HU Berlin WS 2007/08

Problems for BMS Basic Course "Commutative Algebra"

Hand in till 2008, Jan 10th at room 2.304

Advent Problem (50 additional points)

or: Platonic solids - Once more, with feelings (actually, with tensors products)

Please sign each sheet of paper with your name and student ID

Remember that we identified five types of finite subgroups of $SO_3(\mathbb{R})$: Cyclic, dihedral, tetrahedral, octahedral and icosahedral.

By using the natural isomorphism $SO_3(\mathbb{R}) \cong PSU_2(\mathbb{C})$, we can lift these groups to subgroups of doubled order in $SU_2(\mathbb{C})$ (called *binary polyhedral groups*). They are generated by the following matrices (as usual, ζ_n denotes a primitive n-th root of unity):

Binary cyclic group of order n:

$$A = \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}$$

Binary dihedral group of order n:

$$A = \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}, \ B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Binary tetrahedral group:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8 & \zeta_8^3 \\ \zeta_8 & \zeta_8^7 \end{pmatrix}, \ B = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8 & \zeta_8 \\ \zeta_8^3 & \zeta_8^7 \end{pmatrix}$$

Binary octahedral group:

$$A = \begin{pmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8^5 \end{pmatrix}, \ B = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8^7 & \zeta_8^7 \\ \zeta_8^5 & \zeta_8 \end{pmatrix}$$

Binary icosahedral group:

$$A = \frac{1}{\sqrt{5}} \begin{pmatrix} 1-\zeta_5^2 & \zeta_5-\zeta_5^2 \\ \zeta_5^3-\zeta_5^4 & 1-\zeta_5^3 \end{pmatrix}, \ B = \frac{1}{\sqrt{5}} \begin{pmatrix} 1-\zeta_5^2 & 1-\zeta_5^2 \\ \zeta_5^3-\zeta_5^4 & \zeta_5^4-\zeta_5^3 \end{pmatrix}$$

Note that these matrices fulfill the famous relations $A^p = B^q = (AB)^2 = -E_2$ for the respective Platonic pairs (p,q), $p,q \geq 2$, $\frac{1}{p}+\frac{1}{q} > 1$.
Now, let $\Gamma \subseteq SL_2(\mathbb{C})$ be one of this groups. A group homomorphism $\rho : \Gamma \to GL_n(\mathbb{C})$ is called a *representation of* Γ. Note that such a representation corresponds to a group action of Γ on $\mathbb{C}^n$. A representation is called *irreducible* if there are no proper subrepresentations (i.e., a representation $\rho' : \Gamma \to GL(U)$ induced by a proper invariant subspace $U \subset \mathbb{C}^n$). It is easy to see that each representation can be uniquely represented as a sum of irreducible ones.

Denote the set of nontrivial (i.e., not identically 0) irreducible representations of Γ by

$$\mathrm{Irr}^0(\Gamma) := \{\rho_1, \dots, \rho_r\}$$

and the natural representation given by the inculsion $\Gamma \subset \mathrm{GL}_2(\mathbb{C})$ by c.

(a) Determine $\mathrm{Irr}^0(\Gamma)$ for the Platonic groups. (*Hint*: Use the well-known decomposition of Γ into conjugacy classes.)

(b) Compute the coefficients a_{jk} given by $\rho_j \otimes c = \sum_j a_{jk} \rho_k$.

(c) Draw the graphs arising if each representation is represented by a vertex and the j-th vertex is connected with the k-th vertex by a_{jk} directed arrows.