Problems for BMS Basic Course "Commutative Algebra"

Prof. Dr. J. Kramer

Hand in Jan 24th, after the 2nd lecture 4.45 p.m.

Please solve each problem on a different sheet of paper, and sign each sheet with your name and student ID

12th Problem Set (30+10 points)

Problem 1 (10 pts)

(a) Compute the primary decomposition of the ideal $(XZ - Y^2, YW - Z^2)$ in the ring $k[X, Y, Z, W]$, where k is a field. Determine the associated prime ideals and determine the isolated and the embedded prime ideals.

(b) Solve (a) for the ideal $(XW - YZ, X^3 - Y^2, Z^3 - W^2)$.

Problem 2 (10 pts)

(a) Consider the sheaf $\mathcal{F} := (\mathbb{Z}/17\mathbb{Z})^{\sim}$ on $\mathrm{Spec}(\mathbb{Z})$ (see set 11, problem 4). Compute the stalks $\mathcal{F}_{\mathfrak{p}}$ for $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z})$.

(b) Find a sheaf $\mathcal{F}$ on $\mathrm{Spec}(\mathbb{Z})$ such that $\mathcal{F}_{(17)} = 0$, but $\mathcal{F}_{\mathfrak{p}} \neq 0$ for all $\mathfrak{p} \neq (17)$.

Problem 3 (10 + 10 pts)

Let A be a commutative ring with 1 and M an A-module. We call $\mathfrak{p} \in \mathrm{Spec}(A)$ an associated prime of M if there exists an $x \in M$ such that $\mathfrak{p} = \mathrm{Ann}(x)$, and denote the set of such primes as $\mathrm{Ass}_A(M)$.

(a) Let $\mathfrak{a} \subseteq A$ be an ideal. Show that the associated primes of the A-module $A/\mathfrak{a}$ are precisely the associated primes of $\mathfrak{a}$ in the sense of a primary decomposition.

(b) Let M be a finitely generated abelian group. Describe the set $\mathrm{Ass}_{\mathbb{Z}}(M)$.

(c) Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of A-modules. Show that $\mathrm{Ass}_A(M) \subseteq \mathrm{Ass}_A(M') \cup \mathrm{Ass}_A(M'')$.

(d*) Show that $\overline{\mathrm{Ass}_A(M)} = \mathrm{supp}(M) := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_{\mathfrak{p}} \neq 0\}$, where $\overline{\mathrm{Ass}_A(M)}$ is the closure of $\mathrm{Ass}_A(M)$ with respect to the Zariski topology.