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Mathematical Programming with Equilibrium Constraints

Exercise sheet 2

- Exercise 2.H1 is a homework exercises: please turn in your solution via email to the course assistant 36h before the start of the exercise session. The submission must be in one of the following formats: legible scan or photograph of hand-written solutions or typeset solution.
- The exercises 2.P1 to 2.P3 are going to be discussed during the exercise session.

Exercise 2.H1 (based on an example from Michael C. Ferris, UW Madison). Consider the following Nash game:

$$\text{(Player 1)} \qquad \min_{x_1 \geq 0} \quad f_1(x_1, x_2) := \frac{1}{2}x_1^2 + \theta x_1 x_2 - 4x_1$$
$$\text{(Player 2)} \qquad \min_{x_2 \geq 0} \quad f_2(x_1, x_2) := \frac{1}{2}x_2^2 + x_1 x_2 - 3x_2,$$

with $\theta \in \mathbb{R}$ a parameter. Recall that a pair $(x_1^\star, x_2^\star) \in \mathbb{R}_+^2$ is a solution to the Nash game whenever the two conditions

$$f_1(x_1^\star, x_2^\star) \leq f_1(x_1, x_2^\star) \quad \forall x_1 \geq 0$$
$$f_2(x_1^\star, x_2^\star) \leq f_2(x_1^\star, x_2) \quad \forall x_2 \geq 0$$

hold simultaneously.

1. Characterize each players' optimization problem: convexity, existence of solutions, difficulty to solve.

2. Derive the first order optimality conditions as variational inequalities (VI) for each player

3. Find the best response mappings

$$R_1(x_2) := \operatorname*{arg\,min}_{x_1 \geq 0} \; f_1(x_1, x_2)$$
$$R_2(x_1) := \operatorname*{arg\,min}_{x_2 \geq 0} \; f_2(x_1, x_2)$$

(Hint: use the VI and distinguish when the only constraint is active or not).

4. Determine the solution mapping $\mathsf{S}(\theta)$ to the Nash Game. (Hint: check that if $x \in \mathbb{R}_+^2$ is a fixed point of the mapping $R := (R_1, R_2)$, then it is a fixed point to the game)

5. Solve the following MPEC

$$\min_{\theta \in \mathbb{R}} \quad \theta$$
$$\text{s.t.} \quad (x_1, x_2) \in \mathsf{S}(\theta).$$

Exercise 2.P1 Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous convex function. Consider the bilevel problem

$$\min_{(x,y) \in \mathbb{R}^2} \quad f(x, y)$$
$$\text{s.t.} \quad y \in \operatorname*{arg\,max}_{|z| \leq 1} \; xz. \tag{1}$$

1. Transform the lower-level problem (1) into a relationship $y \in \mathsf{S}(x)$, and draw $\operatorname{gph} \mathsf{S}$.

2. Compare this to the feasible set of a (classical) convex optimization problem.

3. Furthermore, suppose that f is *coercive* (that is $\lim_{\|(x,y)\| \to \infty} f(x, y) = \infty$). Devise a strategy to solve the bilevel problem by solving a convex optimization problem 3 times.

Exercise 2.P2 A link between the mathematical economics and mathematical programming views on game theory.

Consider the *Bach or Stravinsky* coordination game: two players wish to synchronize their action, while having conflicting interests. For instance a fictitious partnership/friendship looks to attend a concert, either of Bach or Stravinsky. Each have their own preferred choice, but they value over all attending the same event. Their preferences can be given the form of a *payoff matrix*, where the value associated all the players' choice is given in the cell, see Table ().

		Player 2	
		Bach	Stravinsky
Player 1	Bach	$(3, 2)$	$(0, 0)$
	Stravinsky	$(0, 0)$	$(2, 3)$

Table 1: Payoff matrix

1. What are the Nash equilibria for this game? (Hint: there are pure and mixed strategies)
2. Formulate two maximization problems (one for each player), each with a single binary variable as decision variable and a quadratic objective functional, such that the game formed by the 2 problems is equivalent to the matrix game display in Table (Hint: check that the values in the payoff matrix can be reconstructed the values of the functional for each possible pairs of the decision variables)
3. Check that the pure strategies can be found by solving the game derived at the previous question
4. Consider now the following relaxation of the game: each binary variable is relaxed to take value in $[0, 1]$. Relate this to the mixed strategy concept.
5. Derive the Variational Inequality (VI), or generalized equation (GE) associated with the relaxed game derived at the previous question. (Hint: stack the box-constraint VI for each player).
6. Check that the Nash equilibria found in Question 1 are also solution to the VI (or GE).

Exercise 2.P3 Fitting problems: saddle-point, games, and beyond black and white constraints

We look at the fitting problems, that is given a parametric affine model and some noisy measurement data on both a state and output, find the parameters of the affine function such that it fits best the data. Broadly speaking, this amounts to find $(x, b) \in \mathbb{R}^{n+1}$, such that $Ax + b$ is as close as possible to the output measurement $y \in \mathbb{R}^m$, and the matrix $A \in \mathbb{R}^{m \times n}$ contains all the states.

1. Suppose that we look for an *exact* match of the model output with the observations. Cast this exact linear fitting problem as a feasibility problem with decision variable $(x, b) \in \mathbb{R}^{n+1}$ and an affine constraint involving A, x, b, and y. Is it expected for this problem to have a solution?
2. Write down the Lagrangian corresponding to this problem.
3. Consider the mini-max (or saddle-point) problem associated with the Lagrangian. Check that a solution to the following 2 players game, where Player 1 minimizes the Lagrangian over x, and Player 2 maximizes the Lagrangian over the multiplier variable to the equality constraint, is a solution to the saddle point problem of the Lagrangian.
4. In real-world example, in particular with noisy data, it is not possible to find a solution to the feasibility problem. By looking at the two players game, show that infeasibility manifest itself via the unboundedness of the objective functional for Player 2. In that sense, you can say that Player 2 has too much "power" and drives the value of the Lagrangian to infinity.
5. A classical way to formulate a fitting problem is to minimize an objective functional based on loss functions (also called regularizers or estimators) applied to the residual (or misfit) $r := Ax + b - y$. Check that:
   - If the feasible set Y for Player 2 is restricted to $[-1, 1]^m$, then the game consists in minimizing $\|r\|_1$.
   - If the functional $-\frac{1}{2}\lambda^T\lambda$ is added to the Lagrangian $L(x, \lambda)$, then this corresponds to the classical linear least square case.

- If both the feasible set Y for Player 2 is restricted to $[-\kappa, \kappa]^m$ and $-\frac{1}{2}\lambda^T\lambda$ is added to the Lagrangian, then a solution to the game correspond to fitting the data with the Huber loss function defined componentwise as:

$$L_\kappa(r) = \begin{cases} \frac{1}{2}r^2 & \text{if } |r| \leq \kappa \\ \kappa(|r| - \frac{1}{2}\kappa) & \text{otherwise.} \end{cases}$$

Remark: using the same point-of-view, one can also derive the expression for the following common loss functions: Hinge, Vapnik, Hubnik, elastic net. Loss functions are commonly used when considering machine learning problems like statistical classification.

General remarks/recalls

- The definition of a Nash equilibrium is as follows: consider a game with N players, where each one of them aims at minimizing its cost functions f_i over a feasible set X_i. The classical notation (x_i, x_{-i}) is a convenient way to decompose a vector $x \in X \coloneqq \Pi_j X_j$ in the part corresponding to player's i decision variable $x_i \in X_i$, and all the other players' decision variable $x_{-i} \in \Pi_{j \neq i} X_j$. A point $x^\star \in X$ is a Nash equilibrium if no player has an incentive to *unilaterally* deviate from their strategy. That is for all i, letting $(x_i^\star, x_{-i}^\star) = x^\star$, it holds that

$$f(x_i^\star, x_{-i}^\star) \leq f(x_i, x_{-i}^\star) \qquad \forall x_i \in X_i.$$

  Whenever a player maximizes rather than minimizes, the inequality is reversed.

- In a Nash game, player i only controls the decision variable x_i. All the other players' decision variables are given to him, and considered as constants.

- The Lagrangian function L associated with the optimization problem

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & g(x) \in K \qquad \text{with } K \text{ a closed convex cone, like } \mathbb{R}^m_+,\ \mathbb{R}^m_-, \text{ or } \{0\}, \end{aligned}$$

  is given by $L(x, \lambda) = f(x) + \langle \lambda, g(x) \rangle$, with the multiplier (or dual) variable $\lambda \in K^\circ$, the polar of the cone K. Note that the Lagrangian can also be written as $L(x, \lambda) = f(x) - \langle \lambda, g(x) \rangle$ and $\lambda \in K^D$, the dual of the cone K. See Exercise Sheet 1 for a recall of the notions of polar and dual cones.

- The saddle point problem associate with the Lagrangian $L(x, \lambda)$ over (X, Y), where x is called the primal variable, and λ the dual variable, is to find $(x^\star, y^\star)$ such that

$$L(x^\star, \lambda) \leq L(x^\star, \lambda^\star) \leq L(x, \lambda^\star) \qquad \forall (x, \lambda) \in X \times Y.$$

- For more background on Lagrangian, VI, and Nash games, see *Section 1.4* in "Finite-Dimensional Variational Inequalities and Complementarity Problems" by Facchinei and Pang. This book is avaible through the HU network at https://link.springer.com/book/10.1007%2Fb97543.