Problems for BMS Basic Course "Commutative Algebra"

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Hand in November 22nd, after the 2nd lecture 4.45 p.m.

Please solve each problem on a different sheet of paper, and sign each sheet with your name and student ID

5th Problem Set (30 + 10 points)

Problem 1 (10 pts)

Let A be commutative ring with 1 and let M, M', M'' be A-modules.

(a) Show that a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \tag{1}$$

splits if and only if there exists an isomorphism $M \cong M' \oplus M''$.

(b) Let F be an additive functor mapping the category of A-modules to the category of abelian groups. Show that if the sequence (1) splits, then there is an isomorphism $F(M) \cong F(M') \oplus F(M'')$.

(c) If the functor F of part (b) is covariant and exact on the right, show that the connecting homomorphisms

$$\delta_n : L_n F(M'') \longrightarrow L_{n-1} F(M')$$

in the long exact sequence of left derived functors are zero for all $n \in \mathbb{N}$.

Problem 2 (10 pts)

Let A be commutative ring with 1. Let

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

be a short exact sequence of A-modules and let P be a projective A-module. Show that the sequence

$$0 \longrightarrow \mathrm{Hom}_A(P, N') \longrightarrow \mathrm{Hom}_A(P, N) \longrightarrow \mathrm{Hom}_A(P, N'') \longrightarrow 0$$

is exact.

Problem 3 (10 pts)

Compute the $\mathbb{Z}$-module $\mathrm{Ext}^1_{\mathbb{Z}}\big((\mathbb{Z}/n\mathbb{Z})^2, \mathbb{Z}/n\mathbb{Z}\big)$ in two different ways and give an interpretation of each element as the corresponding extension in the case of $n = 2$.

Problem 4*

Let $p(X) \in \mathbb{C}[X]$ be a non-constant polynomial which is not a perfect square and $f(X,Y) := Y^2 - p(X) \in \mathbb{C}[X,Y]$. We define

$$A := \mathbb{C}[X,Y]/(f);\ x := X + (f) \in A,\ y := Y + (f) \in A.$$

Prove the following claims:

(a) $A \cong \mathbb{C}[x] \oplus y \cdot \mathbb{C}[x]$ and $\mathbb{C}[x] \cong \mathbb{C}[X]$.

(b) (f) is a prime ideal of $\mathbb{C}[X,Y]$.

(c) Let $Z = \left\{ \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathbb{C}^2 \,\middle|\, f(\xi,\eta) = 0 \right\}$ be the zero-set of f. Then, there is a bijection

$$Z \longrightarrow \operatorname{Spec}(A) \setminus \{(0)\},$$

given by

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \langle x - \xi, y - \eta \rangle.$$