HU Berlin WS 2007/08

Problems for BMS Basic Course "Commutative Algebra"

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Hand in Jan 10th, after the 2nd lecture 4.45 p.m.

Please solve each problem on a different sheet of paper, and sign each sheet with your name and student ID

10th Problem Set (40 + 10 points)

Problem 1 (10 pts)

Show directly (without using the Lemma of the course) that the primary ideals $\mathfrak{q} \subseteq \mathbb{Z}$ are precisely of the type $\mathfrak{q} = (0)$ or $\mathfrak{q} = (p^n)$ for a prime number p and $n \in \mathbb{N}_{>0}$.

Problem 2 (10 pts)

Let $A = k[X, Y]$, $\mathfrak{q} = (X, Y^2)$ and $\mathfrak{m} = (X, Y)$. Show that:

(a) The ideal $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal.

(b) There is a chain of strict inclusions $\mathfrak{m}^2 \subsetneq \mathfrak{q} \subsetneq \mathfrak{m}$.

(c) Deduce from (b) that $\mathfrak{q}$ is not a power of a prime ideal of A.

Problem 3 (10 pts)

Let $A = k[X, Y, Z]/(XY - Z^2)$ and $\mathfrak{p} = (\bar{X}, \bar{Z})$, where $\bar{X}, \bar{Z}$ denote the classes of X, Z in A, respectively. Show that $\mathfrak{p}^2$ is not a primary ideal of A.

Problem 4 (10 + 10 pts)

A presheaf $\mathcal{F}$ on X is called a *sheaf* if the following *glueing axiom* holds:

Let $U \subseteq X$ be an open subset and $U = \bigcup_{j \in J} U_j$ an open covering of U. Let $\{s_j\}_{j \in J}$, $s_j \in \mathcal{F}(U_j)$ $(j \in J)$, be a collection of compatible sections, i.e., $\rho_{U_j, U_j \cap U_k}(s_j) = \rho_{U_k, U_j \cap U_k}(s_k)$ for all $j, k \in J$. Then there exists a unique $s \in \mathcal{F}(U)$ such that $\rho_{U, U_j}(s) = s_j$.

Furthermore, for a presheaf $\mathcal{F}$ on X and a point $x \in X$, the *stalk* $\mathcal{F}_x$ is defined as

$$\mathcal{F}_x := \varinjlim_{x \in U,\ U \text{ open}} \mathcal{F}(U).$$

(a) Show that for a commutative ring A with 1, the structure sheaf $\mathcal{O}$ on $\operatorname{Spec}(A)$ is a sheaf.

(b) Compute the stalks $\mathcal{O}_x$ for $x \in \operatorname{Spec}(\mathbb{C}[X])$.

(c*) More generally, compute the stalks $\mathcal{O}_x$ for $x \in \operatorname{Spec}(A)$, again for a commutative ring A with 1.