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Advanced Topics in Optimization: Mathematical Imaging

Exercise sheet I (thanks to Dr. Kostas Papafitsoros for providing some of the problems)

- Exercise consists of standard homework exercise and some problems for open discussion
- The former type usually has a definite answer, while the latter perhaps has no determined or fixed answer, but gives you some freedom to develop or to implement with your own ideas.

Exercise

1. Let X be a normed vector space. Prove the following:

(i) The function defined by the norm $\|\cdot\|_X : X \to \mathbb{R}$ is weakly lower semicontinuous.

(ii) The function defined by the dual norm $\|\cdot\|_{X^*} : X^* \to \mathbb{R}$ is weakly* lower semicontinuous.

2. The *Uniform Boundedness Principle* theorem states that if X is a Banach space, Y is a normed space and $(T_i)_{i\in I}$ is a famility of bounded linear operators from X to Y that are pointwise bounded (i.e., for every $x \in X$ the set $\{T_i(x)\}_{i\in I}$ is bounded) then there exists an $M > 0$ such that $\|T_i\| \leq M$ for every $i \in I$.

Use the theorem above to prove that if K is a weakly compact subset of X then K is bounded with respect to the norm of X.

3. Let X be a topological vector space and let $(F_i)_{i\in I}$ be a family of lower semicontinuous functions where $F_i : X \to \overline{\mathbb{R}}$ for every $i \in I$. Show that the function $F : X \to \overline{\mathbb{R}}$ is also lower semicontinuous where

$$F(x) := \sup_{i\in I} F_i(x).$$

4. Let $\Omega \subset \mathbb{R}^d$ an open bounded domain. Define the function $F : L^1(\Omega) \to \overline{\mathbb{R}}$ as

$$F(u) = \begin{cases} \|u\|_{L^2(\Omega)} & \text{if } u \in L^2(\Omega) \\ +\infty & \text{if } u \in L^1(\Omega) \setminus L^2(\Omega). \end{cases}$$

Show that F is lower semicontinuous with respect to the norm topology of $L^1(\Omega)$.

5. Does the problem

$$\inf_{\substack{u\in W^{1,1}([0,1]) \\ u(0)=0 \\ u(1)=1}} \int_0^1 \sqrt{u^2 + (u')^2}\, dx$$

have a solution?

Problems for open discussion

1. For a given nature image (may be a gray image, no colors, only brightness), think of it as a function $u : \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^2$. Then what will the "gradient" of this image looks like? Can you imagine a proper function space for such functions. Are standard Sobolev spaces appropriate for such type of functions?

2. For a "simple" image $u \in \mathbb{R}^{m\times n}$, corresponding to domain $\Omega = [0,1]^2$, compute the magnitude of its "gradient" (e.g., $|\nabla(u)|$) using finite difference, then plot it. You can use Matlab or whatever programming language you are familiar. Simple here means the image consists of few objectives, e.g, the pepper image data provided. Be careful that there are discretization sizes h_x, h_y (may also refers to resolution scale) involved for the finite differences. What gonna happen if h gets smaller and smaller.