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Advanced Topics in Optimization: Mathematical Image Processing

Exercise sheet V

- Exercise consists of standard homework exercises (Problem 1-3) and some tasks for implementation in Problem 4.

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1. Recall the definition of Moreau–Yosida regularization for non-smooth function $f : X \to \bar{\mathbb{R}}$ and some Hilbert space X:

$$f_\lambda(u) := \inf_{v\in X} \{f(v) + \frac{1}{2\lambda}\|u - v\|_X^2\}$$

Let $u \in L^2(\Omega)$ and f be the indicator function,

$$f(u) = \begin{cases} 0 & a \le u(x) \le b \text{ for almost every } x \in \Omega, \\ \infty & \text{otherwise.} \end{cases}$$

Show that the Moreau–Yosida regularization of the indicator function f is

$$f_\lambda(u) = \frac{1}{2\lambda}\left(\|\min((u-a),0)\|_{L^2(\Omega)}^2 + \|\max((u-b),0)\|_{L^2(\Omega)}^2\right).$$

2. Recall the projection operator $P_{\text{div}} : H_0(\text{div},\Omega) \to H_0(\text{div}0)$ in the lecture, i.e.,

$$\text{for every } p \in H_0(\text{div},\Omega), \text{ there is } \text{div}(P_{\text{div}}p) = 0.$$

How to realize the projection P_{div} for a given $p \in H_0(\text{div},\Omega)$?
Hint: Gradient of every scalar function $u \in H^1(\Omega)$ is curl free. You will find that it leads to solve some partial differential equation to have the projection to be done.

To ease discussion and implementation, we ignore the P_{div} operator in the next two exercises.

3. Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain. Consider the regularized version of the predual TV denoising problem:

$$\min_{p\in H_0^1(\Omega)^2} \frac{1}{2c}\|\nabla p\|_{L^2}^2 + \frac{1}{2}\|\text{div}p + u^d\|_{L^2}^2 + \frac{\gamma}{2}\|p\|_{L^2}^2 + c\mathcal{P}(p,\alpha) \tag{0.1}$$

where ∇ operates on each components of p, i.e., p_1 and p_2, and the term

$$\mathcal{P}(p,\alpha) := \frac{1}{2}\int_\Omega \sum_i |\min(p_i(x)+\alpha,0)|^2 + |\max(p_i(x)-\alpha,0)|^2 dx$$

aims to approximately enforce the constraint $-\alpha \le p_i \le \alpha$, $i = 1,2$, $p = (p_1,p_2)$. Let p^c denote the unique solution of Problem (0.1). Then it satisfies the optimality condition

$$-\frac{1}{c}\Delta p^c - \nabla \text{div} p^c - \nabla u^d + \gamma p^c + \lambda^c = 0;$$
$$\lambda^c = c(\min(p^c+\alpha,0) + \max(p^c-\alpha,0)).$$

Let $c \to \infty$, it was shown in the lecture that $(p^c,\lambda^c) \to (p',\lambda')$ weakly in $H_0(\text{div},\Omega) \times H_0^1(\Omega)^*$, and p' is feasible, i.e., p' satisfies the box constraints. Please based on these results continue to prove that (p',λ') is actually satisfy the optimality of the predual problem before regularization, which is:

$$-\nabla\text{div}p^* - \nabla u^d + \gamma p^* + \lambda^* = 0;$$
$$\langle \lambda^*, p - p^*\rangle_{H_0(\text{div},\Omega)^*,H_0(\text{div},\Omega)} \le 0 \quad \text{for all } p \in H_0(\text{div},\Omega) \text{ with } -\alpha \le p_i \le \alpha.$$

Hint 1: Recall that p^c, λ^c are shown to be uniformly bounded in $H_0(\text{div}) \times H_0^1(\Omega)^*$ which is embedded into $H_0(\text{div}) \times H_0(\text{div})^*$.
Hint 2: We can show strong convergence of $p^c \to p'$ using optimality of p^c and weak lower semicontinuity of norms.
Hint 3: The set $S_2 := \{p \in (C_0^1(\Omega))^2 : p \in [-\alpha,\alpha]^2\}$ is dense in $S_1 := \{p \in H_0(\text{div},\Omega) : p \in [-\alpha,\alpha]^2\}$.

4. Implement the semismooth Newton Method to solve the first order optimality condition of (0.1):

$$F(p) := -\frac{1}{c}\Delta p + \gamma p - \nabla \mathrm{div} p - \nabla u^d + cP(p, \alpha) = 0.$$

with P being the derivative of $\mathcal{P}$ (with respect to p), i.e.,

$$P(p, \alpha) := \min(p + \alpha, 0) + \max(p - \alpha, 0).$$

Instructions:

- The noisy data u^d are provided on the course website (note that there are pepper and noise contained in the data.mat file).
- Define your mesh size h (e.g. $D_x u = \frac{u_{i+1,j} - u_{i,j}}{h}$) for your differential operators as $h = 1/\sqrt{nm}$ where $n \times m$ is the image resolution.
- The Laplacian and the ∇div operator must be constructed with the correct boundary conditions (zero Dirichlet).
- The parameters $\frac{1}{c}, \gamma$ have to be relatively small in order to approximate the original predual TV problem.
- Experiment with different values of the regularization parameter α and observe the differences in the reconstruction result.