Some Remarks on Transversality and Symmetry

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Humboldt-Universität zu Berlin

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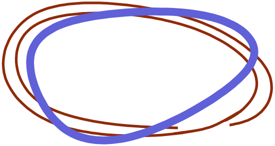

(slides available at `www.math.hu-berlin.de/~wendl/Amsterdam.pdf`)

Prelude

Example 1: Finite dimensions

Say a smooth map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{Z}_2$-**equivariant** if it satisfies

$$f(x, -y) = -f(x, y).$$

Exercise: Try to prove that every such map admits C^∞-close $\mathbb{Z}_2$-**equivariant** perturbations for which 0 is a **regular value**.

Moral: You cannot have transversality and symmetry at the same time. . .

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Example 2: Dynamics

Consider a τ-periodic orbit $\gamma : \mathbb{R} \to M$ of a smooth time-independent vector field X on M. Say γ is **nondegenerate** if

$$1 \notin \text{Spectrum}\,(\text{linearized first-return map along } \gamma)$$

Standard theorem: For generic X, all periodic orbits are **nondegenerate**.

In other words, for generic X, the S^1-**equivariant** section

$$\sigma_X : \mathcal{B} \to \mathcal{E} : (\gamma, \tau) \mapsto \dot{\gamma} - \tau X(\gamma)$$

of the Banach space bundle $\mathcal{E} \to \mathcal{B}$ with $\mathcal{B} = H^1(S^1, M) \times (0, \infty)$ and $\mathcal{E}_{(\gamma,\tau)} = L^2(\gamma^* TM)$ is **transverse to zero**.

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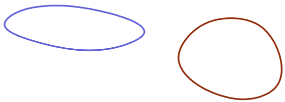

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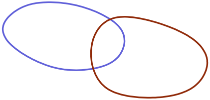

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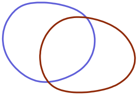

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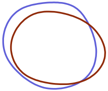

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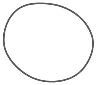

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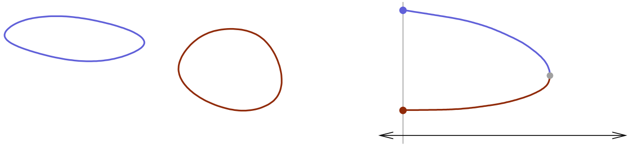

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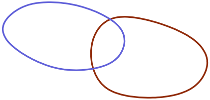

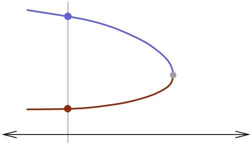

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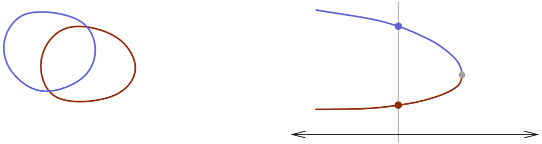

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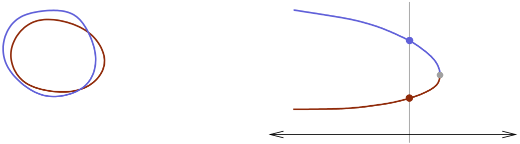

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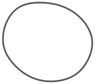

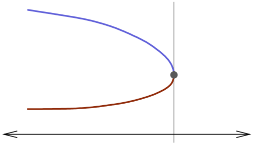

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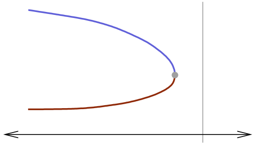

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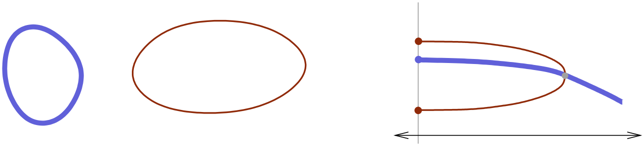

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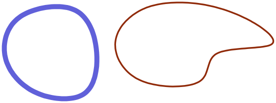

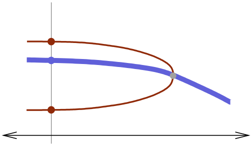

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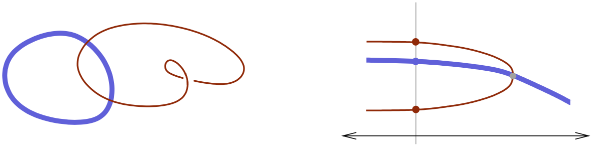

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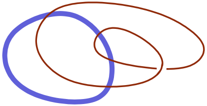

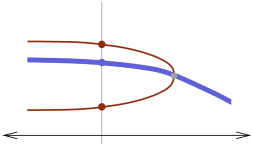

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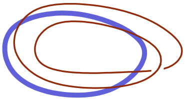

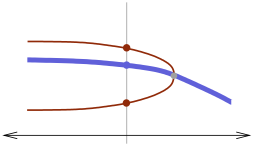

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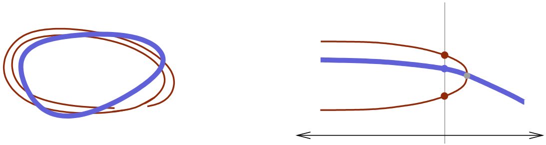

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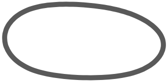

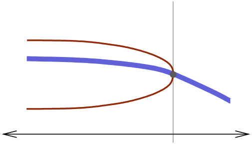

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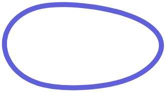

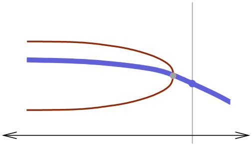

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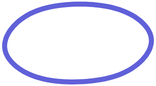

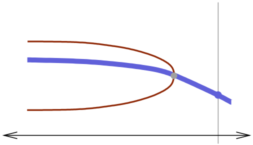

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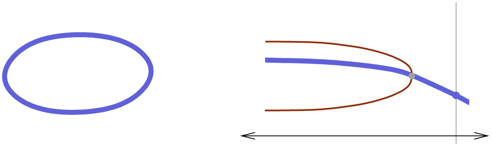

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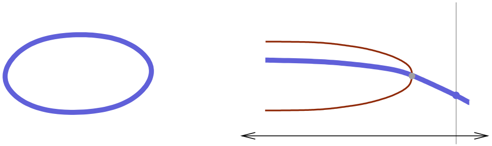

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Example 3: Gromov-Witten theory

(X, J) a symplectic Calabi-Yau 3-fold,

$$N_A^g(X) = \text{“}\#\mathcal{M}_g(A, J)\text{”} \in \mathbb{Q},$$

where $\mathcal{M}_g(A, J) \cong \bar{\partial}_J^{-1}(0)$ is the moduli space of (unparametrized) **J-holomorphic curves** $u : (\Sigma, j) \to (X, J)$ of genus $g \geq 0$ homologous to $A \in H_2(X)$. Here, $c_1(TX) = 0$ and $\dim X = 6$ imply

$$\boxed{\text{vir-dim}\, \mathcal{M}_g(A, J) = \operatorname{ind} D(\bar{\partial}_J) = 0.}$$

Trouble: If $v \in \mathcal{M}_h(A, J)$ and $d \geq 2$, then

$$\mathcal{M}_g(dA, J) \supset \left\{ u = v \circ \varphi \;\middle|\; \varphi : \Sigma_g \xrightarrow{d:1} \Sigma_h \text{ a holomorphic branched cover} \right\},$$

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Trouble: If $v \in \mathcal{M}_h(A, J)$ and $d \geq 2$, then

$$\mathcal{M}_g(dA, J) \supset \left\{ u = v \circ \varphi \;\middle|\; \varphi : \Sigma_g \xrightarrow{d:1} \Sigma_h \text{ a holomorphic branched cover} \right\},$$

which has dimension $2\#\{\text{branch points}\} > 0$ in general, $\Rightarrow \bar{\partial}_J \not\pitchfork 0.$

Prelude

Example 3: Gromov-Witten theory

(X, J) a symplectic Calabi-Yau 3-fold,

$$N_A^g(X) = \text{“}\#\mathcal{M}_g(A, J)\text{”} \in \mathbb{Q},$$

where $\mathcal{M}_g(A, J) \cong \bar{\partial}_J^{-1}(0)$ is the moduli space of (unparametrized) **J-holomorphic curves** $u : (\Sigma, j) \to (X, J)$ of genus $g \geq 0$ homologous to $A \in H_2(X)$. Here, $c_1(TX) = 0$ and $\dim X = 6$ imply

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Conjecture ("super-rigidity", Bryan-Pandharipande 2001)

For generic compatible J in a symplectic Calabi-Yau 3-fold, $\bar{\partial}_J$ intersects zero **cleanly**.

$\Rightarrow$ *For generic J, each $N_A^g(X) \in \mathbb{Q}$ is a sum of Euler numbers of obstruction bundles over the spaces of branched covers of* **finitely many** *disjoint embedded curves.*

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Outline

For the rest of the talk, I will tell you about:

1. The finite-dimensional setting.
   (Mostly elementary, no claim of originality.)
2. The holomorphic curve setting.
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(I'll talk a bit about the **other 10%** too.)

Part 1: Equivariant transversality in finite dimensions

Fix an n-dimensional orbifold M and an orbibundle $E \to M$ of rank m.
Every $x \in M$ has a **finite group** G_x and a neighborhood $\mathcal{U}_x \subset M$ such that

$$E|_{\mathcal{U}_x} \cong (\mathcal{O} \times \mathbb{R}^m) \Big/ G_x$$

for some **linear** action of G_x on $\mathbb{R}^m$ and a neighborhood $\mathcal{O} \subset \mathbb{R}^n$ of 0.

Question: Do generic $\sigma \in \Gamma(E)$ intersect the zero-section **transversely** (or at least **cleanly**)?

Sample theorem 1: If $\dim M = \operatorname{rank} E$ and $|G_x| \leq 3$ for all x, then generic sections of E intersect zero **cleanly**.

Sample theorem 2: Generic smooth functions on an orbifold are **Morse**. (cf. Wasserman '69, Hepworth '09)

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Ingredient A: **Stratification via symmetry**

For any finite group G and representations $\rho : G \to \mathrm{GL}(n, \mathbb{R})$, $\tau : G \to \mathrm{GL}(m, \mathbb{R})$, define the **submanifold**

$$M_{\rho,\tau} = \left\{ x \in M \;\middle|\; G_x \cong G, \text{ acting on } T_x M \text{ as } \rho \text{ and on } E_x \text{ as } \tau \right\}$$

and **subbundle**

$$E_{\rho,\tau} = \left\{ v \in E|_{M_{\rho,\tau}} \;\middle|\; G \text{ acts trivially on } v \right\}.$$

Let $\{\boldsymbol{\theta}_i : G \to \mathrm{Aut}_{\mathbb{R}}(W_i)\}_{i=1}^{N}$ denote the **real irreps** of G, with $\boldsymbol{\theta}_1$ as the trivial representation, and let $m_i(\rho) :=$ multiplicity of $\boldsymbol{\theta}_i$ in ρ. Then

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The orbifold M is thus a countable union of disjoint smooth **submanifolds** $M_{\rho,\tau}$ with **distinguished subbundles** $E_{\rho,\tau} \subset E|_{M_{\rho,\tau}}$.
Notice: For all $\sigma \in \Gamma(E)$, $\sigma(M_{\rho,\tau}) \subset E_{\rho,\tau}$,

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Lemma (standard transversality arguments)

For **generic** $\sigma \in \Gamma(E)$, $\sigma|_{M_{\rho,\tau}}$ is **transverse to the zero-section** of $E_{\rho,\tau} \to M_{\rho,\tau}$ for every G, ρ, τ.

$\Rightarrow$ for generic $\sigma \in \Gamma(E)$, $\mathcal{M}(\sigma) := \sigma^{-1}(0)$ is a countable union of **disjoint smooth manifolds**

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At $x \in \mathcal{M}_{\rho,\tau}(\sigma)$, there is a linearization

$$\mathbf{D}_x := D\sigma(x) : T_x M \to E_x.$$

Recall the irreps $\{\boldsymbol{\theta}_i : G_x \to \mathrm{Aut}_{\mathbb{R}}(W_i)\}_{i=1}^N$, and denote $d_i := \dim W_i$.

Since $\mathbf{D}_x$ is **G_x-equivariant**, Schur's lemma implies that it splits with respect to the **isotypic decompositions** $T_x M = \bigoplus_{i=1}^N T_x M^i$ of ρ and $E_x = \bigoplus_{i=1}^N E_x^i$ of τ, giving

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These operators have **Fredholm indices**

$$\operatorname{ind} \mathbf{D}_x^i = d_i \left[m_i(\rho) - m_i(\tau) \right],$$

and we know $\mathbf{D}_x^1$ **is surjective** if σ is generic.

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Part 1: Equivariant transversality in finite dimensions

Ingredient B: **Splitting the linearization**

At $x \in \mathcal{M}_{\rho,\tau}(\sigma)$, there is a linearization

$$\mathbf{D}_x := D\sigma(x) : T_x M \to E_x.$$

Recall the irreps $\{\boldsymbol{\theta}_i : G_x \to \mathrm{Aut}_{\mathbb{R}}(W_i)\}_{i=1}^N$, and denote $d_i := \dim W_i$.

Since $\mathbf{D}_x$ is **G_x-equivariant**, Schur's lemma implies that it splits with respect to the **isotypic decompositions** $T_x M = \bigoplus_{i=1}^N T_x M^i$ of ρ and $E_x = \bigoplus_{i=1}^N E_x^i$ of τ, giving

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Part 1: Equivariant transversality in finite dimensions

Ingredient C: **Building walls** (in the sense of "crossing")

G_x acts on $\ker \mathbf{D}_x^i$ and $\operatorname{coker} \mathbf{D}_x^i$ as the irrep θ_i with some multiplicities, so their dimensions are **divisible by** d_i.

For nonnegative integers $\mathbf{k} = (k_1, \dots, k_N)$ and $\mathbf{c} = (c_1, \dots, c_N)$, let

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Workhorse theorem

For generic $\sigma \in \Gamma(E)$, for all choices $G, \rho, \tau, \mathbf{k}, \mathbf{c}$,

$$\mathcal{M}_{\rho,\tau}(\sigma; \mathbf{k}, \mathbf{c}) \subset \mathcal{M}_{\rho,\tau}(\sigma)$$

is a **smooth submanifold** with **codimension** $\sum_{i=1}^N t_i k_i c_i$, where $t_i := \dim_{\mathbb{R}} \operatorname{End}_G(W_i) \in \{1, 2, 4\}$.

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Replace $\Gamma(E)$ with a suitable Banach manifold of sections.
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We need to prove $D\Phi(\sigma_0, x_0)$ is **surjective**, then apply the implicit function theorem. Consider $s \in T_{\sigma_0}\Gamma(E)$ with $s(x_0) = 0$, so $(s, 0) \in T_{(\sigma_0,x_0)}\mathcal{M}_{\rho,\tau}$, and

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Given $\mathbf{D} : T_xM \to E_x$, every linear map $\mathbf{A} : \ker \mathbf{D} \to \operatorname{coker} \mathbf{D}$ can be **lifted/extended** to a map $\widetilde{\mathbf{A}} : T_xM \to E_x$.

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Part 1: Equivariant transversality in finite dimensions

Recall **Sample theorem 1**: If $\dim M = \operatorname{rank} E$ and $|G_x| \le 3$ for all x, then generic sections of E intersect zero **cleanly**.

Proof for $|G_x| \le 2$:
For $x \in \mathcal{M}_{\rho,\tau}(\sigma)$ with $G_x = \mathbb{Z}_2$, we have **two irreps $\theta_\pm$** with $d_\pm = 1$, so write $\mathbf{D}_x = \mathbf{D}_x^+ \oplus \mathbf{D}_x^-$. If σ is generic, $\mathcal{M}_{\rho,\tau}(\sigma)$ is a manifold of dimension $m_1(\rho) - m_1(\tau) = \dim\ker \mathbf{D}_x^+$, so it will suffice to prove

$$\dim \mathcal{M}_{\rho,\tau}(\sigma) = \dim\ker \mathbf{D}_x, \quad \text{i.e.} \quad \mathbf{D}_x^- \text{ is } \textbf{injective}!.$$

Useful observation: $\operatorname{ind} \mathbf{D}_x^- = -\operatorname{ind} \mathbf{D}_x^+ = -[m_+(\rho) - m_+(\tau)] \le 0$.
Now suppose $k_- := \dim\ker \mathbf{D}_x^- > 0$, and write $c_- := \dim\operatorname{coker} \mathbf{D}_x^-$ as $k_- + [m_+(\rho) - m_+(\tau)]$. Then $x \in \mathcal{M}_{\rho,\tau}(\sigma; \mathbf{k}, \mathbf{c})$ with

$$\begin{aligned}\dim \mathcal{M}_{\rho,\tau}(\sigma; \mathbf{k}, \mathbf{c}) &= m_+(\rho) - m_+(\tau) - k_- c_- \\ &= -k_-^2 - (k_- - 1)[m_+(\rho) - m_+(\rho)] < 0.\end{aligned}$$

$\square$

What makes this work? Only 2 irreps $\Rightarrow$ **can compute** $\operatorname{ind} \mathbf{D}_x^-$.

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Proof:
Let $E = T^*M$, then we need to show that for **generic** $f : M \to \mathbb{R}$, $df \in \Gamma(E)$ is **transverse to zero**. There are two new features:

1. For $x \in df^{-1}(0)$, the operator $D(df)(x)$ is always **symmetric**, so the previous codimension formula changes to

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Fix a $2n$-dimensional **symplectic cobordism** X with cylindrical stable Hamiltonian ends, assume J is a **compatible** almost complex structure.

Standard transversality result

For generic J, the open set

$$\mathcal{M}^*(J) := \left\{ u \in \mathcal{M}(J) \;\middle|\; u \text{ not multiply covered} \right\}$$

is a **transversely cut-out** manifold of $\dim = \text{vir-dim}$.

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What structure does $\mathcal{M}(J)$ generically have near the **multiple covers**?

Sample theorem 3: If $\dim X = 4$ and J is **generic**, then **unbranched** covers of **immersed** J-holomorphic curves with **trivial normal bundle** and **vanishing CZ-indices** are **cut out transversely**. (cf. Taubes '96)

Let's see how this works in the case of immersed **double** covers. . .

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Ingredient A: **Stratification via symmetry**

$$\mathcal{M}^2(J) := \Big\{ u = v \circ \varphi \;\Big|\; v \in \mathcal{M}^*(J) \text{ immersed, } \varphi \text{ a holomorphic unbranched cover of degree } 2 \Big\}.$$

Observations:

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Ingredient B: **Splitting the linearization**

For $u : (\Sigma, j) \to (X, J)$ immersed, restricting $D\bar{\partial}_J(u)$ to the **normal bundle** $N_u \to \Sigma$ defines a real-linear **Cauchy-Riemann type** operator

$$\mathbf{D}_u^N : \Gamma(N_u) \to \Omega^{0,1}(\Sigma, N_u)$$

such that u is **transversely cut out** $\Leftrightarrow$ $\mathbf{D}_u^N : W^{k,p} \to W^{k-1,p}$ is **surjective** ($k \in \mathbb{N}$, $1 < p < \infty$).

For $u = v \circ \varphi \in \mathcal{M}^2(J)$, the nontrivial deck transformation $\psi \in \mathrm{Aut}(\varphi)$ defines a **splitting** $\Gamma(N_u) = \Gamma_+(N_u) \oplus \Gamma_-(N_u)$, where

$$\Gamma_\pm(N_u) = \left\{ \eta \in \Gamma(N_u) \;\middle|\; \eta \circ \psi = \pm\eta \right\}.$$

Then $\mathbf{D}_u^N$ sends $\Gamma_\pm(N_u)$ to $\Omega^{0,1}_\pm(\Sigma, N_u)$, defining a splitting

$$\mathbf{D}_u^N = \mathbf{D}_u^+ \oplus \mathbf{D}_u^-$$

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For $u : (\Sigma, j) \to (X, J)$ immersed, restricting $D\bar{\partial}_J(u)$ to the **normal bundle** $N_u \to \Sigma$ defines a real-linear **Cauchy-Riemann type** operator

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Brief digression *(why you should believe super-rigidity is true)*

For general branched covers of **arbitrary degree**, there is always a splitting

$$\mathbf{D}_u^N = \bigoplus_{i=1}^{N} (\mathbf{D}_u^{\boldsymbol{\theta}_i})^{\oplus k_i},$$

whose summands are **Cauchy-Riemann type** operators corresponding to the **irreps** of some finite group.

Slightly surprising lemma:
For any $u = v \circ \varphi$ with v a **closed, immersed, simple curve** with $\operatorname{ind} \mathbf{D}_v^N = 0$ and φ a **holomorphic branched cover**, every summand of $\mathbf{D}_u^N$ satisfies

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Ingredient C: **Building walls**

For nonnegative integers $\mathbf{k} = (k_+, k_-)$ and $\mathbf{c} = (c_+, c_-)$, let

$$\mathcal{M}^2(J; \mathbf{k}, \mathbf{c}) = \left\{ u \in \mathcal{M}^2(J) \,\middle|\, \dim \ker \mathbf{D}_u^{\pm} = k_\pm,\ \dim \operatorname{coker} \mathbf{D}_u^{\pm} = c_\pm \right\}$$

Workhorse theorem

For generic J and all choices of $g, A, \mathbf{k}, \mathbf{c}$ satisfying the **workhorse lemma** (to be discussed below),

$$\mathcal{M}^2(J; \mathbf{k}, \mathbf{c}) \subset \mathcal{M}^2(J)$$

is a **smooth submanifold** with **codimension** $k_+c_+ + k_-c_-$.

When this (and its generalization for arbitrary branched covers) holds, it **implies super-rigidity** via dimension counting arguments.

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Proving the workhorse theorem

Perturbing J causes **zeroth-order** perturbations in $\mathbf{D}_u^N$. We thus need to know whether every linear map $\ker \mathbf{D}_u^N \to \operatorname{coker} \mathbf{D}_u^N$ can be realized as

$$\ker \mathbf{D}_u^N \xrightarrow{A} \Omega^{0,1}(\Sigma, N_u) \xrightarrow{\text{proj}} \operatorname{coker} \mathbf{D}_u^N$$

for some **bundle map** $A : N_u \to \Lambda^{0,1} T^*\Sigma \otimes N_u$. If not, then given bases $(\eta_i) \in \ker \mathbf{D}_u^N$ and $(\xi_j) \in \ker(\mathbf{D}_u^N)^* \cong \operatorname{coker} \mathbf{D}_u^N$, there exist nontrivial coefficients $c_{ij} \in \mathbb{R}$ such that

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for **all** zeroth-order perturbations A.

In other words, $\sum_{i,j} c_{ij} \eta_i \otimes \xi_j \in \Gamma(N_u \otimes \Lambda^{0,1} T^*\Sigma \otimes N_u)$ is **identically zero**.

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Part 2: Holomorphic curves

Definition (a "quadratic unique continuation" property)

A linear partial differential operator $\mathbf{D} : \Gamma(E) \to \Gamma(F)$ on Euclidean vector bundles $E, F \to \Sigma$ satisfies **Petri's condition** if the canonical map

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is **injective** for every open subset $\mathcal{U} \subset \Sigma$.

Example 1: Elliptic operators on **1-dimensional** domains
($\Rightarrow$ bifurcation theory for **periodic orbits**)

Example 2: Cauchy-Riemann operators on **trivial line bundles**
($\Rightarrow$ Sample theorem 3)

Meta-theorem (cf. A. Doan and T. Walpuski, in preparation):
Equivariant transversality problems are **tractable** for a large class of elliptic operators that satisfy **Petri's condition**.

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For **generic** J and every J-holomorphic curve u, $\mathbf{D}_u^N$ satisfies **Petri's condition**.

Final remarks:

1. **Unique continuation** $\Rightarrow$ Petri holds whenever $\ker \mathbf{D}_u^N$ or $\operatorname{coker} \mathbf{D}_u^N$ has $\dim \leq 3$.
2. It does not always hold, e.g. for $\mathbf{D} = \bar{\partial}$ and $\mathbf{D}^* = -\partial$:

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(**Achtung**: real tensor products!)

3. For generic J, $\mathbf{D}_u^N$ has **invertible** complex-antilinear part $\Rightarrow$ the above counterexample **never appears**.
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