Problems for BMS Basic Course "Commutative Algebra"

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Hand in December 13th, after the 2nd lecture 4.45 p.m.

Please solve each problem on a different sheet of paper, and sign each sheet with your name and student ID

8th Problem Set (40 points)

Problem 1 (10 pts)

Let A be a commutative ring with 1 and $S \subseteq A$ a multiplicatively closed set.

(a) Show that $S^{-1}A$ can be made into a commutative ring with zero element $\frac{0}{1}$ and unit element $\frac{1}{1}$ by defining

$$\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st} \quad \text{and} \quad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st} \qquad (a, b \in A; s, t \in S).$$

(b) Give an example where the natural ring homomorphism $f : A \longrightarrow S^{-1}A$ given by $a \mapsto \frac{a}{1}$ is not injective.

(c) Show that the prime ideals of $S^{-1}A$ are in bijective correspondence to the prime ideals of A which are disjoint to S.

Problem 2 (10 pts)

Let A be a commutative ring with 1 and $S \subseteq A$ a multiplicatively closed set. Show that the localization with respect to S is an exact functor from the category of A-modules to the category of $S^{-1}A$-modules.

Problem 3 (10 pts)

Let A be a commutative ring with 1, let M, N be A-modules, and $S \subseteq A$ a multiplicatively closed set. Show that there exists a uniquely determined isomorphism of $S^{-1}A$-modules

$$S^{-1}M \otimes_{S^{-1}A} S^{-1}N \cong S^{-1}(M \otimes_A N)$$

given by the assignment

$$\frac{m}{s} \otimes \frac{n}{t} \mapsto \frac{m \otimes n}{st}.$$

Problem 4 (10 pts)

Let $(J, \prec)$ be a partially ordered set such that for all $j, k \in J$ there exists an $\ell \in J$: $j \prec \ell$ and $k \prec \ell$. Let $\{A_j\}_{j \in J}$ be a system of rings together with ring homomorphisms $\varphi_{jk} : A_j \longrightarrow A_k$ for each $j \leq k$ $(j, k \in J)$. Furthermore, assume that $\varphi_{j\ell} = \varphi_{k\ell} \circ \varphi_{jk}$ holds for all $j \leq k \leq \ell$ $(j, k, \ell \in J)$.
We define the *direct (or inductive) limit*

$$A_J = \varinjlim_J A_j$$

of the so-called direct system $\{A_j, \varphi_{jk}\}$ by

$$\varinjlim_J A_j = \coprod_{j \in J} A_j / \sim,$$

where

$$a_j \sim a_k \Longleftrightarrow \exists \ell \in J \ : \ \varphi_{j\ell}(a_j) = \varphi_{k\ell}(a_k).$$

Show that:

(a) A_J is a commutative ring with 1, and there are natural ring homomorphisms $\iota_j : A_j \longrightarrow A_J$ for all $j \in J$ such that $\iota_j = \iota_k \circ \varphi_{jk}$ $(j, k \in J)$.

(b) Formulate and prove a universal mapping property characterizing the direct limit up to ring isomorphism.