Humboldt-Universität zu Berlin
Institut für Mathematik
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Mathematical Programming with Equilibrium Constraints

Exercise sheet 5

- Exercise 5.H1 is a homework exercises: please turn in your solution via email to the course assistant before Tuesday 30.06. The submission must be in one of the following formats: legible scan or photograph of hand-written solutions or typeset solution.
- The exercises 5.P1 to 5.P4 are going to be discussed during the exercise session.

Exercise 5.H1: Variations on convexity and monotonicity

1. Fill the definitions of the concepts in the following table

Function $f\colon \mathbb{R}^n \to \bar{\mathbb{R}}$	Multifunction $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$
Convex	Monotone
Strictly Convex	Strictly Monotone
Strongly Convex	Strongly Monotone

Prove that for each row, the subdifferential of a function enjoying the property of the left column, enjoys the property of the right column.

2. Is there a quadratic mapping $f\colon \mathbb{R}^n \to \mathbb{R}$ that is strictly convex, but not strongly convex?

3. Consider the following optimization problem

$$\min_{x\in\mathbb{R}^2_+} \qquad x^T \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} x$$

(a) Characterize this optimization problem.

(b) Show that the matrix in the objective function is strictly copositive with respect to the feasible set of the minimization problem.

(c) Conclude about the existence and uniqueness of a solution.

Exercise 5.P1: Normal cones computations Compute the Clarke and the limiting (or Mordukhovich) subdifferentials for f at x in the following cases

1. $f(x) = |x|$ at $x = 0$

2. $f(x) = -|x|$ at $x = 0$

3. $f(x) = x^{1/3}$ at $x = 0$ (we only consider the limiting subdifferential, why?)

When possible, compare the Clarke and limiting subdifferentials.

Exercise 5.P2: Coderivative computation For any proper lower semi-continuous $f\colon \mathbb{R}^n \to \bar{\mathbb{R}}$, consider the set-valued mapping $E_f\colon \mathbb{R}^n \to \bar{\mathbb{R}}$ defined as $E_f(x) \coloneqq \{t \in \bar{\mathbb{R}} \mid f(x) \leq t\}$. Let $x \in \operatorname{dom} f$. Show that $D^*E_f(x, f(x))(1) = \partial f(x)$.

Exercise 5.P3: Subdifferential and sums Let $f_1 = \max\{x, 0\}$, and $f_2 = \min\{x, 0\}$. Compute $\partial(f_1 + f_2)(0)$ and $\partial f_1(0) + \partial f_2(0)$, with limiting subdifferential. Does the function $f_1 + f_2$ have a minimum at 0? Conclude!

Exercise 5.P4: Sufficient condition for strong monotonicity Let Ω be a closed convex subset of $\mathbb{R}^n$, and $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 mapping on a open superset of Ω, convex subset of $\mathbb{R}^n$. Prove that a sufficient condition for strongly monotonicity of F is that ∇F is *uniformly positive definite*, that is $\exists \alpha > 0$ such that $\forall y \in \Omega$, $\forall d \in \mathbb{R}^m$, we have

$$\langle d, \nabla F(y)\, d\rangle \geq \alpha \|d\|^2.$$