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Institut für Mathematik
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Mathematical Programming with Equilibrium Constraints

Exercise sheet 4

- Exercise 4.H1 is a homework exercises: please turn in your solution via email to the course assistant 36h before the start of the exercise session. The submission must be in one of the following formats: legible scan or photograph of hand-written solutions or typeset solution.
- The exercises 4.P1 to 4.P4 are going to be discussed during the exercise session.

Exercise 4.H1: Directional derivatives Let $\Delta_t(x; d) \coloneqq \frac{f(x+td)-f(x)}{t}$ and $f'(x; d)$ be the directional derivative of f at x in the direction d.

1. Show that if f is locally Lipschitz, then the values $\liminf_{t\to 0^+} \Delta_t(x; d)$ and $\limsup_{t\to 0^+} \Delta_t(x; d)$ are well defined.
2. Show that if f is convex and finite-valued in a neighborhood of x, then $f'(x; d)$ is well defined for all d, $\partial f(x)$ is compact, and that $f'(x; d) = \max_{v\in\partial f(x)} \langle v, d\rangle$. One may assume the non-emptiness and convexity of $\partial f(x)$.
3. Show that the convex subdifferential is consistent with the Clarke subdifferential at x whenever f is convex and f is finite-valued around x.

Exercise 4.P1: Normal cones computations Compute the Clarke and the limiting (or Mordukhovich) subdifferentials for f at x in the following cases

1. $f(x) = |x|$ at $x = 0$
2. $f(x) = -|x|$ at $x = 0$
3. $f(x) = x^{1/3}$ at $x = 0$ (we only consider the limiting subdifferential, why?)

When possible, compare the Clarke and limiting subdifferentials.

Exercise 4.P2: Coderivative computation

1. Compute the coderivative of $\mathrm{Sgn}(x)$ at $(1, 1)$, $(-1, -1)$, and $(0, u)$, $u \in [1, 1]$.
2. For any proper lower semi-continuous $f\colon \mathbb{R}^n \to \bar{\mathbb{R}}$, consider the set-valued mapping $E_f\colon \mathbb{R}^n \to \bar{\mathbb{R}}$ defined as $E_f(x) = \{t \in \bar{\mathbb{R}} \mid f(x) \le t\}$. Let $x \in \operatorname{dom} f$. Show that $D^*E_f(x, f(x))(1) = \partial f(x)$.

Exercise 4.P3: Convex polyhedral geometry Recall the definition of the tangent cone at a point $x \in C$:

$$T_C(x) \coloneqq \limsup_{\tau\to 0^+} \frac{C - x}{\tau},$$

and the normal cone

$$N_C(x) \coloneqq \{d \in \mathbb{R}^n \mid \langle d, y - x\rangle \le o(\|y - x\|)\ \forall y \in C\}.$$

Let $C \coloneqq \{x \in \mathbb{R}^n \mid Ax \le b\}$, $A \in \mathbb{R}^m \times \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $L \coloneqq \{x \in \mathbb{R}^n \mid Mx = m\}$, with M and m of appropriate dimensions. Let $\mathcal{I}$ be the set of indices of active constraints at a point $x \in C$ (with complement $\bar{\mathcal{I}}$, the set of inactive constraint), and a_i is the i-th row of A.

1. Let $x \in C, z \in L$. Show that the tangent cone $T_C(x) = \{d \in \mathbb{R}^n \mid a_i d \le 0,\ \forall i \in \mathcal{I}\}$, and that $T_L(z) = \ker M$.
2. Using the polarity relationship $T_C(x) = N_C(x)^\circ$, show that the normal cone $N_C(x) = \{A^T\lambda \mid \lambda_{\mathcal{I}} \ge 0, \lambda_{\bar{\mathcal{I}}} = 0\}$, $N_L(z) = L^\perp$.
3. Show that T_C and N_C are polyhedral sets (Hint: Minkowski-Weil theorem).
4. Compute the normal cone and tangent cone to $\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-, \{0\}$, as well as $\mathbb{R}^n, \mathbb{R}^n_+, \mathbb{R}^n_-, \{0^n\}$.

Exercise 3.P4: Subdifferential and sums Let $f_1 = \max\{x, 0\}$, and $f_2 = \min\{x, 0\}$. Compute $\partial(f_1 + f_2)(0)$ and $\partial f_1(0) + \partial f_2(0)$, with limiting subdifferential. Does the function $f_1 + f_2$ have a minimum at 0?