \omega_{D/S}^\vee \otimes \pi^* \mathcal{L}$ for some line bundle $\mathcal{L}$ on S. We may set $c_1(\mathcal{L}) = kh$ for $k \in \mathbf{Z}$, since for very general σ in the divisor, the K3 surface S has Picard number 1. So the first Chern class $c_1(\mathcal{O}(2))$ is equal to $-i^*D + k(\pi^*h)$.

Consider a B-field lifting B' of β'. We compute the twisted Chern class

$$2c_1^{-\pi^*B'}(\mathcal{O}(1)) = c_1^{-2(\pi^*B')}(\mathcal{O}(2)) = c_1(\mathcal{O}(2)) - 2(\pi^*B') = -i^*D + k(\pi^*h) - 2(\pi^*B').$$

The class $c_1^{-\pi^*B'}(\mathcal{O}(1))$ is necessarily integral, so the last term in the equality is divisible by 2. On the other hand, $i^*(D - x_0) = i^*D - 2\pi^*B$ is also divisible by 2 by Lemma 4.7.14. Thus the class $\frac{k}{2}h - B' - B$ is integral in $H^2(S, \mathbf{Z})$, which shows that $\beta' = \beta^{-1} = \beta$. □

Finally, we consider the moduli space of twisted sheaves on S, following [**MS12**, Section 3]. From now on we will always assume that S is of Picard rank 1. We recall the definition of the twisted Mukai lattice. Consider the map

$$\begin{aligned} \eta_B\colon H^2(S, \mathbf{C}) &\longrightarrow H^*(S, \mathbf{C}) \\ u &\longmapsto (0, u, u \cdot B). \end{aligned}$$

The *twisted Mukai lattice* $\widetilde{H}(S, B, \mathbf{Z})$ is given by the usual Mukai lattice $H^*(S, \mathbf{Z}) \coloneqq H^0(S, \mathbf{Z}) \oplus H^2(S, \mathbf{Z}) \oplus H^4(S, \mathbf{Z})$, equipped with the Hodge structure given by η_B, that is, its $(2,0)$-part is the image of $H^{2,0}(S)$ under η_B. We recall that the *Mukai pairing* is given by

$$-\chi\big((r_1, c_1, s_1), (r_2, c_2, s_2)\big) \coloneqq c_1 \cdot c_2 - r_1 s_2 - r_2 s_1.$$

Assuming that S is of Picard rank 1, the twisted Picard lattice $\mathrm{Pic}(S,B) \subset \widetilde{H}(S,B,\mathbf{Z})$ is generated by the classes $(2,2B,0)$, $(0,h,0)$, and $(0,0,1)$. For a β-twisted sheaf E, its *twisted Mukai vector* is defined as

$$v^B(E) \coloneqq \mathrm{ch}^B(E)\cdot\sqrt{\mathrm{td}(S)},$$

where ch^B is the twisted Chern character. Let $M = M(S,v,B)$ be the moduli space of stable β-twisted sheaves $\mathcal{E}$ on S with Mukai vector $v^B(\mathcal{E}) = v \coloneqq (2,2B,0)$. Here $v^2 = 2$, so by the general theory for moduli of twisted sheaves on K3 surfaces, M is a hyperkähler fourfold, with $H^2(M,\mathbf{Z})$ isometric to $v^\perp \subset \widetilde{H}(S,B,\mathbf{Z})$, the orthogonal of v in the twisted Mukai lattice.

Proposition 4.7.16. *For very general $[\sigma] \in \mathcal{D}^{1,6,10}$, there exists a Hodge isometry between $H^2(X_6^\sigma,\mathbf{Z})$ and $H^2(M,\mathbf{Z})$.*

Proof. We take σ to be very general so that the K3 surface S is of Picard rank 1. In this case, the twisted Picard lattice $\mathrm{Pic}(S,B)$ is generated by the classes $(2,2B,0)$, $(0,h,0)$, and $(0,0,1)$. Therefore the lattice $H^2(M,\mathbf{Z})_{\mathrm{trans}} = \mathrm{Pic}(S,B)^\perp$ consists of elements (a,u,b) satisfying

$$\begin{aligned}
-\chi\big((a,u,b),(2,2B,0)\big) &= 2u\cdot B - 2b = 0\\
-\chi\big((a,u,b),(0,h,0)\big) &= u\cdot h = 0\\
-\chi\big((a,u,b),(0,0,1)\big) &= -a = 0
\end{aligned}$$

which are precisely those in the image of $\Lambda_{B,\mathrm{prim}}$ by the map $\eta_B\colon u \mapsto (0,u,u\cdot B)$. Since we have identified $\Lambda_{B,\mathrm{prim}}$ with $H^2(X_6^\sigma,\mathbf{Z})_{\mathrm{trans}}$ via the isometry ι, we can thus define a map

$$\begin{aligned}
\phi\colon H^2(X_6^\sigma,\mathbf{Z})_{\mathrm{trans}} &\longrightarrow H^2(M,\mathbf{Z})_{\mathrm{trans}}\\
x &\longmapsto \eta_B(\iota(x)) = (0,\iota(x),\iota(x)\cdot B)
\end{aligned}$$

This is a Hodge isometry onto its image, since

$$q(x) = \iota(x)\cdot\iota(x) = -\chi\big(\phi(x),\phi(x)\big),$$

where the first equality is proved in Theorem 4.7.11.

For the algebraic part, we may set $\phi(H) = (-2, 2h-2B, 0)$ and $\phi(D) = (2,2B,1)$. It suffices now to extend ϕ to the full lattice. First we notice that the direct sum $H^2(M,\mathbf{Z})_{\mathrm{trans}} \oplus (\mathbf{Z}\phi(H) + \mathbf{Z}\phi(D))$ is of index 24 in $H^2(M,\mathbf{Z})$. We claim that the quotient is $\mathbf{Z}/24\mathbf{Z}$ by finding a primitive class in the direct sum which becomes divisible by 24 in the full lattice. Consider the integral class $h - 12B \in H^2(S,\mathbf{Z})_{\mathrm{prim}}$. Its intersection number with B is not integral, so it is not in $\Lambda_{B,\mathrm{prim}}$. Thus $u_1 \coloneqq 2h - 24B$ is primitive in $\Lambda_{B,\mathrm{prim}}$. We have $\eta_B(u_1) = (0, 2h-24B, -11)$ so $\eta_B(u_1) - \phi(H) + 11\phi(D) = (24,0,0)$ is indeed divisible by 24 in the full lattice.

Denote by $x_1 \in H^2(X_6^\sigma,\mathbf{Z})_{\mathrm{trans}}$ the preimage $\iota^{-1}(u_1)$ of u_1. Recall from Lemma 4.7.14 that there exists a class $x_0 \in H^2(X_6^\sigma,\mathbf{Z})^{\perp D}$ with $\iota(x_0) = 2B$. Moreover, the class $D - x_0$ is divisible by 2. Since $\iota(H+D) = 2h$ by Theorem 4.7.11, we have

$$x_1 = \iota^{-1}(u_1) = \iota^{-1}(2h - 24B) = (H+D) - 12x_0.$$

So the class

$$x_1 - H + 11D = 12(D - x_0)$$

is also divisible by 24, and we may extend the map ϕ to the full lattice by mapping $\frac{1}{2}(D-x_0)$ to $(1,0,0)$. □

Now that we have defined a Hodge isometry between the second cohomologies of the hyperkähler fourfolds X_6^σ and M, we may take advantage of the powerful machinery of the Torelli theorem to obtain the following result.

THEOREM 4.7.17. *A very general Debarre–Voisin fourfold X_6^σ in the family $\mathcal{C}_{24}$ is isomorphic to the moduli space $M = M(S, v, B)$ of twisted sheaves with Mukai vector $(2, 2B, 0)$ on the twisted K3 surface (S, β).*

PROOF. By the Torelli theorem, the existence of a Hodge isometry between second cohomologies shows that X_6^σ and M are birationally isomorphic. Moreover, the number of birational models is given by the number of chambers contained in the movable cone. These chambers are cut out by hyperplanes of the type $\kappa^\perp$, where $\kappa \in \mathrm{Pic}(X_6^\sigma)$ is a primitive class of square -10 and divisibility 2 (see Remark 3.4.6). We show that for a very general X_6^σ with Picard group generated by H and D, there is no such class κ: we may write $\kappa = aH + bD$ and get the equation $22a^2 + 4ab - 2b^2 = -10$. By reduction modulo 5, we verify that this equation has no integral solutions, so no such κ exists. Thus we may conclude that there is only one birational model, and in particular $X_6^\sigma \simeq M$. □

REMARK 4.7.18. As already pointed out in the introduction, the Heegner divisor $\mathcal{D}_{24}$ provides an explicit example for the divisor denoted by $\mathcal{D}^{(1)}_{24,24,\beta}$ in **[KvG21]** (see their Table 1 for the notation): we have $B \cdot B = B \cdot h = \frac{1}{2}$, as shown in Lemma 4.7.13. The intersection matrix appearing in Lemma 4.7.7 is diagonalized in the basis $\langle H+D, D\rangle$, where it becomes $\left(\begin{smallmatrix} 24 & 0 \\ 0 & -2 \end{smallmatrix}\right)$; therefore the class $H + D$ gives the contraction of the conic bundle $D \to S$ by *loc. cit.* Proposition 3.5. Then our Theorem 4.7.17 could also have been deduced by combining *loc. cit.* Proposition 3.5 and Theorem 4.2. Notice finally that a crucial element for our result is Theorem 4.7.11, which is a particular case of *loc. cit.* Proposition 4.6.

4.8. The Heegner divisor of degree 22

We give a description for the singularities of a general singular Debarre–Voisin variety. In Section 4.3.1, we have seen that the class of the trivector σ defining a general such Debarre–Voisin variety X_6^σ lies in the divisor $\mathcal{D}^{3,3,10}$: there exists a unique 3-dimensional subspace $V_3 \subset V_{10}$ such that σ satisfies the degeneracy condition $\sigma(V_3, V_3, V_{10}) = 0$. Under the period map, this divisor is mapped birationally to the Heegner divisor $\mathcal{D}_{22}$ in the period domain. We also obtained a set-theoretical description of the singular locus of X_6^σ in Proposition 4.3.3.

We prove the following stronger result, following the idea in [**Has00**, Lemma 6.3.1], where a similar result is proved for the variety of lines of a nodal cubic hypersurface. We shall see that the two cases share some surprising similarities.

PROPOSITION 4.8.1. *Consider a general* $[\sigma] \in \mathcal{D}^{3,3,10}$*. For the associated Debarre–Voisin variety* X_6^σ*, the singularities along the degree* 22 *K3 surface* S *are codimension-*2 *ordinary double points. More precisely, by blowing up the singular locus* S*, we get a smooth hyperkähler fourfold of* $\mathrm{K3}^{[2]}$*-type, and the exceptional divisor is a conic fibration over* S*.*

PROOF. We briefly recall the argument for the nodal cubic: for a cubic $X \subset \mathbf{P}^5 = \mathbf{P}(V_6)$ containing a node $p := [V_1]$, the projectivized tangent cone $\mathbf{P}\mathcal{T}_pX$ is a quadric hypersurface Q in $\mathbf{P}\mathcal{T}_p\mathbf{P}^5 = \mathbf{P}(V_6/V_1)$, and the varieties of lines $F \subset \mathrm{Gr}(2, V_6)$ is singular along a K3 surface S parametrizing lines in X passing through p. Instead of blowing up S in F, Hassett considered studying the ambient Grassmannian $\mathrm{Gr}(2, V_6)$ and blowing up the Schubert variety $\Sigma := \mathbf{P}(V_6/V_1) \subset \mathrm{Gr}(2, V_6)$, which parametrizes all lines in $\mathbf{P}(V_6)$ passing through p. This gives the following Cartesian diagram

$$\begin{array}{ccc} \widetilde{F} := \mathrm{Bl}_S F & \hookrightarrow & \mathrm{Bl}_\Sigma \mathrm{Gr}(2, V_6) \\ \downarrow & & \downarrow \\ F & \hookrightarrow & \mathrm{Gr}(2, V_6). \end{array}$$

For a given point $x := [V_2] \in S$, we get one distinguished point $y := [V_2/V_1]$ in $\mathbf{P}^4 = \mathbf{P}(V_6/V_1)$ that lies on the quadric Q. The projectivized normal space $\mathbf{P}\mathcal{N}_{\Sigma/\mathrm{Gr}(2,V_6),x}$ can be identified with $\mathbf{P}(V_6/V_2)$, which parametrizes lines in $\mathbf{P}^4 = \mathbf{P}(V_6/V_1)$ passing through the point y, and the projectivized normal cone $\mathbf{P}C_{S,x}F$ is given by the subscheme parametrizing such lines that are also entirely contained in the quadric threefold Q, in other words, lines in Q passing through a given point. This condition gives a smooth conic curve, so the singularities of F along S are indeed codimension-2 ordinary double points.

We use a similar argument to study the singular Debarre–Voisin variety X_6^σ. By assumption, the hyperplane section X_3^σ admits an ordinary double point at $[V_3]$, so its tangent cone at $[V_3]$ is a smooth quadric hypersurface Q in the projectivization of the tangent space

$$\mathbf{P}\mathcal{T}_{[V_3]} \mathrm{Gr}(3, V_{10}) \simeq \mathbf{P}\,\mathrm{Hom}(V_3, V_{10}/V_3) =: \mathbf{P}(T_{21}) = \mathbf{P}^{20}.$$

For a given $x := [V_6] \in S$, the projective space $\mathbf{P}\,\mathrm{Hom}(V_3, V_6/V_3) := \mathbf{P}(T_9) = \mathbf{P}^8$ gives a distinguished linear subspace contained in Q.

Following the proof of Hassett, instead of blowing up S in X_6^σ, we consider the ambient Grassmannian $\mathrm{Gr}(6, V_{10})$ and blow up the entire Schubert variety

$$\Sigma := \{[V_6] \in \mathrm{Gr}(6, V_{10}) \mid V_6 \supset V_3\} \simeq \mathrm{Gr}(3, V_{10}/V_3),$$

which is smooth of codimension 12. We have the following description for its normal bundle in $\mathrm{Gr}(6, V_{10})$:

$$\mathcal{N}_{\Sigma/\mathrm{Gr}(6,V_{10})} = \mathrm{Hom}(\mathcal{U}_6, \mathcal{Q}_{10/6})/\mathrm{Hom}(\mathcal{U}_6/V_3, \mathcal{Q}_{10/6}) \simeq \mathrm{Hom}(V_3, \mathcal{Q}_{10/6}),$$

where we denote by $\mathcal{U}_6$ and $\mathcal{Q}_{10/6}$ the restrictions to Σ of the two tautological bundles on $\mathrm{Gr}(6, V_{10})$. For the given point $x \in S$, the projectivization of the normal space is therefore an 11-dimensional projective space

$$\mathbf{P}\mathcal{N}_{\Sigma/\mathrm{Gr}(6,V_{10}),x} \simeq \mathbf{P}\,\mathrm{Hom}(V_3, V_{10}/V_6) \simeq \mathbf{P}(T_{21}/T_9),$$

where we recall that T_{21} is the tangent space of $\mathrm{Gr}(3, V_{10})$ at $[V_3]$, and T_9 is the tangent space of $\mathrm{Gr}(3, V_6)$ at $[V_3]$, viewed as a subspace of T_{21}.

Consider the proper transform of X_6^σ denoted by $\widetilde{X}_6^\sigma$. We have the following Cartesian diagram

$$\begin{array}{ccc} \widetilde{X}_6^\sigma & \hookrightarrow & \mathrm{Bl}_\Sigma \mathrm{Gr}(6, V_{10}) \\ \downarrow & & \downarrow \\ X_6^\sigma & \hookrightarrow & \mathrm{Gr}(6, V_{10}). \end{array}$$

Consequently we get a natural closed embedding of the projectivized normal cone

$$\mathbf{P}C_{S,x}X_6^\sigma \hookrightarrow \mathbf{P}\mathcal{N}_{\Sigma/\mathrm{Gr}(6,V_{10}),x} \simeq \mathbf{P}(T_{21}/T_9).$$

The total projective space $\mathbf{P}(T_{21}/T_9)$ parametrizes 9-dimensional linear subspaces of $\mathbf{P}(T_{21})$ that contains the distinguished $\mathbf{P}^8 = \mathbf{P}(T_9)$, and the projectivized normal cone $\mathbf{P}C_{S,x}X_6^\sigma$ can then be identified with the subscheme that parametrizes such $\mathbf{P}^9$ that are also contained in the quadric Q. In other words, it parametrizes 9-dimensional linear subspaces in a 19-dimensional quadric containing a fixed $\mathbf{P}^8$. This is again a smooth conic curve, just like in the nodal cubic case. Thus the singularities of X_6^σ along S are indeed codimension-2 ordinary double points, and $\widetilde{X}_6^\sigma$ is smooth.

Finally, we show that the resolution $\widetilde{X}_6^\sigma$ that we obtained has trivial canonical class. Since X_6^σ is birational to the Hilbert square $S^{[2]}$, this will then force $\widetilde{X}_6^\sigma$ to be a smooth hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type.

We denote by E the exceptional divisor for the blowup $\mathrm{Bl}_\Sigma \mathrm{Gr}(6, V_{10}) \to \mathrm{Gr}(6, V_{10})$, and by D the exceptional divisor for the blowup $\widetilde{X}_6^\sigma \to X_6^\sigma$. The divisor D can be identified with the projectivized normal cone $\mathbf{P}C_S X_6^\sigma$, so the morphism $D \to S$ is a conic fibration by the above analysis. By construction, the Zariski open subset $\widetilde{X}_6^\sigma \smallsetminus D$ is isomorphic to the smooth locus $X_6^\sigma \smallsetminus S$. The latter has trivial canonical class since it is the regular zero-locus of σ viewed as a section of the vector bundle $\bigwedge^3 \mathcal{U}_6^\vee$. Therefore, the canonical divisor $K_{\widetilde{X}_6^\sigma}$ is linearly equivalent to some multiple of D. We write $K_{\widetilde{X}_6^\sigma} = mD$, and it remains to show that $m = 0$.

Since $D \to S$ is a smooth conic fibration in the projectivized normal bundle $E \to \Sigma$, the relative $\mathcal{O}(-1)$ of $E \to \Sigma$ restricts to the relative canonical bundle of $D \to S$. Note that by the Leray–Hirsch theorem, this bundle is necessarily non-trivial. Since E is the exceptional divisor, the relative $\mathcal{O}(-1)$ on E is given by $\mathcal{O}_E(E)$, so we have

$$\omega_{D/S} \simeq \mathcal{O}_E(E)|_D.$$

Using the fact that S is a K3 surface and that $\mathcal{O}_E(E)|_{\widetilde{X}_6^\sigma} \simeq \mathcal{O}_{\widetilde{X}_6^\sigma}(D)$, this gives

$$\omega_D \simeq \omega_{D/S} \simeq \mathcal{O}_{\widetilde{X}_6^\sigma}(D)|_D.$$

Hence we obtain

$$K_D = D|_D,$$

which in particular must be non-trivial since $\omega_{D/S}$ is non-trivial.

On the other hand, by the adjunction formula we have

$$K_D \simeq (K_{\widetilde{X}_6^\sigma} + D)|_D = (m+1)D|_D.$$

Thus we may conclude that $m = 0$, and $K_{\widetilde{X}_6^\sigma}$ is indeed trivial. □

REMARK 4.8.2. Contrary to the nodal cubic case, the resolution $\widetilde{X}_6^\sigma$ obtained is not isomorphic to the Hilbert scheme $S^{[2]}$, even for a generic member of the family. This can be seen by studying the chamber decomposition for a generic $S^{[2]}$ with Picard rank 2: one may find exactly two chambers in the movable cone, corresponding to $S^{[2]}$ and a second birational model; the Plücker polarization pulled back to $S^{[2]}$ via the birational map is equal to $10H - 33\delta$ and is not nef (see for example [**DHOV20**, Table 1]), so we may conclude that $\widetilde{X}_6^\sigma$ is the second birational model. The two models are related by a Mukai flop. It would be interesting to see this flop geometrically.

CHAPTER 5

A special Debarre–Voisin variety

In this chapter, we study a special Debarre–Voisin fourfold with a large automorphism group, using the general results obtained in Chapter 4.

The results of this chapter have appeared in [**Son21**].

5.1. Introduction

Let $\mathbf{G}$ be the finite simple group $\mathrm{PSL}(2, \mathbf{F}_{11})$. Mongardi discovered a special Eisenbud–Popescu–Walter sextic with a faithful $\mathbf{G}$-action, from which one can construct a double EPW sextic that is a smooth hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type and is highly symmetric (see [**Mon13**] and [**DM21**]).

One can ask the same question for hyperkähler fourfolds of Debarre–Voisin type. Let V_{10} be a 10-dimensional complex vector space. We recall that a *Debarre–Voisin variety* X_6^σ is defined inside the Grassmannian $\mathrm{Gr}(6, V_{10})$ from the datum of a trivector $\sigma \in \bigwedge^3 V_{10}^\vee$. By studying the representations of the group $\mathbf{G}$, it is not hard to find a candidate for σ: denote by V_{10} one of the two 10-dimensional irreducible representations of $\mathbf{G}$; there exists a unique (up to multiplication by a scalar) trivector $\sigma_0 \in \bigwedge^3 V_{10}^\vee$ that is $\mathbf{G}$-invariant.

Using the general results obtained in Chapter 4 on the geometry of Debarre–Voisin varieties and associated Peskine varieties, one can study in detail the geometry of this special Debarre–Voisin variety $X_6^{\sigma_0}$. We prove the following results.

THEOREM 5.1.1. *Let $\sigma_0 \in \bigwedge^3 V_{10}^\vee$ be the special $\mathbf{G}$-invariant trivector.*

(1) (Proposition 5.3.1) The Debarre–Voisin variety $X_6^{\sigma_0} \subset \mathrm{Gr}(6, V_{10})$ is smooth of dimension 4.
(2) (Proposition 5.4.1) The associated Peskine variety $X_1^{\sigma_0} \subset \mathbf{P}(V_{10})$ has 55 *isolated singular points. The group $\mathbf{G}$ acts transitively on them.*
(3) (Proposition 5.5.5) The group $\mathrm{Aut}_H^s(X_6^{\sigma_0})$ of symplectic automorphisms that fix the polarization H on $X_6^{\sigma_0}$ is isomorphic to $\mathbf{G}$.
(4) One can give an explicit description of the Picard lattice of $X_6^{\sigma_0}$, which has maximal rank 21 *(see* (5.4) *for the Gram matrix and Proposition 5.5.8 for the isomorphism type).*

Notation. We use σ to denote a general trivector and σ_0 to denote the special $\mathbf{G}$-invariant trivector.

5.2. The special trivector

We first give the construction of the special trivector $\sigma_0 \in \bigwedge^3 V_{10}^\vee$.

The finite simple group $\mathbf{G} := \mathrm{PSL}(2, \mathbf{F}_{11})$ of order 660 admits 8 different irreducible complex representations: two of them are of dimension 5 and will be denoted by V_5 and $V_5^\vee$. They are the dual to each other.

A classical result is that the symmetric power $\mathrm{Sym}^3 V_5^\vee$—the space of cubic polynomials on V_5—admits an irreducible subrepresentation of dimension 1: for a suitable choice of basis $(y_0, \dots, y_4)$ of $V_5^\vee$, this corresponds to the Klein cubic with equation

$$y_0^2 y_1 + y_1^2 y_2 + y_2^2 y_3 + y_3^2 y_4 + y_4^2 y_0 \in \mathrm{Sym}^3 V_5^\vee.$$

Adler showed in **[Adl78]** that the automorphism group of this smooth cubic is precisely the group $\mathbf{G}$.

The wedge product $\bigwedge^2 V_5$ gives another irreducible representation, of dimension 10, which is self-dual and will be denoted by V_{10}. We consider elements of $\bigwedge^3 V_{10}^\vee$. A computation of characters tells us that this representation of $\mathbf{G}$ also admits one irreducible subrepresentation of dimension 1, generated by a $\mathbf{G}$-invariant trivector σ_0.

Below we provide the character table for the irreducible complex representations of $\mathbf{G}$.[1] Note that the other irreducible representation V'_{10} of dimension 10 does not provide $\mathbf{G}$-invariant trivectors.

Conj. class	[Id]	$[(\begin{smallmatrix}1&1\\0&1\end{smallmatrix})]$	$[(\begin{smallmatrix}1&2\\0&1\end{smallmatrix})]$	$[(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix})]$	$[(\begin{smallmatrix}0&1\\-1&-1\end{smallmatrix})]$	$[(\begin{smallmatrix}2&-2\\2&4\end{smallmatrix})]$	$[(\begin{smallmatrix}4&0\\0&3\end{smallmatrix})]$	$[(\begin{smallmatrix}5&0\\0&9\end{smallmatrix})]$
Size	1	60	60	55	110	110	132	132
$\mathbf{C}$	1	1	1	1	1	1	1	1
V_5	5	$\frac{1}{2}\sqrt{-11} - \frac{1}{2}$	$-\frac{1}{2}\sqrt{-11} - \frac{1}{2}$	1	-1	1	0	0
$V_5^\vee$	5	$-\frac{1}{2}\sqrt{-11} - \frac{1}{2}$	$\frac{1}{2}\sqrt{-11} - \frac{1}{2}$	1	-1	1	0	0
V_{10}	10	-1	-1	-2	1	1	0	0
V'_{10}	10	-1	-1	2	1	-1	0	0
V_{11}	11	0	0	-1	-1	-1	1	1
V_{12}	12	1	1	0	0	0	$\frac{1}{2}\sqrt{5} - \frac{1}{2}$	$-\frac{1}{2}\sqrt{5} - \frac{1}{2}$
V'_{12}	12	1	1	0	0	0	$-\frac{1}{2}\sqrt{5} - \frac{1}{2}$	$\frac{1}{2}\sqrt{5} - \frac{1}{2}$
$\bigwedge^3 V_{10}^\vee$	120	-1	-1	8	3	-1	0	0

TABLE 4. Character table of $\mathbf{G} = \mathrm{PSL}(2, \mathbf{F}_{11})$

We now give a concrete description of the special trivector σ_0 in terms of coordinates in a suitable basis. The subgroup $\mathbf{B}$ of $\mathbf{G}$ of upper triangular matrices can be generated by the elements

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix},$$

[1] The character table can be easily computed using `Sage` or `GAP`.

of respective orders 11 and 5. Write $\zeta = e^{2\pi i/11}$ and $\rho\colon \mathbf{G} \to \mathrm{GL}(V_5^\vee)$ for the representation $V_5^\vee$. In a suitable basis $(y_0, \dots, y_4)$ of $V_5^\vee$, the matrices of P and R are

$$\rho(P) = \begin{pmatrix} \zeta^1 & 0 & 0 & 0 & 0 \\ 0 & \zeta^9 & 0 & 0 & 0 \\ 0 & 0 & \zeta^4 & 0 & 0 \\ 0 & 0 & 0 & \zeta^3 & 0 \\ 0 & 0 & 0 & 0 & \zeta^5 \end{pmatrix} \quad \text{and} \quad \rho(R) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{5.1}$$

Note that one can already identify the equation of the $\mathbf{G}$-invariant Klein cubic using only these two elements, instead of the whole group $\mathbf{G}$.

The elements $y_{ij} \coloneqq y_i \wedge y_j$ form a basis of $V_{10}^\vee$. In this basis, we see that P acts diagonally and R as a permutation (see Table 5; note that we have chosen a particular order in which the action of R is very simple).

	y_{01}	y_{12}	y_{23}	y_{34}	y_{40}	y_{02}	y_{13}	y_{24}	y_{30}	y_{41}
Eigenvalues of $\bigwedge^2\rho(P)$	ζ^{10}	ζ^2	ζ^7	ζ^8	ζ^6	ζ^5	ζ^1	ζ^9	ζ^4	ζ^3
Action of $\bigwedge^2\rho(R)$	y_{12}	y_{23}	y_{34}	y_{40}	y_{01}	y_{13}	y_{24}	y_{30}	y_{41}	y_{02}

TABLE 5. The action of P and R in the basis (y_{ij})

We may easily verify that the space of trivectors invariant under the action of P and R is of dimension 2 and is spanned by the $\mathbf{B}$-invariant trivectors

$$\sigma_1 \coloneqq y_{01} \wedge y_{23} \wedge y_{02} + y_{12} \wedge y_{34} \wedge y_{13} + y_{23} \wedge y_{40} \wedge y_{24} + y_{34} \wedge y_{01} \wedge y_{30} + y_{40} \wedge y_{12} \wedge y_{41},$$

$$\sigma_2 \coloneqq y_{01} \wedge y_{41} \wedge y_{24} + y_{12} \wedge y_{02} \wedge y_{30} + y_{23} \wedge y_{13} \wedge y_{41} + y_{34} \wedge y_{24} \wedge y_{02} + y_{40} \wedge y_{30} \wedge y_{13}.$$

To identify the unique $\mathbf{G}$-invariant trivector, we must look at some elements in $\mathbf{G} \smallsetminus \mathbf{B}$. Since the explicit description for the representation V_5 is available at the ATLAS of finite group representations [**ATLAS**], we will pick one such element and compute explicitly its matrix.

The group $\mathbf{G}$ admits a presentation with two generators a, b and relations $a^2 = b^3 = (ab)^{11} = [a, babab]^2 = 1$, which can be take to be $a = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$. One may check that $ab = P$ while $bbabababbababab b = R$. Matrices for $\rho(a)$ and $\rho(b)$ are provided by the ATLAS, so the representation is completely determined. Choose a suitable basis of $V_5^\vee$ consisting of eigenvectors of $\rho(ab) = \rho(P)$. In this basis, the matrices of P and R are as in (5.1). Since the element a does not lie in the subgroup $\mathbf{B}$, we use its matrix in this new basis to verify that the unique (up to multiplication by a scalar) $\mathbf{G}$-invariant trivector is $\sigma_0 \coloneqq \sigma_1 + \sigma_2$.

From now on, we will rewrite the basis (y_{ij}) as $(x_0, \dots, x_9)$ in the order chosen in Table 5, so the trivector σ_0 is given by

$$\begin{aligned} \sigma_0 = {} & x_0 \wedge x_2 \wedge x_5 + x_1 \wedge x_3 \wedge x_6 + x_2 \wedge x_4 \wedge x_7 + x_3 \wedge x_0 \wedge x_8 + x_4 \wedge x_1 \wedge x_9 \\ & + x_0 \wedge x_9 \wedge x_7 + x_1 \wedge x_5 \wedge x_8 + x_2 \wedge x_6 \wedge x_9 + x_3 \wedge x_7 \wedge x_5 + x_4 \wedge x_8 \wedge x_6, \end{aligned}$$

or more succinctly,

$$\sigma_0 = [025] + [136] + [247] + [308] + [419] + [097] + [158] + [269] + [375] + [486]. \tag{5.2}$$

We have therefore shown the following result.

PROPOSITION 5.2.1. *Up to multiplication by a scalar, the trivector σ_0 in* (5.2) *is the unique* $\mathbf{G}$*-invariant trivector in* $\bigwedge^3 V_{10}^\vee$, *where* V_{10} *is the* 10*-dimensional irreducible* $\mathbf{G}$*-representation given in Table 4.*

5.3. The Debarre–Voisin fourfold

The Debarre–Voisin variety associated with a non-zero trivector σ is the scheme

$$X_6^\sigma \coloneqq \{[V_6] \in \mathrm{Gr}(6, V_{10}) \mid \sigma|_{V_6} = 0\}$$

in the Grassmannian $\mathrm{Gr}(6, V_{10})$ parametrizing those $[V_6]$ on which σ vanishes. Its expected dimension is 4. For σ general, it is shown in **[DV10]** that X_6^σ is a smooth hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type. The Plücker polarization on $\mathrm{Gr}(6, V_{10})$ induces a polarization H on X_6^σ, which is primitive and of Beauville–Bogomolov–Fujiki square 22 and divisibility 2.

The variety

$$X_3^\sigma \coloneqq \{[V_3] \in \mathrm{Gr}(3, V_{10}) \mid \sigma|_{V_3} = 0\}$$

is the Plücker hyperplane section associated with σ. It has dimension 20. Recall that we have obtained the criterion for the smoothness of X_3^σ and X_6^σ in Proposition 4.1.1.

In the case of σ_0, the smoothness of $X_3^{\sigma_0}$ can be verified directly using computer algebra (thanks to Frédéric Han for his help with this computation).

PROPOSITION 5.3.1. *For the special trivector σ_0 defined in* (5.2), *the hyperplane section $X_3^{\sigma_0}$ and hence the special Debarre–Voisin $X_6^{\sigma_0}$ are both smooth.*

PROOF. A direct check of the smoothness of $X_3^{\sigma_0}$ using its ideal is not feasible, since there are too many variables and equations. Instead, we can check the smoothness on each chart of $\mathrm{Gr}(3, V_{10})$ where it is defined by one single cubic polynomial in an affine space $\mathbf{A}^{21}$, using the Jacobian criterion. We provide the `Macaulay2` code.

```
-- the trivector has ten components
comps = {(0,2,5),(1,3,6),(2,4,7),(3,0,8),(4,1,9),
         (0,9,7),(1,5,8),(2,6,9),(3,7,5),(4,8,6)};
sigma = (u,v,w) -> sum(for idx in comps list det submatrix(u||v||w, idx));
-- check that the hyperplane section is smooth on each chart of Gr(3,10)
S = QQ[g_0..g_20]; -- each chart has 21 coordinates
I3 = id_(S^3); M = genericMatrix(S,3,7);
-- generate the coordinates matrix for the chart ijk
chart = (i,j,k) ->
    M_{0..i-1}|I3_{0}|M_{i..j-2}|I3_{1}|M_{j-1..k-3}|I3_{2}|M_{k-2..6};
isSmooth = true;
for ijk in subsets(10,3) do (Mijk = chart toSequence ijk;
```

```
    if dim singularLocus ideal sigma(Mijk^{0},Mijk^{1},Mijk^{2}) >= 0
        then isSmooth = false;);
assert(isSmooth);
```

Since X_3^σ is smooth, there is no V_3 satisfying the degeneracy condition $\sigma(V_3, V_3, V_{10}) = 0$, hence X_6^σ is smooth as well by Proposition 4.1.1, which concludes the proof. □

The action of $\mathbf{G}$ on V_{10} induces an action of $\mathbf{G}$ on $X_6^{\sigma_0}$ that preserves the polarization H, and hence a homomorphism of groups $\mathbf{G} \to \mathrm{Aut}_H(X_6^{\sigma_0})$. Since $\mathbf{G}$ is simple and non-abelian, we may deduce that the image is contained in the subgroup $\mathrm{Aut}_H^s(X_6^{\sigma_0})$ of symplectic automorphisms. We shall see that this is an isomorphism onto $\mathrm{Aut}_H^s(X_6^{\sigma_0})$.

5.4. The Peskine variety

With a trivector σ, we can associate yet another variety: the Peskine variety

$$X_1^\sigma := \{[V_1] \in \mathbf{P}(V_{10}) \mid \operatorname{rank} \sigma|_{V_1} \leq 6\}.$$

More precisely, for each $[V_1] \in \mathbf{P}(V_{10})$, the skew-symmetric 2-form $\sigma(V_1, -, -)$ generically has rank 8, and the Peskine variety X_1^σ is the locus where this rank drops to 6 or less. Equivalently, given a basis (e_i) of V_{10}, we can identify σ with a 10×10 skew-symmetric matrix with entries $f_{ij} := \sigma(e_i, e_j, -)$. Then X_1^σ is defined in $\mathbf{P}(V_{10})$ by all the 8×8-Pfaffians of this matrix. It has expected dimension 6 and degree 15. We showed in Proposition 4.3.5 that X_1^σ is singular if and only if σ lies in the union of the two divisors $\mathcal{D}^{1,6,10}$ and $\mathcal{D}^{3,3,10}$.

In the case of the special trivector σ_0, since $X_3^{\sigma_0}$ was shown to be smooth, the second case does not happen by Proposition 4.3.3. Therefore, the singular locus of $X_1^{\sigma_0}$ is precisely the locus where the rank of σ_0 drops even less. Equivalently, it is defined by all the 6×6 Pfaffians of σ_0 seen as a skew-symmetric matrix. This allows us to explicitly compute the ideal of the singular subscheme. In particular, we may verify that the rank-4 locus $\mathrm{Sing}(X_1^\sigma)$ is a subscheme of dimension 0 and length 55, while the rank-2 locus is empty. Also, the rank-6 locus $X_1^{\sigma_0}$ is indeed of expected dimension 6 and degree 15.

Here is the `Macaulay2` code that verifies these claims. The variables `I2`, `I4`, and `I6` are the ideals generated by 4×4, 6×6, and 8×8 Pfaffians respectively, hence they give the loci of rank 2, 4, and 6.

```
comps = {(0,2,5),(1,3,6),(2,4,7),(3,0,8),(4,1,9),
         (0,9,7),(1,5,8),(2,6,9),(3,7,5),(4,8,6)};
-- we study the singular locus of the Peskine X1 in P^9
S = QQ[x_0..x_9];
-- compute the skew-symmetric matrix of sigma
delta = (x,y,ex) -> (table(10,10,(i,j) -> if i==x and j==y then ex else 0));
skew = (i,j,k) -> (delta(i,j,x_k)+delta(j,k,x_i)+delta(k,i,x_j)
                  -delta(j,i,x_k)-delta(k,j,x_i)-delta(i,k,x_j));
sigmaskew = matrix sum(for idx in comps list skew(idx));
(I2,I4,I6) = toSequence for k in {4,6,8} list pfaffians_k sigmaskew;
print (dim I2-1, degree I2); -- (-1, 11) irrelevant ideal
print (dim I4-1, degree I4); -- (0, 55)  55 points
print (dim I6-1, degree I6); -- (6, 15)  the Peskine X1
```

We now show that $\mathrm{Sing}(X_1^\sigma)$ is reduced.

PROPOSITION 5.4.1. *For the special trivector σ_0, the singular locus of the Peskine variety $X_1^{\sigma_0}$ consists of the* 55 *distinct points*

$$(p_{i,j})_{0\le i\le 4, 0\le j\le 10}$$

where the rank of σ_0 is equal to 4 *instead of* 6*. The group* **G** *acts transitively on these* 55 *points.*

PROOF. Since we have already obtained the ideal of the rank-4 locus, to verify that there are 55 distinct points, we can compute the radical in `Macaulay2` to check that it is indeed reduced. Here the variable `I4` is the ideal that we have already obtained above. Since it is a homogeneous ideal, we dehomogenize it by letting the first coordinate to be 1.

```
print(radical(I4 + (x_0-1)) == I4 + (x_0-1)); -- true
```

Alternatively, we can use a Gröbner bases computation to obtain the explicit coordinates for the underlying points, and verify that there are 55 distinct solutions over the splitting field (the author wrote a `Macaulay2` package, `RationalPoints2`, that can perform this computation to produce the explicit coordinates).

```
needsPackage "RationalPoints2";
assert(#unique rationalPoints(I4, Projective=>true, Split=>true) == 55);
```

But to better understand the action of the group **G** on these points, we will explain another step-by-step procedure to solve the system using this group action.

We first consider the hyperplane $x_0 + x_1 + x_2 + x_3 + x_4 = 0$. The intersection of this hyperplane with the singular locus is a subscheme of length 5. To compute the coordinates of these 5 points, we can use elimination and obtain a degree-5 equation for $X \coloneqq x_1/x_0$

$$1 - 4X + 2X^2 + 5X^3 - 2X^4 - X^5 = 0.$$

This polynomial splits in the cyclotomic field $\mathbf{Q}(\zeta)$ and all the roots are real, so its splitting field is the real subextension of $\mathbf{Q}(\zeta)/\mathbf{Q}$ of degree 5. We take one real root $\zeta^7 + \zeta^6 + \zeta^5 + \zeta^4$, which allows us to recover the coordinates of one point $p_{0,0}$. The action of the order-5 element R now recovers all the five points on the hyperplane. We denote these by $p_{0,0}, \ldots, p_{4,0}$. They are all real points.

We then consider the action of the order-11 element P, which acts as in Table 5. This allows us to recover the other 50 points which have coordinates in $\mathbf{Q}(\zeta)$ and are complex points. We write $p_{i,j}$ for $P^j(p_{i,0})$. One may then verify that all 55 points are distinct and thus the subgroup **B** generated by P and R acts transitively on them.

Here is the `Macaulay2` code. The variable `I4` is the ideal that we obtained above.

```
-- use the hyperplane x_0+x_1+x_2+x_3+x_4 to identify 5 points
fivePts = trim(I4 + (x_0+x_1+x_2+x_3+x_4));
-- get a degree 5 polynomial in x_0 and x_1 using elimination
pol5 = (gens gb sub(fivePts, QQ[x_0..x_9, Weights=>(2:0)|(8:1)]))_(0,0);
-- the polynomial will split in Q(zeta) so we take a field extension
F = toField(QQ[z]/((z^11-1)//(z-1))); S' = F[x_0..x_9];
root = ideal(x_1 - (z^7+z^6+z^5+z^4) * x_0); -- take a root
assert zero(sub(pol5, S') % root); -- verify that it is a root
fivePts' = sub(fivePts, S');
-- ideal of one of the points, saturated at the irrelevant ideal
Ip = saturate(trim(fivePts' + root), ideal gens S');
-- recover the coordinates of p by solving a linear system in x_0,...,x_9
coeffs = f->first entries transpose(coefficients(f, Monomials=>gens S'))_1;
mat = matrix({{1,9:0}}|apply(first entries gens Ip, coeffs));
p = first entries transpose sub((inverse mat)_{0}, F);
-- finally compute the orbit of p
coord = p -> apply(p, x->x//p_0); -- compute the coordinate (1:x_1:x_2:...:x_9)
Peigen = (for a in (10,2,7,8,6,5,1,9,4,3) list z^a);
P = (j, p) -> coord apply(10, i->p_i*Peigen_i^j); -- P acts by scaling
R = p -> coord{p_1,p_2,p_3,p_4,p_0,p_6,p_7,p_8,p_9,p_5}; -- R acts by permuting
pts = flatten apply({p, R p, R R p, R R R p, R R R R p}, p->apply(11, j -> P_j p));
assert(#unique pts == 55); -- G acts transitively on all 55 singular points
```

Here the variable `pts` contains the coordinates of all 55 singular points. □

5.5. Automorphism group and Picard group

We consider again the general case. In Section 4.3.2, we showed that for a trivector σ such that X_6^σ is smooth, each isolated singular point $p = [V_1]$ of X_1^σ where $\sigma(V_1, -, -)$ has rank 4 leads to a divisor

$$D \coloneqq \{[U_6] \in X_6^\sigma \mid U_6 \supset V_1\}$$

in X_6^σ. Then we showed in Lemma 4.7.7 that the intersection matrix between the Plücker polarization H and the class D with respect to the Beauville–Bogomolov–Fujiki form q is

$$\begin{pmatrix} 22 & 2 \\ 2 & -2 \end{pmatrix}.$$

Moreover, the class D has divisibility 1 by Corollary 4.7.8.

Divisors induced by distinct isolated singular points are also distinct. This can be proved by computing their intersection numbers as follows.

Proposition 5.5.1. *Let σ be a trivector such that X_6^σ is a smooth hyperkähler fourfold. Let $p = [V_1]$ and $p' = [V_1']$ be two different isolated singular points on X_1^σ. Write D and D' for the divisors on X_6^σ that they define. If $\sigma(V_1, V_1', -) = 0$, then $q(D, D') = 1$; otherwise we have $q(D, D') = 0$. In particular, the classes D and D' are distinct.*

PROOF. The Beauville–Bogomolov–Fujiki form q has the property

$$H^2 \cdot D \cdot D' = q(H,H)q(D,D') + 2q(H,D)q(H,D') = 22q(D,D') + 8. \tag{5.3}$$

So we shall calculate the degree of the intersection $D \cap D'$ with respect to the polarization H.

If $\sigma(V_1, V_1', -) = 0$, the intersection $D \cap D'$ can be identified with the locus in $\mathrm{Gr}(2, V_6/(V_1+V_1')) \times \mathrm{Gr}(2, V_6'/(V_1+V_1'))$ where σ vanishes. A simple computation with `Macaulay2` shows that its degree is 30: in fact when $D \cap D'$ is smooth it is a K3 surface admitting (at least) two polarizations with intersection matrix $\left(\begin{smallmatrix} 6 & 9 \\ 9 & 6 \end{smallmatrix}\right)$ and the class H is their sum which has degree 30. We may then conclude that $q(D, D') = 1$ using the relation (5.3).

```
needsPackage "Schubert2";
G = flagBundle{2,2}; U1 = first bundles G;
GxG = flagBundle({2,2}, 4*OO_G); U2 = first bundles GxG;
X = sectionZeroLocus dual(det U1*(1+U2) + det U2*(1+U1));
(h1, h2) = chern_1 \ (dual U1*OO_X, dual U2*OO_X);
print (integral \ (h1^2, h1*h2, h2^2)); -- (6, 9, 6)
```

If $\sigma(V_1, V_1', -) \neq 0$, the kernel of this linear form is a subspace V_9 such that $V_6 + V_6' \subset V_9$. We first show that we have $V_6 + V_6' = V_9$. If the inclusion were strict, we would get a subspace $V_6 \cap V_6'$ of dimension ≥ 4 which satisfies the vanishing condition $\sigma(V_1 + V_1', V_6 \cap V_6', V_{10}) = 0$. But the condition $\sigma(V_1, V_1', -) \neq 0$ shows that $V_1 \not\subset V_6'$ and $V_1' \not\subset V_6$ so the intersection of $V_1 + V_1'$ and $V_6 \cap V_6'$ is 0. This means that for every V_1'' contained in $V_1 + V_1'$, the kernel of $\sigma(V_1'', -, -)$ contains both V_1'' and $V_6 \cap V_6'$ and is therefore of dimension at least 5. In particular, we have $\operatorname{rank} \sigma|_{V_1''} \leq 4$ so the entire line $\mathbf{P}(V_1 + V_1')$ is singular in X_1^σ, contradicting the hypothesis that $[V_1]$ and $[V_1']$ are isolated singular points.

So we get $V_6 + V_6' = V_9$ and therefore $\dim V_6 \cap V_6' = 3$. A point $[U_6]$ in the intersection $D \cap D'$ can be given by the following data: first choose a 2-plane U_2 in $V_6 \cap V_6'$, then choose a 1-dimensional subspace of $V_6/(V_1 + U_2)$ and another 1-dimensional subspace of $V_6'/(V_1' + U_2)$. In other words, the intersection $D \cap D'$ can be identified as a certain zero-locus in the fiber product of two projective bundles over $\mathrm{Gr}(2, V_6 \cap V_6')$. A computation with `Macaulay2` shows that this is a surface of degree 8 with respect to the polarization H. We can thus conclude that $q(D, D') = 0$ in the second case, again using the relation (5.3).

```
needsPackage "Schubert2";
G = flagBundle{2,1}; (U,Q) = bundles G;
P1 = flagBundle({1,2}, 2+Q); (U1,Q1) = bundles P1;
P2 = flagBundle({1,2}, 2+Q*OO_P1); (U2,Q2) = bundles P2;
X = sectionZeroLocus dual(det U*(U1+U2)+U*U1*U2);
h = chern_1 (dual(U+U1+U2)*OO_X);
print integral h^2; -- 8
```

Since $q(D, D) = -2$, we immediately see that different isolated singular points p induce different divisor classes D. $\square$

In the case of the special trivector σ_0, we get 55 distinct divisors $D_{i,j}$ on $X_6^{\sigma_0}$, where $D_{i,j}$ is induced by the isolated singular point $p_{i,j}$ as given in Proposition 5.4.1. Since the subgroup $\mathbf{B}$ acts transitively on the 55 singular points, we see that $\mathbf{B}$ injects into $\mathrm{Aut}^s_H(X_6^{\sigma_0})$. By the simplicity of $\mathbf{G}$, this holds for the whole group $\mathbf{G}$. Alternatively, the injectivity also follows from the more general result in Corollary 4.4.3.

COROLLARY 5.5.2. *The automorphism group* $\mathrm{Aut}^s_H(X_6^{\sigma_0})$ *admits* $\mathbf{G}$ *as a subgroup.*

In particular, the element $P \in \mathbf{G}$ gives a symplectic automorphism of $X_6^{\sigma_0}$ of order 11.

We now study the Picard group of $X_6^{\sigma_0}$. We will write $\mathrm{Pic}(X_6^{\sigma_0})$ for the Picard group and $\mathrm{Tr}(X_6^{\sigma_0})$ for the transcendental lattice, which is the orthogonal complement of $\mathrm{Pic}(X_6^{\sigma_0})$ in $H^2(X_6^{\sigma_0}, \mathbf{Z})$. We mention that Mongardi studied the Picard group of a hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type that admits a symplectic birational automorphism of order 11 and proved the following general result (see [**DM21**, Theorem A.3]).

THEOREM 5.5.3 (Mongardi). *Let X be a hyperkähler fourfold of* $\mathrm{K3}^{[2]}$*-type that admits a symplectic birational automorphism g of order* 11*. The Picard rank of such a fourfold is equal to the maximal value* 21*. There are two possibilities for the g-invariant sublattice of* $H^2(X, \mathbf{Z})$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 22 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 6 & 2 & 2 \\ 2 & 8 & -3 \\ 2 & -3 & 8 \end{pmatrix}.$$

However, we will not use this result directly. Instead, we will compute explicitly the Picard group of $X_6^{\sigma_0}$ and confirm these statements.

Since we have the explicit coordinates for the 55 singular points, Proposition 5.5.1 allows us to compute the Gram matrix between their corresponding divisors. In fact, it suffices to consider the first 21 singular points $p_{0,0}, \dots, p_{0,10}, p_{1,0}, \dots, p_{1,9}$, that is, the entire $\langle P \rangle$-orbit of $p_{0,0}$ plus another 10 points in the $\langle P \rangle$-orbit of $p_{1,0}$. We compute the 21×21 Gram matrix for the corresponding classes $D_{0,0}, \dots, D_{0,10}, D_{1,0}, \dots, D_{1,9}$ using Proposition 5.5.1.

```
sigma = (u,v,w) -> sum(for idx in comps list det submatrix(u||v||w, idx));
M21 = matrix table(21,21,(i,j)->(if i == j then -2 else
  if sigma(matrix{pts_i},matrix{pts_j},genericMatrix(S',1,10))==0 then 1 else 0));
```

We obtain the following

$$\begin{pmatrix}
-2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & -2 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & -2 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & -2
\end{pmatrix}. \tag{5.4}$$

This matrix is of determinant 22, a square-free integer, hence the given 21 classes are linearly independent and generate the whole Picard group. So $X_6^{\sigma_0}$ indeed has maximal Picard rank 21. Using the condition $q(H, D) = 2$, we may express H in terms of the classes $D_{i,j}$: we obtain

$$H = D_{0,0} + \cdots + D_{0,10}.$$

In other words, the polarization H is the sum of the class of 11 divisors obtained using the cyclic action of P.

Write $H^\perp$ for the orthogonal complement of H in $\mathrm{Pic}(X_6^{\sigma_0})$, which is of rank 20. Its Gram matrix (in a suitable basis) can be explicitly computed.

```
H = matrix{(11:{1})|(10:{0})}; K = gens gb ker(transpose H*M21);
M = sub(transpose K*M21*K, ZZ); print M; -- the Gram matrix of H^perp
```

Again using computer algebra, we can verify the following.

Lemma 5.5.4. *The lattice $H^\perp$ is of discriminant* 121 *and $\widetilde{\mathrm{O}}(H^\perp)$—the subgroup of isometries of $H^\perp$ acting trivially on the discriminant group $D(H^\perp)$—is isomorphic to* $\mathbf{G}$.

Moreover, as a $\mathbf{G}$*-representation, $H^\perp$ is isomorphic to $(V'_{10})^{\oplus 2}$, where V'_{10} is the other* 10*-dimensional irreducible representation of* $\mathbf{G}$ *(see Table 4).*

Proof. We use `Sage` to check directly that the group of isometries $\widetilde{\mathrm{O}}(H^\perp)$ is isomorphic to $\mathbf{G}$. We use the variable `M` for the Gram matrix of $H^\perp$, which has determinant 121 and is negative definite.

```
M = matrix([
[-6,-2,-4,-2,-3,-3,-3,-4,-2,-4,-3,-3,-4,-2,-3,-3,-3,-3,-3,-4],
[-2,-4,-2,-2,-1,-2,-2,-3,-2,-2,-1,-2,-3,-2,-1,-2,-2,-2,-2,-3],
[-4,-2,-6,-2,-3,-2,-3,-4,-3,-4,-3,-2,-4,-3,-3,-2,-3,-3,-3,-4],
[-2,-2,-2,-4,-1,-2,-1,-3,-2,-3,-2,-2,-2,-2,-2,-2,-1,-2,-2,-3],
[-3,-1,-3,-1,-4,-1,-2,-2,-2,-3,-2,-2,-3,-1,-2,-2,-2,-1,-2,-3],
[-3,-2,-2,-2,-1,-4,-1,-3,-1,-3,-2,-2,-3,-2,-1,-2,-2,-2,-1,-3],
[-3,-2,-3,-1,-2,-1,-4,-2,-2,-2,-2,-2,-3,-2,-2,-1,-2,-2,-2,-2],
[-4,-3,-4,-3,-2,-3,-2,-6,-2,-4,-3,-3,-4,-3,-3,-3,-2,-3,-3,-4],
[-2,-2,-3,-2,-2,-1,-2,-2,-4,-2,-1,-2,-3,-2,-2,-2,-2,-1,-2,-3],
[-4,-2,-4,-3,-3,-3,-2,-4,-2,-6,-3,-2,-4,-3,-3,-3,-3,-3,-2,-4],
[-3,-1,-3,-2,-2,-2,-2,-3,-1,-3,-4,-2,-2,-2,-2,-1,-1,-2,-2,-2],
[-3,-2,-2,-2,-2,-2,-2,-3,-2,-2,-2,-4,-3,-1,-2,-2,-1,-1,-2,-3],
[-4,-3,-4,-2,-3,-3,-3,-4,-3,-4,-2,-3,-6,-3,-2,-3,-3,-2,-2,-4],
[-2,-2,-3,-2,-1,-2,-2,-3,-2,-3,-2,-1,-3,-4,-2,-1,-2,-2,-1,-2],
[-3,-1,-3,-2,-2,-1,-2,-3,-2,-3,-2,-2,-2,-2,-4,-2,-1,-2,-2,-2],
[-3,-2,-2,-2,-2,-2,-1,-3,-2,-3,-1,-2,-3,-1,-2,-4,-2,-1,-2,-3],
[-3,-2,-3,-1,-2,-2,-2,-2,-2,-3,-1,-1,-3,-2,-1,-2,-4,-2,-1,-3],
[-3,-2,-3,-2,-1,-2,-2,-3,-1,-3,-2,-1,-2,-2,-2,-1,-2,-4,-2,-2],
[-3,-2,-3,-2,-2,-1,-2,-3,-2,-2,-2,-2,-2,-1,-2,-2,-1,-2,-4,-3],
[-4,-3,-4,-3,-3,-3,-2,-4,-3,-4,-2,-3,-4,-2,-2,-3,-3,-2,-3,-6]])
```

We first compute the full orthogonal group $\mathrm{O}(H^\perp)$ of the lattice $H^\perp$, which is a matrix group of order $15840 = 24 \times 660$ given by 5 generators.[2] Then we compute $\widetilde{\mathrm{O}}(H^\perp)$ as the kernel of the natural homomorphism $\chi\colon\ \mathrm{O}(H^\perp) \to \mathrm{O}(D(H^\perp))$.[3]

```
L = IntegralLattice(M)
OL = L.automorphisms()
D = L.discriminant_group()
OD = D.orthogonal_group()
a, b = L.dual_lattice().gens()[0:2]
# we choose u,v so that r(u),r(v) generate the discriminant group D
u, v = (4*a+9*b, 9*a+8*b)
r = L.dual_lattice().hom(D) # r(u)=(1,0), r(v)=(0,1)
chi = OL.Hom(OD)([matrix((r(u*g), r(v*g))) for g in OL.gens()])
G = chi.kernel()
print(G.structure_description()) # PSL(2,11)
```

One may then proceed to compute the character of this $\mathbf{G}$-representation and use the character table (see Table 4) to deduce that $(H^\perp)_{\mathbf{C}} = (V'_{10})^{\oplus 2}$ (which is in fact defined over $\mathbf{Q}$).

```
ch = G.character(matrix(x).trace() for x in G.conjugacy_classes_representatives())
[(m,c.values()) for (m,c) in ch.decompose()] # [(2, [10, 2, 1, 0, 0, -1, -1, -1])]
```

A geometric interpretation of this last fact would be interesting. □

We now show that the group $\mathrm{Aut}^s_H(X_6^{\sigma_0})$ of symplectic automorphisms fixing the polarization H is exactly $\mathbf{G}$.

PROPOSITION 5.5.5. *We have* $\mathrm{Aut}^s_H(X_6^{\sigma_0}) \simeq \mathbf{G}$.

PROOF. The second cohomology group $\Lambda \coloneqq H^2(X_6^{\sigma_0}, \mathbf{Z})$ is a lattice with discriminant 2. The Picard group is a primitive sublattice of Λ of discriminant 22, which contains the sublattice $H^\perp$ of discriminant 121. The orthogonal complement T of $H^\perp$ in Λ must then have discriminant 242. It is the saturation lattice of the direct sum $\mathrm{Tr}(X_6^{\sigma_0}) \oplus \langle H\rangle$. In particular, we have $\left|\Lambda/(T \oplus H^\perp)\right| = 121$ which is equal to $|D(H^\perp)|$.

The transcendental lattice $\mathrm{Tr}(X_6^{\sigma_0})$ is of rank 2 and is contained in $H^{2,0}(X_6^{\sigma_0}) \oplus H^{0,2}(X_6^{\sigma_0})$. Therefore, each symplectic automorphism of $X_6^{\sigma_0}$ fixes $\mathrm{Tr}(X_6^{\sigma_0})$, and an element of $\mathrm{Aut}^s_H(X_6^{\sigma_0})$ fixes the sublattice T. Denote by $\mathrm{O}(\Lambda, T)$ the subgroup of isometries of Λ fixing the sublattice T, that is,

$$\mathrm{O}(\Lambda, T) \coloneqq \{\phi \in \mathrm{O}(\Lambda) \mid \phi|_T = \mathrm{Id}_T\}.$$

We get homomorphisms

$$\mathbf{G} \hookrightarrow \mathrm{Aut}^s_H(X_6^{\sigma_0}) \hookrightarrow \mathrm{O}(\Lambda, T) \xrightarrow{\mathrm{res}} \mathrm{O}(H^\perp).$$

[2]Internally, this is computed by the library PARI/GP.

[3]Internally, this is computed by GAP.

In the last homomorphism, since we have the equality $\left|\Lambda/(T \oplus H^{\perp})\right| = |D(H^{\perp})|$, we may apply [**GHS10**, Corollary 3.4] to show that the image is contained in the subgroup $\widetilde{\mathrm{O}}(H^{\perp})$, which is isomorphic to $\mathbf{G}$ by Lemma 5.5.4. So all the inclusions are equalities. $\square$

Remark 5.5.6.

(1) By viewing $\Lambda_{\mathbf{Q}} \coloneqq H^2(X_6^{\sigma_0}, \mathbf{Q})$ as a rational $\mathbf{G}$-representation, we just saw that it decomposes into

$$\Lambda_{\mathbf{Q}} = \mathrm{Tr}(X_6^{\sigma_0})_{\mathbf{Q}} \oplus \mathbf{Q}H \oplus (H^{\perp})_{\mathbf{Q}},$$

where $\mathbf{G}$ acts trivially on the first two components. For the third component, we have seen that it is the direct sum of two copies of V'_{10} by Lemma 5.5.4.

(2) Mongardi defined a rank-20 lattice $\mathbf{S}$ with explicit Gram matrix and he showed that for a hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type admitting a symplectic birational automorphism g of order 11, the orthogonal of the g-invariant sublattice is isomorphic to $\mathbf{S}$ (see [**Mon13**, Example 2.5.9] or [**DM21**, Appendix A.3]). One may verify that the lattice $H^{\perp}$ in our case is indeed isomorphic to $\mathbf{S}$.

(3) Mongardi also showed that the fixed locus of such an automorphism g consists of 5 isolated points, by using the holomorphic Lefschetz–Riemann–Roch formula. As an example, for the automorphism on $X_6^{\sigma_0}$ given by the element P, using the eigenvalues of $\bigwedge^2 \rho(P)$ given in Table 5, we can explicitly determine the fixed locus as the 5 points

$$[024568], [235679], [125689], [015789], [046789],$$

where the symbol $[abcdef]$ means the 6-dimensional subspace $V_6 = \langle e_a, \dots, e_f \rangle$ of V_{10}.

We would now like to determine the structures of the various lattices: the Picard group $\mathrm{Pic}(X_6^{\sigma_0})$, the transcendental lattice $\mathrm{Tr}(X_6^{\sigma_0})$ and the $\mathbf{G}$-invariant sublattice T which is the saturation of the direct sum $\mathrm{Tr}(X_6^{\sigma_0}) \oplus \langle H \rangle$. We recall the following results from lattice theory [**Nik79**, Corollary 1.13.3 and Corollary 1.13.5].

Proposition 5.5.7 (Nikulin). *Let L be an even lattice of signature (p, q). Let l be the minimal number of generators of the discriminant group $D(L)$.*

(1) If $p \geq 1, q \geq 1$, and $p + q \geq l + 2$, then L is uniquely determined by its discriminant form.

(2) If $p \geq 1, q \geq 1$, and $p + q \geq l + 3$, then L decomposes into $U \oplus L'$.

(3) If $p \geq 1, q \geq 8$, and $p + q \geq l + 9$, then L decomposes into $E_8(-1) \oplus L'$.

Here, U denotes the hyperbolic plane, and $E_8(-1)$ denotes the E_8-lattice with negative definite form.

PROPOSITION 5.5.8. *Consider the lattice*

$$L_{11} := \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}.$$

We have the following isomorphisms of lattices

$$\mathrm{Tr}(X_6^{\sigma_0}) \simeq L_{11}, \qquad T = \mathrm{Tr}(X_6^{\sigma_0}) \oplus \langle H \rangle \simeq L_{11} \oplus (22),$$
$$\mathrm{Pic}(X_6^{\sigma_0}) \simeq U \oplus E_8(-1)^{\oplus 2} \oplus L(-1),$$

where we can take the component L to be $\left(\begin{smallmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 8 \end{smallmatrix}\right)$ or $L_{11} \oplus (2)$ (by this we mean the isomorphism holds for both values of L).

PROOF. Since $\mathrm{Pic}(X_6^{\sigma_0})$ has discriminant 22, its orthogonal $\mathrm{Tr}(X_6^{\sigma_0})$ has discriminant either 11 or 44. In the second case, the direct sum $\mathrm{Tr}(X_6^{\sigma_0}) \oplus \langle H \rangle$ would have index 2 in its saturation T, so there would exist a class $x \in \mathrm{Tr}(X_6^{\sigma_0})$ such that $\frac{1}{2}(H + x)$ is integral. But then we would have $q\big(H, \frac{1}{2}(H + x)\big) = 11$, contradicting the fact that $\mathrm{div}(H) = 2$.

So $\mathrm{Tr}(X_6^{\sigma_0})$ has discriminant 11. Every rank-2 positive definite lattice has a reduced form (see for instance [**CS98**, Chapter 15.3.2]). For discriminant 11, the lattice L_{11} is the only one that is even. Thus we may conclude that $\mathrm{Tr}(X_6^{\sigma})$ is isomorphic to L_{11}. Since the direct sum $\mathrm{Tr}(X_6^{\sigma_0}) \oplus \langle H \rangle$ is primitive, we have $T = \mathrm{Tr}(X_6^{\sigma_0}) \oplus \langle H \rangle \simeq L_{11} \oplus (22)$.

Finally we determine the structure of $\mathrm{Pic}(X_6^{\sigma_0})$. By using (2) and (3) of Proposition 5.5.7, it decomposes into a direct sum

$$\mathrm{Pic}(X_6^{\sigma_0}) \simeq U \oplus E_8(-1)^{\oplus 2} \oplus L(-1), \tag{5.5}$$

where L is positive definite of rank 3 and discriminant 22 and also even. There are two possibilities: either L is indecomposable, then by [**CS98**, Table 15.6] it is unique and is given by

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 8 \end{pmatrix};$$

or L is decomposable, then it must be the direct sum $L_{11} \oplus (2)$. By comparing discriminant forms and using (1) of Proposition 5.5.7, we may conclude that the isomorphism (5.5) holds for both values of L. □

REMARK 5.5.9.

(1) We determined the sublattice T that is $\mathbf{G}$-invariant. In particular, for a given element $g \in \mathbf{G}$ of order 11, T is contained in the g-invariant sublattice. Notice that this is indeed one of the two possibilities listed in Theorem 5.5.3.
(2) In [**DM21**], it is shown that for a double EPW sextic with a symplectic birational automorphism of order 11, its transcendental lattice is isomorphic to $(22)^{\oplus 2}$. So the special Debarre–Voisin fourfold $X_6^{\sigma_0}$ is not isomorphic to the special double EPW studied in [**DM21**].

(3) When σ is such that X_1^σ admits a unique singular point $[V_1]$, there is an associated K3 surface S of degree 6, and the divisor D admits a $\mathbf{P}^1$-fibration over S. In the case of σ_0, if the K3 surface associated to one $D_{i,j}$ is still smooth of dimension 2 (which we were not able to prove), then all 55 are isomorphic under the action of $\mathbf{G}$. Moreover, the Picard number of this K3 surface would be the maximal value 20, and we can obtain its Gram matrix using $\mathrm{Pic}(X_6^{\sigma_0})$ (in particular, the period of this K3 surface is uniquely determined).

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