LECTURE 10

Cylindrical contact homology and the tight 3-tori

Contents

We've now developed enough of the technical machinery of holomorphic curves to be able to give a rigorous construction of the most basic version of SFT and apply it to a problem in contact topology.

1. Contact structures on $\mathbb{T}^3$ and Giroux torsion

As a motivating goal in this lecture, we will prove a result about the classification of contact structures on $\mathbb{T}^3 = S^1 \times S^1 \times S^1$. Denote the three global coordinates on $\mathbb{T}^3$ valued in $S^1 = \mathbb{R}/\mathbb{Z}$ by (ρ, ϕ, θ), and for any $k \in \mathbb{N}$, consider the contact structure

$$\xi_k := \ker \alpha_k, \qquad \text{where} \qquad \alpha_k := \cos(2\pi k \rho)\, d\theta + \sin(2\pi k \rho)\, d\phi.$$

It is an easy exercise to verify that these all satisfy the contact condition $\alpha_k \wedge d\alpha_k > 0$; see Figure 1 for a visual representation. The following result is originally due to Giroux [Gir94] and Kanda [Kan97].

Theorem 10.1. *For each pair of positive integers $k \neq \ell$, the contact manifolds $(\mathbb{T}^3, \xi_k)$ and $(\mathbb{T}^3, \xi_\ell)$ are not contactomorphic.*

One of the reasons this result is interesting is that it cannot be proved using any so-called "classical" invariants, i.e. invariants coming from algebraic topology. An example of a classical invariant would be the Euler class of the oriented vector bundle $\xi_k \to \mathbb{T}^3$, or anything else that depends only on the isomorphism class of this

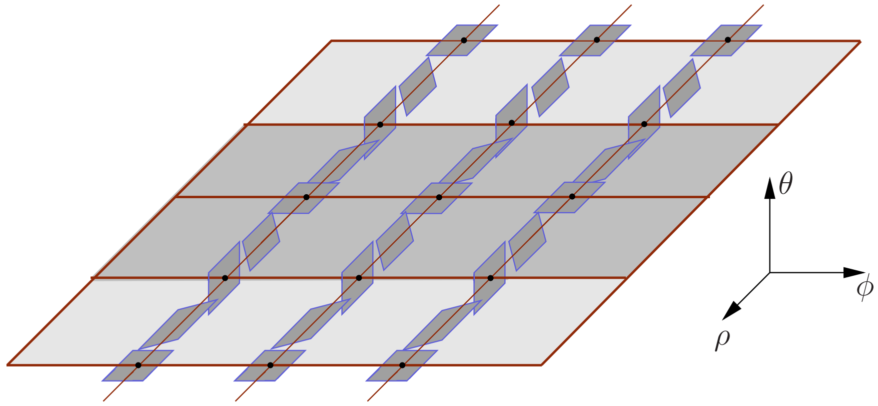

FIGURE 1. The contact structures ξ_k on $\mathbb{T}^3$ can be constructed by gluing k copies of the same model $[0,1] \times \mathbb{T}^2$ to each other cyclically.

bundle. The following observation shows that such invariants will never distinguish ξ_k from ξ_ℓ.

PROPOSITION 10.2. *For every $k, \ell \in \mathbb{N}$, ξ_k and ξ_ℓ are homotopic through a smooth family of oriented 2-plane fields on $\mathbb{T}^3$.*

PROOF. In fact, all the ξ_k can be deformed smoothly to $\ker d\rho$, via the homotopy

$$\ker\left[(1-s)\,\alpha_k + s\,d\rho\right], \qquad s \in [0,1].$$

□

REMARK 10.3. One can check in fact that the 1-form in the homotopy given above is contact for every $s \in [0,1)$, so Gray's stability theorem implies that every ξ_k is isotopic to an arbitrarily small perturbation of the foliation $\ker d\rho$. In [Gir94], Giroux used this observation to show that all of them are what we now call *weakly symplectically fillable*. If $\ker d\rho$ were also contact, then Gray's theorem would imply that ξ_k and ξ_ℓ are always isotopic. Thus Theorem 10.1 indicates the impossibility of modifying a homotopy from ξ_k to ξ_ℓ into one that passes only through contact structures.

Let us place this discussion in a larger context. Using the coordinates (ρ, ϕ, θ) on $\mathbb{R} \times \mathbb{T}^2$, a pair of smooth functions $f, g : \mathbb{R} \to \mathbb{R}$ gives rise to a contact form

$$\alpha = f(\rho)\,d\theta + g(\rho)\,d\phi$$

whenever the function $D(\rho) := f(\rho)g'(\rho) - f'(\rho)g(\rho)$ is everywhere positive. Indeed, we have $\alpha \wedge d\alpha = D(\rho)\,d\rho \wedge d\phi \wedge d\theta$, and one easily derives a similar formula for the Reeb vector field,

$$R_\alpha = \frac{1}{D(\rho)}\left[g'(\rho)\,\partial_\theta - f'(\rho)\,\partial_\phi\right].$$

The condition $D > 0$ means geometrically that the path $(f,g) : \mathbb{R} \to \mathbb{R}^2$ winds counterclockwise around the origin with its angular coordinate strictly increasing. The simplest special case is the contact form

$$\alpha_{\mathrm{GT}} := \cos(2\pi\rho)\,d\theta + \sin(2\pi\rho)\,d\phi,$$

which matches the formula for α_1 on $\mathbb{T}^3$ given above. Let $\xi_{\mathrm{GT}} := \ker \alpha_{\mathrm{GT}}$ on $\mathbb{R} \times \mathbb{T}^2$.

DEFINITION 10.4. The **Giroux torsion** $\mathrm{GT}(M, \xi) \in \mathbb{N} \cup \{0, \infty\}$ of a contact 3-manifold (M, ξ) is the supremum of the set of positive integers k such that there exists a contact embedding

$$\left([0, k] \times \mathbb{T}^2, \xi_{\mathrm{GT}}\right) \hookrightarrow (M, \xi).$$

We write $\mathrm{GT}(M, \xi) = 0$ if no such embedding exists for any k, and $\mathrm{GT}(M, \xi) = \infty$ if it exists for all k.

EXAMPLE 10.5. The tori $(\mathbb{T}^3, \xi_k)$ for $k \geq \mathbb{Z}$ are contactomorphic to $(\mathbb{R}\times\mathbb{T}^2, \xi_{\mathrm{GT}})/k\mathbb{Z}$, with $k\mathbb{Z}$ acting by translation of the ρ-coordinate. Thus $\mathrm{GT}(\mathbb{T}^3, \xi_k) \geq k - 1$.

A 2-torus $T \subset (M, \xi)$ embedded in a contact 3-manifold is called **pre-Lagrangian** if a neighborhood of T in (M, ξ) admits a contactomorphism to a neighborhood of $\{0\} \times \mathbb{T}^2$ in $(\mathbb{R} \times \mathbb{T}^2, \xi_{\mathrm{GT}})$, identifying T with $\{0\} \times \mathbb{T}^2$. The neighborhood in $\mathbb{R} \times \mathbb{T}^2$ can be arbitrarily small, thus the existence of a pre-Lagrangian torus does not imply $\mathrm{GT}(M, \xi) > 0$; in fact, pre-Lagrangian tori always exist in abundance, e.g. as boundaries of neighborhoods of transverse knots (using the contact model provided by the transverse neighborhood theorem). But given any pre-Lagrangian torus $T \subset (M, \xi)$, one can make a local modification of ξ near T to produce a new contact structure (up to isotopy) with positive Giroux torsion. Define (M', ξ') from (M, ξ) by replacing the small neighborhood $((-\epsilon, \epsilon) \times \mathbb{T}^2, \xi_{\mathrm{GT}})$ with $((-\epsilon, 1+\epsilon) \times \mathbb{T}^2, \xi_{\mathrm{GT}})$, then identify M' with M by a choice of compactly supported diffeomorphism $(-\epsilon, 1+\epsilon) \to (-\epsilon, \epsilon)$. There is now an obvious contact embedding of $([0, 1] \times \mathbb{T}^2, \xi_{\mathrm{GT}})$ into (M, ξ'), hence $\mathrm{GT}(M, \xi') \geq 1$. Moreover, one can adapt the proof of Prop. 10.2 above to show that ξ' is homotopic to ξ through a smooth family of oriented 2-plane fields. The operation changing ξ to ξ' is known as a **Lutz twist** along T. In this language, we see that for each $k \in \mathbb{N}$, $(\mathbb{T}^3, \xi_{k+1})$ is obtained from $(\mathbb{T}^3, \xi_k)$ by performing a Lutz twist along $\{0\} \times \mathbb{T}^2$.

The invariant $\mathrm{GT}(M, \xi)$ is easy to define, but hard to compute in general. The natural guess,

$$\mathrm{GT}(\mathbb{T}^3, \xi_k) = k - 1,$$

turns out to be correct, as was shown in [Gir00], so this is one way to prove Theorem 10.1, but not the approach we will take. The following example shows that one must in any case be careful with such guesses.

EXAMPLE 10.6. For each $k \in \mathbb{N}$, define a model of $S^1 \times S^2$ by

$$S^1 \times S^2 \cong \left([0, k + 1/2] \times \mathbb{T}^2\right) / \sim$$

where the equivalence relation identifies $(\rho, \phi, \theta) \sim (\rho, \phi', \theta)$ for $\rho \in \{0, k+1/2\}$ and every $\theta, \phi, \phi' \in S^1$. Near $\rho = 0$ and $\rho = k+1/2$, this means thinking of (ρ, ϕ) as polar coordinates, so the two subsets $\{\rho = 0\}$ and $\{\rho = k + 1/2\}$ become circles of the form $S^1 \times \{\text{const}\}$ embedded in $S^1 \times S^2$. Since the ϕ-coordinate is singular at these two circles, the contact form α_{GT} needs to be modified slightly in this region before it will descend to a smooth contact form on $S^1 \times S^2$: this can be done by a C^0-small modification of the form $f(\rho)\,d\theta + g(\rho)\,d\phi$, and the resulting contact structure is then uniquely determined up to isotopy. We shall call this contact manifold

$$(S^1 \times S^2, \xi_k).$$

Now observe that for each $k \in \mathbb{N}$, $(S^1 \times S^2, \xi_{k+1})$ is obtained from $(S^1 \times S^2, \xi_k)$ by a Lutz twist. However, both contact manifolds are also **overtwisted**: recall that a contact 3-manifold (M, ξ) is overtwisted whenever it contains an embedded closed 2-disk $\mathcal{D} \subset M$ such that $T(\partial\mathcal{D}) \subset \xi$ but $T\mathcal{D}|_{\partial\mathcal{D}} \pitchfork \xi$. (Exercise: find a disk with this property in $(S^1 \times S^2, \xi_k)$!) Eliashberg's flexibility theorem for overtwisted contact structures [**Eli89**] implies that whenever ξ and ξ' are two contact structures on a closed 3-manifold that are both overtwisted and are homotopic as oriented 2-plane fields, they are actually isotopic. As a consequence, the contact structures ξ_k on $S^1 \times S^2$ defined above for every $k \in \mathbb{N}$ are all isotopic to each other. As tends to be the case with most interesting h-principles, the isotopy is very hard to see concretely, but it must exist.

EXERCISE 10.7. Show that if (M, ξ) is a closed overtwisted contact 3-manifold, then $\mathrm{GT}(M, \xi) = \infty$.

In contrast to the $S^1 \times S^2$ example above, the contact manifolds $(\mathbb{T}^3, \xi_k)$ are not overtwisted, they are **tight**—in fact, the classification of contact structures on $\mathbb{T}^3$ by Giroux [**Gir94**, **Gir99**, **Gir00**] and Kanda [**Kan97**] states that these are *all* of the tight contact structures on $\mathbb{T}^3$ up to contactomorphism. We will use cylindrical contact homology to show that they are not contactomorphic to each other. The reader should keep Example 10.6 in mind and try to spot the reason why the same argument cannot work for $(S^1 \times S^2, \xi_k)$.

REMARK 10.8. It has been conjectured that the converse of Exercise 10.7 might also hold, so every closed tight contact 3-manifold would have finite Giroux torsion. This conjecture is wide open.

2. Definition of cylindrical contact homology

2.1. Preliminary remarks. Cylindrical contact homology is the natural "first attempt" at using holomorphic curves in symplectizations to define a Floer-type invariant of contact manifolds (M, ξ). The idea is to define a chain complex generated by Reeb orbits in M and a differential ∂ that counts holomorphic cylinders in $\mathbb{R} \times M$. We already know some pretty good reasons why this idea cannot work in general: in order to prove $\partial^2 = 0$, we need to be able to identify the space of rigid "broken" holomorphic cylinders (these are what is counted by ∂^2) with the boundary of the compactified 1-dimensional space of index 2 cylinders (up to $\mathbb{R}$-translation). But this compactified boundary has more than just broken cylinders in it, see Figure 2. In order to define cylindrical contact homology, one must therefore restrict to situations in which complicated pictures like Figure 2 cannot occur. The first useful remark in this direction is that since we are working with a stable Hamiltonian structure of the form $(d\alpha, \alpha)$ for a contact form α, a certain subset of the scenarios allowed by the SFT compactness theorem can be excluded immediately. Indeed:

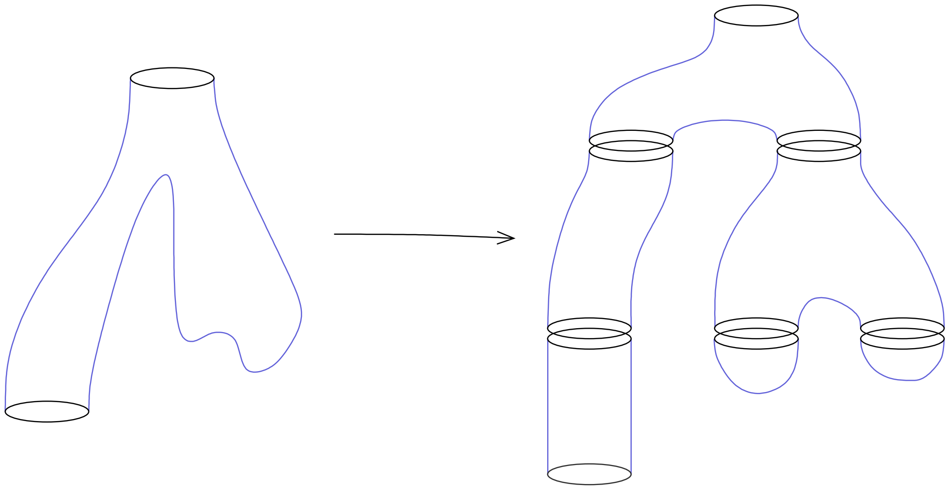

Figure 2. A family of holomorphic cylinders can converge in the SFT topology to buildings that include more complicated curves than cylinders—this is why cylindrical contact homology is not well defined for all contact manifolds.

Proposition 10.9. *If $J \in \mathcal{J}(\alpha)$ and $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ is an asymptotically cylindrical J-holomorphic curve, then u has at least one positive puncture.*

Let us give two proofs of this result, since both contain useful ideas. As preparation for the first proof, recall the definition of energy for curves in symplectizations of contact manifolds that we wrote down in Lecture 1:

$$E(u) := \sup_{f \in \mathcal{T}} \int_{\dot{\Sigma}} u^* d(e^{f(r)} \alpha),$$

where

$$\mathcal{T} := \left\{ f \in C^\infty(\mathbb{R}, (-1,1)) \;\middle|\; f' > 0 \right\}.$$

This formula is not identical to the definition of energy used in Lecture 9, but it is equivalent in the sense that any uniform bounds on one imply similar uniform bounds on the other.

First proof of Proposition 10.9. Denote the positive and negative punctures of $u : \dot{\Sigma} \to \mathbb{R} \times M$ by Γ^+ and Γ^- respectively, and suppose u is asymptotic at $z \in \Gamma^\pm$ to the orbit γ_z with period $T_z > 0$. Choose any $f \in \mathcal{T}$ and denote $f_\pm := \lim_{r \to \pm\infty} f(r) \in [-1, 1]$. Since $d(e^{f(r)} \alpha)$ tames $J \in \mathcal{J}(\alpha)$, Stokes' theorem gives

$$0 \le E(u) = e^{f_+} \sum_{z \in \Gamma^+} T_z - e^{f_-} \sum_{z \in \Gamma^-} T_z, \tag{10.1}$$

hence Γ^+ cannot be empty. □

Remark 10.10. The proof via Stokes' theorem works just as well if instead of $\mathbb{R} \times M$, u lives in the completion of an exact symplectic cobordism (W, ω) with concave boundary $(M_-, \xi_- = \ker \alpha_-)$ and convex boundary $(M_+, \xi_+ = \ker \alpha_+)$.

Recall that this means $\partial W = -M_- \sqcup M_+$, and $\omega = d\lambda$ for a 1-form λ that restricts to positive contact forms $\lambda|_{TM_\pm} = \alpha_\pm$. As in Lecture 1, we will write

$$\mathcal{J}(W, \omega, \alpha_+, \alpha_-) \subset \mathcal{J}(\widehat{W})$$

for the space of almost complex structures J on $\widehat{W} := ((-\infty, 0] \times M_-) \cup_{M_-} W \cup_{M_+} ([0, \infty) \times M_+)$ that are compatible with ω on W and belong to $\mathcal{J}(\alpha_\pm)$ on the cylindrical ends. The energy of a J-holomorphic curve $u : (\dot{\Sigma}, j) \to (\widehat{W}, J)$ is then

$$E(u) := \sup_{f \in \mathcal{T}} \int_{\dot{\Sigma}} u^* d\lambda_f,$$

where $\mathcal{T} := \{f \in C^\infty(\mathbb{R}, (-1, 1)) \mid f' > 0 \text{ and } f(r) = r \text{ near } r = 0\}$ and

$$\lambda_f := \begin{cases} e^{f(r)}\alpha_+ & \text{on } [0, \infty) \times M_+, \\ \lambda & \text{on } W, \\ e^{f(r)}\alpha_- & \text{on } (-\infty, 0] \times M_-. \end{cases}$$

The above proof now generalizes verbatim to show that u must always have a positive puncture. Notice that in both settings, the argument also gives a uniform bound for the energy in terms of the periods of the positive asymptotic orbits.

REMARK 10.11. We can also prove Prop. 10.9 using the fact that $u^* d\alpha \geq 0$ for any $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ with $J \in \mathcal{J}(\alpha)$. Indeed, Stokes' theorem then gives

$$0 \leq \int_{\dot{\Sigma}} u^* d\alpha = \sum_{z \in \Gamma^+} T_z - \sum_{z \in \Gamma^-} T_z. \tag{10.2}$$

The quantity $\int_{\dot{\Sigma}} u^* d\alpha$ is sometimes called the **contact area** of u. This version of the argument however does not easily generalize to arbitrary exact cobordisms.

The second proof is based on the maximum principle for subharmonic functions.

PROPOSITION 10.12. *Suppose $J \in \mathcal{J}(\alpha)$ and $u = (u_\mathbb{R}, u_M) : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ is J-holomorphic, where $\dot{\Sigma}$ has no boundary. Then $u_\mathbb{R} : \dot{\Sigma} \to \mathbb{R}$ has no local maxima.*

PROOF. In any local holomorphic coordinates (s, t) on a region in $\dot{\Sigma}$, the nonlinear Cauchy-Riemann equation for u is equivalent to the system of equations

$$\begin{aligned} \partial_s u_\mathbb{R} - \alpha(\partial_t u_M) &= 0, \\ \partial_t u_\mathbb{R} + \alpha(\partial_s u_M) &= 0, \\ \pi_\xi \, \partial_s u_M + J \pi_\xi \, \partial_t u_M &= 0, \end{aligned}$$

where $\pi_\xi : TM \to \xi$ denotes the projection along the Reeb vector field. This gives

$$\begin{aligned} -\Delta u_\mathbb{R} = -\partial_s^2 u_\mathbb{R} - \partial_t^2 u_\mathbb{R} &= -\partial_s \left[\alpha(\partial_t u_M)\right] + \partial_t \left[\alpha(\partial_s u_M)\right] \\ &= -d\alpha(\partial_s u_M, \partial_t u_M) = -d\alpha(\pi_\xi \partial_s u_M, J \pi_\xi \partial_s u_M) \leq 0 \end{aligned}$$

since $J|_\xi$ is tamed by $d\alpha|_\xi$, hence $u_\mathbb{R}$ is subharmonic. The result thus follows from the maximum principle, see e.g. [**Eva98**]. □

SECOND PROOF OF PROPOSITION 10.9. If $u = (u_{\mathbb{R}}, u_M) : \dot{\Sigma} \to \mathbb{R} \times M$ has no positive puncture then $u_{\mathbb{R}} : \dot{\Sigma} \to \mathbb{R}$ is a proper function bounded above, and therefore has a local maximum, contradicting Proposition 10.12. □

REMARK 10.13. The proof via the maximum principle does not generalize to arbitrary exact cobordisms $(W, d\lambda)$, but it does work in *Stein* cobordisms, i.e. if λ_f and J are related by $\lambda_f = -dF \circ J$ for some plurisubharmonic function $F : \widehat{W} \to \mathbb{R}$, then $F \circ u : \dot{\Sigma} \to \mathbb{R}$ is subharmonic (cf. [**CE12**]).

With these preliminaries understood, the next two exercises reveal one natural setting in which breaking of cylinders can be kept under control. Both exercises are essentially combinatorial.

EXERCISE 10.14. Suppose **u** is a stable J-holomorphic building in a completed symplectic cobordism $\widehat{W}$ with the following properties:

(1) **u** has arithmetic genus 0 and exactly one positive puncture;
(2) every connected component of **u** has at least one positive puncture.

Show that **u** has no nodes, and all of its connected components have *exactly* one positive puncture.

EXERCISE 10.15. Suppose that in addition to the conditions of Exercise 10.14, **u** has exactly one negative puncture and no connected component of **u** is a plane. Show that every level of **u** then consists of a single cylinder with one positive and one negative end.

Exercise 10.15 makes it reasonable to define a Floer-type theory counting only cylinders in any setting where planes can be excluded, for instance because the Reeb vector field has no contractible orbits. This is not always possible, e.g. Hofer [**Hof93**] proved that on overtwisted contact manifolds, there is *always* a plane (which is why the Weinstein conjecture holds). So the invariant we construct will not be defined in such settings, but it happens to be ideally suited to the study of $(\mathbb{T}^3, \xi_k)$.

2.2. A compactness result for cylinders. Fix a closed contact manifold (M, ξ) of dimension $2n-1$ and a primitive homotopy class of loops $h \in [S^1, M]$. By **primitive**, we mean that h is not equal to Nh' for any $h' \in [S^1, M]$ and an integer $N > 1$, and this assumption will be crucial for technical reasons in the following.[1] Given a contact form α for ξ, let

$$\mathcal{P}_h(\alpha)$$

denote the set of closed Reeb orbits homotopic to h, where two Reeb orbits are identified if they differ only by parametrization.

DEFINITION 10.16. Given a contact manifold (M, ξ) and a primitive homotopy class $h \in [S^1, M]$, we will say that a contact form α for ξ is **h-admissible** if:

(1) All orbits in $\mathcal{P}_h(\alpha)$ are nondegenerate;

[1] It is to be expected that cylindrical contact homology can be defined also for non-primitive homotopy classes, but this would require more sophisticated methods to address transversality problems. The assumption that h is primitive allows us to assume that all holomorphic curves in the discussion are somewhere injective, hence they are always regular if J is generic.

(2) There are no contractible closed Reeb orbits.

Similarly, we will say that (M, ξ) is **h-admissible** if a contact form with the above properties exists.

DEFINITION 10.17. Given $h \in [S^1, M]$ and an h-admissible contact form α on (M, ξ), we will say that an almost complex structure $J \in \mathcal{J}(\alpha)$ is **h-regular** if every J-holomorphic cylinder in $\mathbb{R} \times M$ with a positive and a negative end both asymptotic to orbits in $\mathcal{P}_h(\alpha)$ is Fredholm regular.

PROPOSITION 10.18. *If $h \in [S^1, M]$ is a primitive homotopy class of loops and α is h-admissible on (M, ξ), then the space of h-regular almost complex structures is comeager in $\mathcal{J}(\alpha)$.*

PROOF. Since h is primitive, the asymptotic orbits for the relevant holomorphic cylinders cannot be multiply covered, hence all of these cylinders are somewhere injective. The result therefore follows from the standard transversality results proved in Lecture 8 for somewhere injective curves in symplectizations. □

PROPOSITION 10.19. *Given an h-admissible contact form α, an h-regular almost complex structure $J \in \mathcal{J}(\alpha)$ and an orbit $\gamma \in \mathcal{P}_h(\alpha)$, suppose u_k is a sequence of J-holomorphic cylinders in $\mathbb{R} \times M$ with one positive puncture at γ and one negative puncture. Then u_k has a subsequence convergent in the SFT topology to a broken J-holomorphic cylinder, i.e. a stable building $\mathbf{u}_\infty$ whose levels $u_\infty^1, \ldots, u_\infty^{N_+}$ are each cylinders with one positive and one negative puncture. Moreover, each level satisfies $\operatorname{ind}(u_\infty^N) \geq 1$, thus for large k in the convergent subsequence,*

$$\operatorname{ind}(u_k) = \sum_{N=1}^{N_+} \operatorname{ind}(u_\infty^N) \geq N_+.$$

PROOF. Let's start with some bad news: the standard SFT compactness theorem is not applicable in this situation, because we have not assumed that α is nondegenerate, nor even Morse Bott—there is no assumption at all about Reeb orbits in homotopy classes other than h and 0. This fairly loose set of hypotheses is very convenient in applications, as nondegeneracy of a contact form is generally a quite difficult condition to check. The price we pay is that we will have to prove compactness manually instead of applying the big theorem (see Remark 10.20). Fortunately, it is not that hard: the crucial point is that in the situation at hand, there can be no bubbling at all.

Indeed, we claim that the given sequence $u_k : (\mathbb{R} \times S^1, i) \to (\mathbb{R} \times M, J)$ must satisfy a uniform bound

$$|du_k| \leq C$$

with respect to any translation-invariant Riemannian metrics on $\mathbb{R} \times S^1$ and $\mathbb{R} \times M$. To see this, note first that since all the u_k have the same positive asymptotic orbit γ, their energies are uniformly bounded via (10.1). Thus if $|du_k(z_k)| \to \infty$ for some sequence $z_k \in \mathbb{R} \times S^1$, we can perform the usual rescaling trick from Lecture 9 and deduce the existence of a nonconstant finite-energy plane $v_\infty : \mathbb{C} \to \mathbb{R} \times M$. Its singularity at ∞ cannot be removable since this would produce a nonconstant J-holomorphic sphere, violating Proposition 10.9. It follows that v_∞ is asymptotic to a

Reeb orbit at ∞, but this is also impossible since α does not admit any contractible orbits, and the claim is thus proved.

Suppose now that γ has period $T_+ > 0$, and observe that by nondegeneracy, the set

$$\mathcal{P}_h(\alpha, T_+) := \{\gamma \in \mathcal{P}_h(\alpha) \mid \gamma \text{ has period at most } T_+\}$$

is finite. Let

$$\mathcal{A}_h(\alpha), \mathcal{A}_h(\alpha, T_+) \subset (0, \infty)$$

denote the set of all periods of orbits in $\mathcal{P}_h(\alpha)$ and $\mathcal{P}_h(\alpha, T_+)$ respectively. By (10.2), the negative asymptotic orbit of each u_k is in $\mathcal{P}_h(\alpha, T_+)$, so we can take a subsequence and assume that these are all the same orbit; call it $\gamma_- \in \mathcal{P}_h(\alpha, T_+)$ and its period $T_- \in \mathcal{A}_h(\alpha, T_+)$. If $T_- = T_+$ then $u_k^* d\alpha \equiv 0$ for all k, implying that all u_k are the trivial cylinder over γ and thus trivially converge. Assume therefore $T_- < T_+$. Then since $u_k^* d\alpha \geq 0$, Stokes' theorem implies that for each k, the function

$$\mathbb{R} \to \mathbb{R} : s \mapsto \int_{S^1} u_k(s, \cdot)^* \alpha$$

is increasing and is a surjective map onto (T_-, T_+). The uniform bound on the derivatives implies that for any sequences $s_k, r_k \in \mathbb{R}$ with $u_k(s_k, 0) \in \{r_k\} \times M$, the sequence[2]

$$v_k : \mathbb{R} \times S^1 \to \mathbb{R} \times M : (s,t) \mapsto \tau_{-r_k} \circ u_k(s + s_k, t)$$

has a subsequence convergent in $C^\infty_{\text{loc}}(\mathbb{R} \times S^1)$ to some finite-energy J-holomorphic cylinder

$$v_\infty : \mathbb{R} \times S^1 \to \mathbb{R} \times M,$$

which necessarily satisfies

$$\int_{S^1} v_\infty(s, \cdot)^* \alpha = \lim_{k \to \infty} \int_{S^1} u_k(s + s_k, \cdot)^* \alpha \in [T_-, T_+]$$

for every $s \in \mathbb{R}$. This proves that v_∞ is nonconstant, with a positive puncture at $s = \infty$ and negative puncture at $s = -\infty$, and both of its asymptotic orbits are in $\mathcal{P}_h(\alpha, T_+)$.[3] If v_∞ is not a trivial cylinder, then it therefore satisfies

$$\int_{\mathbb{R} \times S^1} v_\infty^* d\alpha \geq \delta,$$

where δ is any positive number less than the smallest distance between neighboring elements of $\mathcal{A}_h(\alpha, T_+)$.

Let us call a sequence $s_k \in \mathbb{R}$ *nontrivial* whenever the limiting cylinder v_∞ obtained by the above procedure is not a trivial cylinder, and call two such sequences s_k and s'_k *compatible* if $s_k - s'_k$ is not bounded. We claim now that if $s_k^1, \ldots, s_k^m$ is a collection of nontrivial sequences that are all compatible with each other, then

$$m < \frac{2(T_+ - T_-)}{\delta}.$$

[2] Recall from Lecture 9 that we denote the $\mathbb{R}$-translation action on $\mathbb{R} \times M$ by $\tau_c(r, x) := (r + c, x)$.

[3] For an alternative argument that v_∞ must have a positive puncture at $s = \infty$ and negative at $s = -\infty$, see Figure 3.

Indeed, we can assume after ordering our collection appropriately and restricting to a subsequence that $s_k^{N+1} - s_k^N \to \infty$ for each $N = 1, \dots, m-1$, and let $v_\infty^N : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ denote the limits of the corresponding convergent subsequences. Then we can find $R > 0$ such that

$$\int_{[-R,R]\times S^1} (v_\infty^N)^* d\alpha > \frac{\delta}{2}$$

and thus

$$\int_{[s_k^N - R, s_k^N + R]\times S^1} u_k^* d\alpha > \frac{\delta}{2}$$

for each $N = 1, \dots, m$ for sufficiently large k. But these domains are also all disjoint for sufficiently large k, implying

$$T_+ - T_- = \int_{\mathbb{R}\times S^1} u_k^* d\alpha \geq \sum_{N=1}^{m} \int_{[s_k^N - R, s_k^N + R]\times S^1} u_k^* d\alpha > \frac{\delta m}{2}.$$

We've shown that there exists a maximal collection of nontrivial sequences $s_k^1, \dots, s_k^{N_+} \in \mathbb{R}$ satisfying $s_k^{N+1} - s_k^N \to \infty$ for each N, such that if $u_k(s_k^N, 0) \in \{r_k^N\} \times M$, then after restricting to a subsequence, the cylinders

$$v_k^N(s,t) := \tau_{-r_k^N} \circ u_k(s + s_k^N, t)$$

each converge in $C^\infty_{\text{loc}}(\mathbb{R} \times S^1)$ as $k \to \infty$ to a nontrivial J-holomorphic cylinder $u_\infty^N : \mathbb{R} \times S^1 \to \mathbb{R} \times M$. Let $\gamma_N^\pm$ denote the asymptotic orbit of u_∞^N at $s = \pm\infty$. We claim,

$$\gamma_N^+ = \gamma_{N+1}^- \quad \text{for each } N = 1, \dots, N_+ - 1.$$

If $\gamma_N^+ \neq \gamma_{N+1}^-$ for some N, choose a neighborhood $\mathcal{U} \subset M$ of the image of γ_N^+ that does not intersect any other orbit in $\mathcal{P}_h(\alpha, T_+)$. Then since each u_k is continuous, there must exist a sequence $s_k' \in \mathbb{R}$ with

$$s_k' - s_k^N \to \infty \quad \text{and} \quad s_k^{N+1} - s_k' \to \infty$$

such that $u_k(s_k', 0)$ lies in $\mathcal{U}$ for all k but stays a positive distance away from the image of γ_N^+. A subsequence of $(s,t) \mapsto u_k(s + s_k', t)$ then converges after suitable $\mathbb{R}$-translations to a cylinder $u_\infty' : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ that cannot be trivial since $u_\infty'(0,0)$ is not contained in any orbit in $\mathcal{P}_h(\alpha, T_+)$. This contradicts the assumption that our collection $s_k^1, \dots, s_k^{N_+}$ is maximal. A similar argument shows

$$\gamma_1^- = \gamma^- \quad \text{and} \quad \gamma_{N_+}^+ = \gamma,$$

so the curves $u_\infty^1, \dots, u_\infty^{N_+}$ form the levels of a stable holomorphic building $\mathbf{u}_\infty$. A similar argument by contradiction also shows that the sequence u_k must converge in the SFT topology to $\mathbf{u}_\infty$.

Finally, note that since all the breaking orbits in $\mathbf{u}_\infty$ are homotopic to h and J is h-regular, the levels u_∞^N are Fredholm regular. Since all of them also come in 1-parameter families of distinct curves related by the $\mathbb{R}$-action, this implies $\operatorname{ind}(u_\infty^N) \geq 1$ for each $N = 1, \dots, N_+$. $\square$

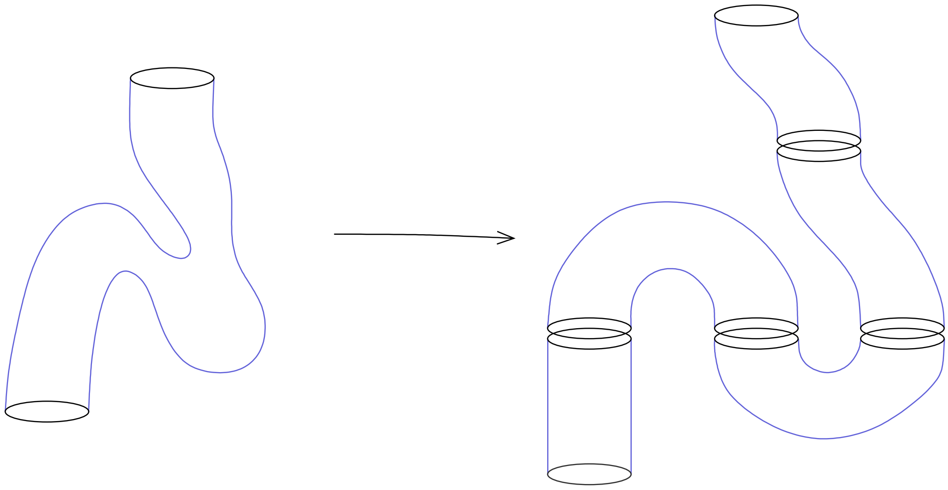

FIGURE 3. A degenerating sequence of holomorphic cylinders $u_k : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ cannot have a limiting level with a puncture of the "wrong" sign unless u_k violates the maximum principle for large k.

REMARK 10.20. Nondegeneracy or Morse-Bott conditions are required for several reasons in the proof of SFT compactness, and indeed, the theorem is not true in general without some such assumption. One can see this by considering what happens to a sequence u_k of J_k-holomorphic curves where $J_k \to J_\infty$ is compatible with a sequence of nondegenerate contact forms α_k converging to one that is only Morse-Bott. A compactness theorem for this scenario is proved in [**Bou02**], but it requires more general limiting objects than holomorphic buildings. On the other hand, it is useful for certain kinds of applications to know when one can do without nondegeneracy assumptions and prove compactness anyway. There are two main advantages to knowing that all Reeb orbits are nondegenerate or belong to Morse-Bott families:

(1) It implies that the set of all periods of closed orbits, the so-called **action spectrum** of α, is a *discrete* subset of $(0, \infty)$; in fact, for any $T > 0$, the set of all periods less than T is finite. Using the relations (10.1) and (10.2), this implies lower bounds on the possible energies of limiting components and thus helps show that only finitely many such components can arise.
(2) Curves asymptotic to nondegenerate or Morse-Bott orbits also satisfy exponential convergence estimates proved in [**HWZ96**, **HWZ01**, **HWZ96**, **Bou02**], and similar asymptotic estimates yield a result about "long cylinders with small area" (see [**HWZ02**] and [**BEH**$^+$**03**, Prop. 5.7]) which helps in proving that neighboring levels connect to each other along breaking orbits.

Our situation in Proposition 10.19 was simple enough to avoid using the "long cylinder" lemma, and we did use the discreteness of the action spectrum, but only needed it for orbits in $\mathcal{P}_h(\alpha)$ since we were able to rule out bubbling in the first step. An alternative would have been to assume that all orbits (in all homotopy classes) with period up to the period of γ are nondegenerate: then (10.2) implies that degenerate orbits never play any role in the main arguments of [BEH+03], so the big theorem becomes safe to use.

2.3. The chain complex. We now define a $\mathbb{Z}_2$-graded chain complex with coefficients in $\mathbb{Z}_2$ and generators $\langle\gamma\rangle$ for $\gamma \in \mathcal{P}_h(\alpha)$, i.e.

$$CC_*^h(M,\alpha) := \bigoplus_{\gamma\in\mathcal{P}_h(\alpha)} \mathbb{Z}_2.$$

The degree of each generator $\langle\gamma\rangle \in CC_*^h(M,\alpha)$ is defined by

$$|\langle\gamma\rangle| = n - 3 + \mu_{\mathrm{CZ}}(\gamma) \in \mathbb{Z}_2,$$

where $\mu_{\mathrm{CZ}}(\gamma) \in \mathbb{Z}_2$ denotes the parity of the Conley-Zehnder index with respect to any choice of trivialization. The choice to write $n-3$ in front of this is a convention that will make no difference at all in this lecture, but it is consistent with a $\mathbb{Z}$-grading that we will be able to define under suitable assumptions in Lecture 12. To define the differential on $CC_*^h(M,\alpha)$, choose an h-regular almost complex structure $J \in \mathcal{J}(\alpha)$. Given Reeb orbits $\gamma^+,\gamma^- \in \mathcal{P}_h(\alpha)$ and a number $I \in \mathbb{Z}$, let

$$\mathcal{M}^I(J,\gamma^+,\gamma^-)$$

denote the space of all $\mathbb{R}$-equivalence classes of index I holomorphic cylinders in $(\mathbb{R}\times M, J)$ asymptotic to $\gamma^\pm$ at $\pm\infty$, i.e. the union of all components $\mathcal{M}_{0,0}(J,A,\gamma^+,\gamma^-)/\mathbb{R}$ for which $\operatorname{vir-dim}\mathcal{M}_{0,0}(J,A,\gamma^+,\gamma^-) = I$. Since J is h-regular, all the curves in $\mathcal{M}^I(J,\gamma^+,\gamma^-)$ are Fredholm regular, so if $I \ge 1$, $\mathcal{M}^I(J,\gamma^+,\gamma^-)$ is a smooth manifold with

$$\dim \mathcal{M}^I(J,\gamma^+,\gamma^-) = I - 1.$$

Similarly, $\mathcal{M}^0(J,\gamma^+,\gamma^-)$ only contains trivial cylinders and is thus empty unless $\gamma^+ = \gamma^-$, and $\mathcal{M}^I(J,\gamma^+,\gamma^-)$ is always empty for $I < 0$. In particular, $\mathcal{M}^1(J,\gamma^+,\gamma^-)$ is a discrete set whenever $\gamma^+ \ne \gamma^-$, and by Proposition 10.19, it is also compact, hence finite. We can therefore define

$$\partial\langle\gamma\rangle = \sum_{\gamma'\in\mathcal{P}_h(\alpha)} \#_2\mathcal{M}^1(J,\gamma,\gamma')\langle\gamma'\rangle,$$

where for any set X, we denote by $\#_2 X$ the cardinality of X modulo 2. The operator ∂ has odd degree with respect to the grading since every index 1 holomorphic cylinder u with asymptotic orbits γ^+ and γ^- satisfies

$$\operatorname{ind}(u) = 1 = \mu_{\mathrm{CZ}}^\tau(\gamma^+) - \mu_{\mathrm{CZ}}^\tau(\gamma^-)$$

for suitable choices of the trivialization τ.

2.4. The homology. Following the standard Floer theoretic prescription, the relation $\partial^2 = 0$ should arise by viewing the compactification $\overline{\mathcal{M}}^2(J,\gamma^+,\gamma^-)$ for each $\gamma^+,\gamma^- \in \mathcal{P}_h(\alpha)$ as a compact 1-manifold whose boundary is identified with the set of rigid broken cylinders, as these are what is counted by ∂^2. Here $\overline{\mathcal{M}}^2(J,\gamma^+,\gamma^-)$ is defined as the closure of $\mathcal{M}^2(J,\gamma^+,\gamma^-)$ in the space of all J-holomorphic buildings in $\mathbb{R}\times M$ modulo $\mathbb{R}$-translation. Proposition 10.19 gives a natural inclusion

$$\overline{\mathcal{M}}^2(J,\gamma^+,\gamma^-)\setminus\mathcal{M}^2(J,\gamma^+,\gamma^-)\subset\coprod_{\gamma_0\in\mathcal{P}_h(\alpha)}\mathcal{M}^1(J,\gamma^+,\gamma_0)\times\mathcal{M}^1(J,\gamma_0,\gamma^-).$$

We therefore need an inclusion in the other direction, and for this we need to say a word about gluing. We have not had time to discuss gluing in earnest in these notes, and we will not do so now either, but the basic idea should be familiar from Floer homology: given $u_+\in\mathcal{M}^1(J,\gamma^+,\gamma_0)$ and $u_-\in\mathcal{M}^1(J,\gamma_0,\gamma^-)$, one would like to show that there exists a unique (up to $\mathbb{R}$-translation) one-parameter family $\{u_R\in\mathcal{M}^2(J,\gamma^+,\gamma^-)\}_{R\in[R_0,\infty)}$ such that u_R converges as $R\to\infty$ to the building $\mathbf{u}_\infty$ with bottom level u_- and top level u_+. One starts by constructing a family of *preglued* maps

$$\tilde{u}_R:\mathbb{R}\times S^1\to\mathbb{R}\times M,$$

meaning a smooth family of maps which converge in the SFT topology as $R\to\infty$ to $\mathbf{u}_\infty$ but are only *approximately* J-holomorphic. More precisely, fix parametrizations of u_- and u_+ and a parametrization of the orbit $\gamma_0:\mathbb{R}/T\mathbb{Z}\to M$ such that

$$\begin{aligned} u_+(s,t)&=\exp_{(Ts,\gamma_0(Tt))}h_+(s,t) &&\text{for } s\ll 0,\\ u_-(s,t)&=\exp_{(Ts,\gamma_0(Tt))}h_-(s,t) &&\text{for } s\gg 0,\end{aligned}$$

where $h_\pm$ are vector fields along the trivial cylinder satisfying $\lim_{s\to\mp\infty}h_\pm(s,t)=0$. By interpolating between suitable reparametrizations of h_+ and h_-, one can now define $\tilde{u}_R$ such that

$$\begin{aligned} \tilde{u}_R(s,t)&=\tau_{2RT}\circ u_+(s-2R,t) &&\text{for } s\geq R,\\ \tilde{u}_R(s,t)&\approx(Ts,\gamma_0(Tt)) &&\text{for } s\in[-R,R],\\ \tilde{u}_R(s,t)&=\tau_{-2RT}\circ u_-(s+2R,t) &&\text{for } s\leq -R,\\ \bar\partial_J\tilde{u}_R&\to 0 &&\text{as } R\to\infty.\end{aligned}$$

Given regularity of u_+ and u_-, one can now use a quantitative version of the implicit function theorem (cf. [**MS04**, §3.5]) to show that a distinguished J-holomorphic cylinder u_R close to $\tilde{u}_R$ exists for all R sufficiently large. For a more detailed synopsis of the analysis involved, see [**Nel13**, Chapter 7], and [**AD14**, Chapters 9 and 13] for the analogous story in Floer homology. The result is:

PROPOSITION 10.21. *For an h-admissible α, an h-regular $J\in\mathcal{J}(\alpha)$ and any two orbits $\gamma^+,\gamma^-\in\mathcal{P}_h(\alpha)$, the space $\overline{\mathcal{M}}^2(J,\gamma^+,\gamma^-)$ admits the structure of a compact 1-dimensional manifold with boundary, where its boundary points can be identified naturally with $\coprod_{\gamma_0\in\mathcal{P}_h(\alpha)}\mathcal{M}^1(J,\gamma^+,\gamma_0)\times\mathcal{M}^1(J,\gamma_0,\gamma^-)$.* □

COROLLARY 10.22. *The homomorphism $\partial: CC_*^h(M,\alpha)\to CC_{*-1}^h(M,\alpha)$ satisfies $\partial^2=0$.* □

We shall denote the homology of this chain complex by

$$HC_*^h(M,\alpha,J):=H_*\big(CC_*^h(M,\alpha),\partial\big).$$

The goal of the rest of this section is to prove that up to natural isomorphisms, $HC_*^h(M, \alpha, J)$ depends on (M, ξ) and h but not on the auxiliary data α and J.

2.5. Chain maps. For any constant $c > 0$, there is an obvious bijection between the generators of $CC_*^h(M, \alpha)$ and $CC_*^h(M, c\alpha)$, as the rescaling changes periods of orbits but not the set of closed orbits itself. Moreover, if $J \in \mathcal{J}(\alpha)$ and $J_c \in \mathcal{J}(c\alpha)$ are defined to match on ξ, then there is a biholomorphic diffeomorphism

$$(\mathbb{R} \times M, J) \to (\mathbb{R} \times M, J_c) : (r, x) \mapsto (cr, x),$$

thus giving a bijective correspondence between the moduli spaces of J-holomorphic and J_c-holomorphic curves. It follows that our bijection of chain complexes is also a chain map and therefore defines a canonical isomorphism

$$HC_*^h(M, \alpha, J) = HC_*^h(M, c\alpha, J_c). \tag{10.3}$$

Next suppose α_- and α_+ are two distinct contact forms for ξ, hence

$$\alpha_\pm = e^{f_\pm} \alpha$$

for some fixed contact form α and a pair of smooth functions $f_\pm : M \to \mathbb{R}$. After rescaling α_+ by a constant, we are free to assume $f_+ > f_-$ everywhere. Fix h-regular almost complex structures $J_\pm \in \mathcal{J}(\alpha_\pm)$ and let

$$\partial_\pm : CC_*^h(M, \alpha_\pm) \to CC_{*-1}^h(M, \alpha_\pm)$$

denote the resulting differentials on the two chain complexes. The region

$$W := \left\{ (r, x) \in \mathbb{R} \times M \;\middle|\; f_-(x) \le r \le f_+(x) \right\}$$

now defines an exact symplectic cobordism from (M, ξ) to itself: more precisely, setting

$$M_\pm := \left\{ (f_\pm(x), x) \in W \;\middle|\; x \in M \right\}$$

gives $\partial W = -M_- \sqcup M_+$, and the Liouville form $\lambda := e^r \alpha$ satisfies $\lambda|_{TM_\pm} = \alpha_\pm$. Choose a generic $d\lambda$-compatible almost complex structure J on the completion $\widehat{W}$ that restricts to $J_\pm$ on the cylindrical ends. Now given $\gamma^+ \in \mathcal{P}_h(\alpha_+)$ and $\gamma^- \in \mathcal{P}_h(\alpha_-)$ and a number $I \in \mathbb{Z}$, we shall denote by

$$\mathcal{M}^I(J, \gamma^+, \gamma^-)$$

the union of all components $\mathcal{M}_{0,0}(J, A, \gamma^+, \gamma^-)$ that have virtual dimension I. Note that we are not dividing by any $\mathbb{R}$-action here since J need not be $\mathbb{R}$-invariant. Since $\gamma^\pm$ are still guaranteed to be simply covered, curves in $\mathcal{M}^I(J, \gamma^+, \gamma^-)$ are again always somewhere injective and therefore regular, hence $\mathcal{M}^I(J, \gamma^+, \gamma^-)$ is a smooth manifold with

$$\dim \mathcal{M}^I(J, \gamma^+, \gamma^-) = I$$

if $I \ge 0$, and $\mathcal{M}^I(J, \gamma^+, \gamma^-) = \emptyset$ for $I < 0$. The compactification $\overline{\mathcal{M}}^I(J, \gamma^+, \gamma^-)$ is described via the following straightforward generalization of Proposition 10.19:

PROPOSITION 10.23. *For J as described above, suppose u_k is a sequence of J-holomorphic cylinders in $\widehat{W}$ with one positive puncture at an orbit $\gamma \in \mathcal{P}_h(\alpha_+)$ and one negative puncture. Then u_k has a subsequence convergent in the SFT topology to a broken J-holomorphic cylinder, i.e. a stable building $\mathbf{u}_\infty$ whose levels u_∞^N for $N = -N_-, \dots, -1, 0, 1, \dots, N_+$ are each cylinders with one positive and one negative puncture, living in $\mathbb{R} \times M^\pm$ for $\pm N > 0$ and $\widehat{W}$ for $N = 0$. Moreover, the levels satisfy* $\operatorname{ind}(u_\infty^0) \geq 0$ *and* $\operatorname{ind}(u_\infty^N) \geq 1$ *for $N \neq 0$, thus for large k in the convergent subsequence,*

$$\operatorname{ind}(u_k) = \sum_{N=-N_-}^{N_+} \operatorname{ind}(u_\infty^N) \geq N_- + N_+.$$

□

It follows that the set $\mathcal{M}^0(J, \gamma^+, \gamma^-)$ is always finite, and we use this to define a map

$$\Phi_J : CC_*^h(M, \alpha_+) \to CC_*^h(M, \alpha_-) : \langle\gamma\rangle \mapsto \sum_{\gamma' \in \mathcal{P}_h(\alpha_-)} \#_2 \mathcal{M}^0(J, \gamma, \gamma')\langle\gamma'\rangle.$$

This map preserves degrees since it counts index 0 curves, and we claim that it is a chain map:

$$\Phi_J \circ \partial_+ = \partial_- \circ \Phi_J.$$

This follows from the fact that by Proposition 10.23 (in conjunction with a corresponding gluing theorem), $\overline{\mathcal{M}}^1(J, \gamma^+, \gamma^-)$ is a compact 1-manifold whose boundary consists of two types of broken cylinders, depending whether the index 1 curve appears in an upper or lower level:

$$\partial\overline{\mathcal{M}}^1(J, \gamma^+, \gamma^-) = \bigsqcup_{\gamma_0 \in \mathcal{P}_h(\alpha_+)} \left(\mathcal{M}^1(J_+, \gamma^+, \gamma_0) \times \mathcal{M}^0(J, \gamma_0, \gamma^-)\right)$$
$$\cup \bigsqcup_{\gamma_0 \in \mathcal{P}_h(\alpha_-)} \left(\mathcal{M}^0(J, \gamma^+, \gamma_0) \times \mathcal{M}^1(J_-, \gamma_0, \gamma^-)\right).$$

Counting broken cylinders of the first type gives the coefficient in front of $\langle\gamma^-\rangle$ in $\Phi_J \circ \partial_+(\langle\gamma^+\rangle)$, and the second type gives $\partial_- \circ \Phi_J(\langle\gamma^+\rangle)$.

It follows that Φ_J descends to a homomorphism

$$\Phi_J : HC_*^h(M, \alpha_+, J_+) \to HC_*^h(M, \alpha_-, J_-). \tag{10.4}$$

2.6. Chain homotopies. We claim that the map Φ_J in (10.4) does not depend on J. To see this, suppose J_0 and J_1 are two generic choices of compatible almost complex structures on $\widehat{W}$ that both match $J_\pm$ on the cylindrical ends. The space of almost complex structures with these properties is contractible, so we can find a smooth path

$$\{J_s\}_{s\in[0,1]}$$

connecting them. For $I \in \mathbb{Z}$, consider the parametric moduli space

$$\mathcal{M}^I(\{J_s\}, \gamma^+, \gamma^-) := \left\{(s, u) \;\middle|\; s \in [0,1],\ u \in \mathcal{M}^I(J_s, \gamma^+, \gamma^-)\right\}.$$

As we observed in Remark 7.4 of Lecture 7, a generic choice of the homotopy $\{J_s\}$ makes $\mathcal{M}^I(\{J_s\})$ a smooth manifold with

$$\dim \mathcal{M}^I(\{J_s\}, \gamma^+, \gamma^-) = I + 1$$

whenever $I \geq -1$, and $\mathcal{M}^I(\{J_s\}, \gamma^+, \gamma^-) = \emptyset$ when $I < -1$. Adapting Proposition 10.23 to allow for a converging sequence of almost complex structures, it implies that $\mathcal{M}^{-1}(\{J_s\}, \gamma^+, \gamma^-)$ is compact and thus finite, so we can use it to define a homomorphism of odd degree by

$$H : CC_*^h(M, \alpha_+) \to CC_{*+1}^h(M, \alpha_-) : \langle\gamma\rangle \mapsto \sum_{\gamma' \in \mathcal{P}_h(\alpha_-)} \#\mathcal{M}^{-1}(\{J_s\}, \gamma, \gamma')\langle\gamma'\rangle.$$

We claim that this is a chain homotopy between Φ_{J_0} and Φ_{J_1}, i.e.

$$\Phi_{J_1} - \Phi_{J_0} = \partial_- \circ H + H \circ \partial_+.$$

This follows by looking at the boundary of the compactified 1-dimensional space $\overline{\mathcal{M}}^0(\{J_s\}, \gamma^+, \gamma^-)$, which consists of four types of objects:

(1) Pairs $(0, u)$ with $u \in \mathcal{M}^0(J_0, \gamma^+, \gamma^-)$, which are counted by Φ_{J_0}.
(2) Pairs $(1, u)$ with $u \in \mathcal{M}^0(J_1, \gamma^+, \gamma^-)$, which are counted by Φ_{J_1}.
(3) Pairs $(s, \mathbf{u})$ with $\mathbf{u}$ a broken cylinder with upper level $u_+ \in \mathcal{M}^1(J_+, \gamma^+, \gamma_0)$ and main level $u_0 \in \mathcal{M}^{-1}(J_s, \gamma_0, \gamma^-)$ for some $s \in (0, 1)$; these are counted by $H \circ \partial_+$.
(4) Pairs $(s, \mathbf{u})$ with $\mathbf{u}$ a broken cylinder with lower level $u_- \in \mathcal{M}^1(J_-, \gamma_0, \gamma^-)$ and main level $u_0 \in \mathcal{M}^{-1}(J_s, \gamma^+, \gamma_0)$ for some $s \in (0, 1)$; these are counted by $\partial_- \circ H$.

The sum $\Phi_{J_0} + \Phi_{J_1} + \partial_- \circ H + H \circ \partial_+$ therefore counts (modulo 2) the boundary points of a compact 1-manifold, so it vanishes.

Since the action of Φ_J on homology no longer depends on J, we will denote it from now on by

$$\Phi : HC_*^h(M, \alpha_+, J_+) \to HC_*^h(M, \alpha_-, J_-).$$

It is well defined for any pair of h-admissible contact forms $\alpha_\pm$ and h-regular $J_\pm \in \mathcal{J}(\alpha_\pm)$ since one can first rescale α_+ to assume $\alpha_\pm = e^{f_\pm}\alpha$ with $f_+ > f_-$, using the canonical isomorphism (10.3).

2.7. Proof of invariance. We claim that for any h-admissible α and h-regular $J \in \mathcal{J}(\alpha)$, the cobordism map

$$\Phi : HC_*^h(M, \alpha, J) \to HC_*^h(M, \alpha, J)$$

is the identity. Indeed, the literal meaning of this statement is that for any $c > 1$, the composition of the canonical isomorphism (10.3) with the map

$$\Phi : HC_*^h(M, c\alpha, J_c) \to HC_*^h(M, \alpha, J)$$

defined by counting index 0 cylinders in a trivial cobordism from (M, α, J) to $(M, c\alpha, J_c)$ is the identity. Writing $c = e^a$ for $a > 0$, the Liouville cobordism in question is simply

$$(W, d\lambda) = ([0, a] \times M, d(e^r\alpha)),$$

and one can choose a compatible almost complex structure on this which matches J and J_c on ξ while taking ∂_r to $g(r)R_\alpha$ for a suitable function g with $g(r) = 1$ near $r = 0$ and $g(r) = 1/c$ near $r = a$. The resulting almost complex manifold is biholomorphically diffeomorphic to the usual symplectization $(\mathbb{R} \times M, J)$, so our count of index 0 cylinders is equivalent to the count of such cylinders in $(\mathbb{R} \times M, J)$. The latter are simply the trivial cylinders, all of which are Fredholm regular, so counting these defines the identity map on the chain complex.

Finally, we need to show that for any three h-admissible pairs (α_i, J_i) with $i = 0, 1, 2$, the cobordism maps $\Phi_{ij} : HC_*^h(M, \alpha_j, J_j) \to HC_*^h(M, \alpha_i, J_i)$ satisfy

$$\Phi_{21} \circ \Phi_{10} = \Phi_{20}. \tag{10.5}$$

We will only sketch this part: the idea is to use a stretching construction. After rescaling, suppose without loss of generality that $\alpha_i = e^{f_i}\alpha$ with $f_2 > f_1 > f_0$. Then the cobordism

$$W_{20} := \big\{ (r, x) \ \big|\ f_0(x) \le r \le f_2(x) \big\}$$

contains a contact-type hypersurface

$$M_1 := \big\{ (f_1(x), x) \ \big|\ x \in M \big\} \subset W_{20}.$$

As described at the end of Lecture 9, one can now choose a sequence of compatible almost complex structures $\{J_{20}^N\}_{N\in\mathbb{N}}$ on $\widehat{W}_{20}$ that are fixed outside a neighborhood of M_1 but degenerate in this neighborhood as $N \to \infty$, equivalent to replacing a small tubular neighborhood of M_1 with increasingly large collars $[-N, N] \times M$ in which J_{20}^N belongs to $\mathcal{J}(\alpha_1)$. The resulting chain maps

$$\Phi_{J_{20}^N} : CC_*^h(M, \alpha_2, J_2) \to CC_*^h(M, \alpha_0, J_0)$$

are chain homotopic for all N, but as $N \to \infty$, the index 0 cylinders counted by these maps converge to buildings with two levels, the top one an index 0 cylinder in the completion of a cobordism from (M, α_1, J_1) to (M, α_2, J_2), while the bottom one also has index 0 and lives in a cobordism from (M, α_0, J_0) to (M, α_1, J_1). The composition $\Phi_{21} \circ \Phi_{10}$ counts these broken cylinders, so this proves (10.5).

In particular, we conclude now that each of the cobordism maps

$$\Phi : HC_*^h(M, \alpha_+, J_+) \to HC_*^h(M, \alpha_-, J_-)$$

is an isomorphism, since composing it with a cobordism map in the opposite direction must give the identity. The isomorphism class of $HC_*^h(M, \alpha, J)$ is therefore independent of the auxiliary data (α, J), and will be denoted by

$$HC_*^h(M, \xi).$$

This is the **cylindrical contact homology** of (M, ξ) in the homotopy class h. It is defined for any primitive homotopy class $h \in [S^1, M]$ and closed contact manifold that is h-admissible in the sense of Definition 10.16. It is also invariant under contactomorphisms in the following sense:

Proposition 10.24. *Suppose $\varphi : (M_0, \xi_0) \to (M_1, \xi_1)$ is a contactomorphism with $\varphi_* h_0 = h_1$, where $h_0 \in [S^1, M]$ is a primitive homotopy class of loops, and (M_1, ξ_1) is h_1-admissible. Then (M_0, ξ_0) is h_0-admissible, and $HC_*^{h_0}(M_0, \xi_0) \cong HC_*^{h_1}(M_1, \xi_1)$.*

PROOF. Given an h_1-admissible contact form α_1 on (M_1, ξ_1) and an h_1-regular $J_1 \in \mathcal{J}(\alpha_1)$, the contact form $\alpha_0 := \varphi^*\alpha_1$ on M_0 is h_0-admissible since φ defines a bijection from $\mathcal{P}_{h_0}(\alpha_0)$ to $\mathcal{P}_{h_1}(\alpha_1)$ and also a bijection between the sets of contractible Reeb orbits for α_0 and α_1. Since $\varphi_*\xi_0 = \xi_1$, α_0 is a contact form for (M_0, ξ_0), hence the latter is h_0-admissible. The diffeomorphism $\tilde{\varphi} := \mathrm{Id} \times \varphi : \mathbb{R} \times M_0 \to \mathbb{R} \times M_1$ then maps ∂_r to ∂_r, R_{α_0} to R_{α_1} and ξ_0 to ξ_1, thus $J_0 := \tilde{\varphi}^* J_1 \in \mathcal{J}(\alpha_0)$, so $\tilde{\varphi}$ defines a biholomorphic map $(\mathbb{R} \times M_0, J_0) \to (\mathbb{R} \times M_1, J_1)$ and thus a bijection between the sets of holomorphic cylinders in each. It follows that J_0 is h_0-regular, and the bijection $\mathcal{P}_{h_0}(\alpha_0) \to \mathcal{P}_{h_1}(\alpha_1)$ defines an isomorphism between the chain complexes defining $HC_*^{h_0}(M_0, \alpha_0, J_0)$ and $HC_*^{h_1}(M_1, \alpha_1, J_1)$. □

3. Computing $HC_*(\mathbb{T}^3, \xi_k)$

3.1. The Morse-Bott setup. The contact form α_k on $\mathbb{T}^3$ defined at the beginning of this lecture has Reeb vector field

$$R_k(\rho, \phi, \theta) = \cos(2\pi k\rho)\,\partial_\theta + \sin(2\pi k\rho)\,\partial_\phi.$$

Its Reeb orbits therefore preserve and define linear foliations on each of the tori $\{\rho\} \times \mathbb{T}^2$. In particular, none of the closed orbits are contractible, though all of them are also degenerate, as they all come in S^1-parametrized families foliating $\{\mathrm{const}\} \times \mathbb{T}^2$. For certain homotopy classes $h \in [S^1, \mathbb{T}^3]$, this yields a very easy computation of $HC_*^h(\mathbb{T}^3, \xi_k)$, namely whenever h contains no periodic orbits:

THEOREM 10.25. *Suppose $h \in [S^1, \mathbb{T}^3]$ is any primitive homotopy class of loops such that the projection $p : \mathbb{T}^3 \to S^1 : (\rho, \phi, \theta) \mapsto \rho$ satisfies $p_*h \neq 0 \in [S^1, S^1]$. Then α_k is h-admissible and the resulting contact homology $HC_*^h(\mathbb{T}^3, \xi_k)$ is trivial.* □

Now for the interesting part. Every primitive class $h \in [S^1, \mathbb{T}^3]$ not covered by Theorem 10.25 contains closed orbits of R_k, all of them degenerate since they come in S^1-parametrized families foliating the tori $\{\mathrm{const}\} \times \mathbb{T}^2$. This makes it not immediately clear whether $(\mathbb{T}^3, \xi_k)$ is h-admissible, though the following observation in conjunction with Proposition 10.24 shows that if $HC_*^h(\mathbb{T}^3, \xi_k)$ can be defined, it will be the same for all the homotopy classes under consideration.

LEMMA 10.26. *Suppose $h_0, h_1 \in [S^1, \mathbb{T}^3]$ are primitive homotopy classes that are both mapped to the trivial class under the projection $\mathbb{T}^3 \to S^1 : (\rho, \phi, \theta) \mapsto \rho$. Then there exists a contactomorphism $\varphi : (\mathbb{T}^3, \xi_k) \to (\mathbb{T}^3, \xi_k)$ satisfying $\varphi_* h_0 = h_1$.*

PROOF. We can represent h_i for $i = 0, 1$ by loops of the form $\gamma_i(t) = (0, \beta_i(t)) \in S^1 \times \mathbb{T}^2$, where the loops $\beta_i : S^1 \to \mathbb{T}^2$ are embedded and thus represent generators of $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$. One can thus find a matrix $\begin{pmatrix} m & n \\ p & q \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ such that the diffeomorphism

$$\varphi : \mathbb{T}^3 \to \mathbb{T}^3 : (\rho, \phi, \theta) \mapsto (\rho, m\phi + n\theta, p\phi + q\theta)$$

satisfies $\varphi_* h_0 = h_1$. We have

$$\begin{aligned} \varphi^*\alpha_k &= [q\cos(2\pi k\rho) + n\sin(2\pi k\rho)]\; d\theta + [p\cos(2\pi k\rho) + m\sin(2\pi k\rho)]\; d\phi \\ &=: F(\rho)\, d\theta + G(\rho)\, d\phi. \end{aligned}$$

The loop $(F,G) : S^1 \to \mathbb{R}^2$ satisfies

$$\begin{pmatrix} F(\rho) \\ G(\rho) \end{pmatrix} = \begin{pmatrix} q & n \\ p & m \end{pmatrix} \begin{pmatrix} \cos(2\pi k\rho) \\ \sin(2\pi k\rho) \end{pmatrix},$$

where $\begin{pmatrix} q & n \\ p & m \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, thus (F,G) winds k times about the origin. Any choice of homotopy from (F,G) to $(\cos(2\pi k\rho), \sin(2\pi k\rho))$ through loops $(F_s, G_s) : S^1 \to \mathbb{R}^2$ winding k times about the origin with positive rotational velocity then gives rise to a homotopy from $\varphi^*\alpha_k$ to α_k through contact forms $F_s(\rho)\,d\theta + G_s(\rho)\,d\phi$. Gray's stability theorem therefore yields a contactomorphism $\psi : (\mathbb{T}^3, \xi_k) \to (\mathbb{T}^3, \ker \varphi^*\alpha_k)$ with ψ smoothly isotopic to the identity. The map $\varphi \circ \psi$ is thus a contactomorphism of $(\mathbb{T}^3, \xi_k)$ with $(\varphi \circ \psi)_* h_0 = \varphi_* \psi_* h_0 = \varphi_* h_0 = h_1$. □

In light of the lemma, we are free from now on to restrict our attention to the particular homotopy class

$$h := [t \mapsto (0,0,t)],$$

which is the homotopy class of the 1-periodic orbits foliating the k tori

$$T_m := \{m/k\} \times \mathbb{T}^2, \qquad m = 0, \dots, k-1$$

since $R_k(m/k, \phi, \theta) = \partial_\theta$. Though the orbits on these tori are degenerate, it is not hard to show that they all satisfy the Morse-Bott condition; in fact, α_k is a Morse-Bott contact form. We will explain a self-contained computation of $HC_*^h(\mathbb{T}^3, \xi_k)$ in the next two sections without using the Morse-Bott condition—but first, it seems worthwhile to sketch how one can guess the answer using Morse-Bott data.

Bourgeois's thesis [**Bou02**] gives a prescription for calculating contact homology in Morse-Bott settings, i.e. for deducing what orbits and what holomorphic curves will appear under certain standard ways of perturbing the Morse-Bott contact form to make it nondegenerate. Notice first that the only orbits in $\mathcal{P}_h(\alpha_k)$ are the ones that foliate the k tori $T_0, \dots, T_{k-1}$, and they all have period 1. By (10.2), it follows that for any $J \in \mathcal{J}(\alpha_k)$, there can be no nontrivial J-holomorphic cylinders connecting two orbits in $\mathcal{P}_h(\alpha_k)$. This makes the calculation of $HC_*^h(\mathbb{T}^3, \xi_k)$ sound trivial, but of course there is more to the story since α_k is not admissible; indeed, the chain complex $CC_*(\mathbb{T}^3, \alpha_k)$ is not even well defined. The prescription in [**Bou02**] now gives the following. Each of the families of orbits in $T_0, \dots, T_{k-1}$ is parametrized by S^1, and by a standard perturbation technique, any choice of a Morse function $f_m : S^1 \to \mathbb{R}$ for $m = 0, \dots, k-1$ yields a contact form α_k' that is C^∞-close to α_k, matches it outside a neighborhood of T_m, but has a nondegenerate Reeb orbit on T_m for each critical point of f_m, while every other closed orbit in the perturbed region can be assumed to have arbitrarily large period. Moreover, there is a corresponding perturbation from $J \in \mathcal{J}(\alpha_k)$ to $J' \in \mathcal{J}(\alpha_k')$ such that every gradient flow line of the function $f_m : S^1 \to \mathbb{R}$ gives rise to a J'-holomorphic cylinder in $\mathbb{R} \times \mathbb{T}^3$ connecting the corresponding nondegenerate Reeb orbits along T_m. In the present situation, since no J-holomorphic cylinders of the relevant type exist before the perturbation, the only ones after the perturbation are those that come from gradient flow lines.

Now imagine performing a similar perturbation near every $T_0, \dots, T_{k-1}$, using Morse functions $f_0, \dots, f_{k-1} : S^1 \to \mathbb{R}$ that each have exactly two critical points.

For the perturbed contact form α'_k, $\mathcal{P}_h(\alpha'_k)$ now consists of exactly $2k$ orbits

$$\gamma_0^\pm, \ldots, \gamma_{k-1}^\pm \in \mathcal{P}_h(\alpha'_k),$$

where we denote by γ_m^+ and γ_m^- the orbits on T_m corresponding to the maximum and minimum of f_m respectively. For the obvious choice of trivialization τ for the contact bundle along $\gamma_m^\pm$, one can relate the Conley-Zehnder indices to the Morse indices of the corresponding critical points, giving

$$\mu_{\mathrm{CZ}}^\tau(\gamma_m^+) = 0, \qquad \mu_{\mathrm{CZ}}^\tau(\gamma_m^-) = 1, \qquad m = 0, \ldots, k-1.$$

Moreover, the two gradient flow lines connecting maximum and minimum for each f_m give rise two exactly two holomorphic cylinders in $\mathcal{M}^1(J', \gamma_m^-, \gamma_m^+)$ for each $m = 0, \ldots, k-1$, and these are all the curves that are counted for the differential on $CC_*^h(\mathbb{T}^3, \alpha'_k, J')$. Counting modulo 2, we thus have

$$\partial\langle\gamma_m^\pm\rangle = 0 \quad \text{for all} \quad m = 0, \ldots, k-1,$$

implying

$$HC_*^h(\mathbb{T}^3, \alpha'_k, J') = \begin{cases} \mathbb{Z}_2^k & * = \text{odd}, \\ \mathbb{Z}_2^k & * = \text{even}. \end{cases}$$

Let us state this as a theorem.

THEOREM 10.27. *Suppose $h \in [S^1, \mathbb{T}^3]$ is a primitive homotopy class that maps to the trivial class under the projection $\mathbb{T}^3 \to S^1 : (\rho, \phi, \theta) \mapsto \rho$. Then $(\mathbb{T}^3, \xi_k)$ is h-admissible and*

$$HC_*^h(\mathbb{T}^3, \xi_k) \cong \begin{cases} \mathbb{Z}_2^k & * = \textit{odd}, \\ \mathbb{Z}_2^k & * = \textit{even}. \end{cases}$$

Theorem 10.1 is an immediate corollary of this: indeed, if $\varphi : (\mathbb{T}^3, \xi_k) \to (\mathbb{T}^3, \xi_\ell)$ is a contactomorphism, choose any $h \in [S^1, \mathbb{T}^3]$ for which Theorem 10.27 applies, and let $h_0 := \varphi^* h \in [S^1, \mathbb{T}^3]$. Then $HC_*^h(\mathbb{T}^3, \xi_\ell) \cong \mathbb{Z}_2^{2\ell}$ implies via Proposition 10.24 that $HC_*^{h_0}(\mathbb{T}^3, \xi_k) \cong \mathbb{Z}_2^{2\ell}$. But Theorems 10.25 and 10.27 imply that the latter is also either 0 or $\mathbb{Z}_2^{2k}$, hence $k = \ell$.

3.2. A digression on the Floer equation. In preparation for giving a self-contained proof of Theorem 10.27, we now explain a general procedure for relating holomorphic cylinders in a symplectization to solutions of the Floer equation. This idea is loosely inspired by arguments in [**EKP06**].

To motivate what follows, notice that on a neighborhood of $T_0 = \{0\} \times \mathbb{T}^2 \subset (\mathbb{T}^3, \xi_k)$, we can write

$$\alpha_k = \cos(2\pi k\rho)\,(d\theta + \beta)\,,$$

where $\beta := \tan(2\pi k\rho)\, d\phi$ defines a Liouville form on the annulus $\mathbb{A} := [-1/8, 1/8] \times S^1$ with coordinates (ρ, ϕ). This makes the neighborhood $\mathbb{A} \times S^1 \subset (\mathbb{T}^3, \xi_k)$ a special case of the following general construction.

DEFINITION 10.28. Suppose V is a $2n$-dimensional manifold with an exact symplectic form $d\beta$. The contact manifold $(V \times S^1, \ker(d\theta + \beta))$ is then called the **contactization** of (V, β).[4] Here θ denotes the coordinate on the S^1 factor.

It's easy to check that $d\theta + \beta$ is indeed a contact form on $V \times S^1$ whenever $d\beta$ is symplectic on V: the latter means $(d\beta)^n > 0$ on V, so

$$(d\theta + \beta) \wedge [d(d\theta + \beta)]^n = (d\theta + \beta) \wedge (d\beta)^n = d\theta \wedge (d\beta)^n > 0.$$

Now here's a cute trick one can play with contactizations. For the rest of this subsection, assume

$$(V, d\beta)$$

is an arbitrary compact $2n$-dimensional exact symplectic manifold with boundary. Fix a smooth function

$$H : V \times S^1 \to \mathbb{R},$$

which we shall think of in the following as a time-dependent Hamiltonian $H_\theta := H(\cdot, \theta) : V \to \mathbb{R}$ on $(V, d\beta)$. The 2-form on $V \times S^1$ defined by

$$\Omega = d\beta + d\theta \wedge dH = d(\beta - H\, d\theta)$$

is then *fiberwise symplectic*, meaning its restriction to each of the fibers of the projection map $V \times S^1 \to S^1$ is symplectic. We claim that for every $\epsilon > 0$ sufficiently small,

$$\lambda_\epsilon := d\theta + \epsilon(\beta - H\, d\theta)$$

defines a contact form on $V \times S^1$. This is a variation on the construction that was used by Thurston and Winkelnkemper [**TW75**] to define contact forms out of open book decompositions, and the proof is simple enough: since $d\lambda_\epsilon = \epsilon\Omega$, we just need to check that $\lambda_\epsilon \wedge \Omega^n > 0$ for $\epsilon > 0$ sufficiently small, and indeed,

$$\lambda_\epsilon \wedge \Omega^n = d\theta \wedge (d\beta)^n + \epsilon(\beta - H\, d\theta) \wedge \Omega^n > 0$$

since the first term is a volume form and ϵ is small. To see the relation between λ_ϵ and the contactization, we can write

$$\lambda_\epsilon = (1 - \epsilon H)\, d\theta + \epsilon\beta = (1 - \epsilon H) \left(d\theta + \frac{\epsilon}{1 - \epsilon H} \beta \right)$$

and observe that $\frac{\epsilon}{1-\epsilon H}\beta$ is also a Liouville form on V whenever H is θ-independent and $\epsilon > 0$ is sufficiently small.

The Reeb vector fields R_ϵ for λ_ϵ vary with ϵ, but their directions do not, since $d\lambda_\epsilon = \epsilon\Omega$ has the same kernel for every ϵ. Moreover, while λ_ϵ ceases to be a contact form when $\epsilon \to 0$, the Reeb vector fields still have a well-defined limit: they converge as $\epsilon \to 0$ to the unique vector field R_0 satisfying

$$d\theta(R_0) \equiv 1 \quad \text{and} \quad \Omega(R_0, \cdot) \equiv 0.$$

The latter can be written more explicitly as

$$R_0 = \partial_\theta + X_\theta,$$

[4] Elsewhere in the literature, the contactization is also often defined as $V \times \mathbb{R}$ instead of $V \times S^1$. The usage here is consistent with [**MNW13**].

where X_θ is the time-dependent Hamiltonian vector field determined by H_θ, i.e. via the condition

$$d\beta(X_\theta, \cdot) = -dH_\theta.$$

As one can easily compute, the reason for this nice behavior as $\epsilon \to 0$ is that the R_ϵ are also the Reeb vector fields for a smooth family of stable Hamiltonian structures:

PROPOSITION 10.29. *The pairs $\mathcal{H}_\epsilon := (\Omega, \lambda_\epsilon)$ for $\epsilon \ge 0$ sufficiently small define a smooth family of stable Hamiltonian structures whose Reeb vector fields are R_ϵ.* □

We shall write the hyperplane distributions induced by $\mathcal{H}_\epsilon$ as

$$\Xi_\epsilon := \ker \lambda_\epsilon \subset T(V \times S^1).$$

These are contact structures for $\epsilon > 0$ small, and the space $\mathcal{J}(\mathcal{H}_\epsilon)$ of $\mathbb{R}$-invariant almost complex structures on $\mathbb{R} \times (V \times S^1)$ compatible with $\mathcal{H}_\epsilon$ is then identical to $\mathcal{J}(\lambda_\epsilon)$. On the other hand for $\epsilon = 0$, $\Xi_0 = \ker d\theta$ is a foliation, namely it is the vertical subbundle of the trivial fibration $V \times S^1 \to S^1$. To interpret $\mathcal{H}_0$, notice that its closed Reeb orbits in the homotopy class of $\gamma : S^1 \to V \times S^1 : t \mapsto (\text{const}, t)$ are all of the form $\gamma(t) = (x(t), t)$ where $x : S^1 \to V$ is a contractible 1-periodic orbit of X_θ. Moreover, suppose $J \in \mathcal{J}(\mathcal{H}_0)$, which is equivalent to a choice of compatible complex structure on the symplectic bundle $(\Xi_0, \Omega|_{\Xi_0})$, or in other words, an S^1-parametrized family of $d\beta$-compatible almost complex structures $\{J_\theta\}_{\theta \in S^1}$ on V. Then if

$$u = (f, v, g) : \mathbb{R} \times S^1 \to \mathbb{R} \times (V \times S^1)$$

is a J-holomorphic cylinder asymptotic at $\{\pm\infty\} \times S^1$ to two orbits of the form described above, the nonlinear Cauchy-Riemann equation for u turns out to imply that $(f, g) : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$ is a holomorphic map with degree 1 sending $\{\pm\infty\} \times S^1$ to $\{\pm\infty\} \times S^1$, and we can therefore choose a unique biholomorphic reparametrization of u so that (f, g) becomes the identity map. Having done this, the equation satisfied by $v : \mathbb{R} \times S^1 \to V$ is now

$$\partial_s v + J_t(v)(\partial_t v - X_t(v)) = 0,$$

in other words, the Floer equation for the data $\{J_\theta\}_{\theta \in S^1}$ and $\{H_\theta\}_{\theta \in S^1}$.

To complete the analogy, notice that since Ω is exact, we can write down a natural symplectic action functional with respect to each $\mathcal{H}_\epsilon$ as

$$\mathcal{A}_\epsilon : C^\infty(S^1, V \times S^1) \to \mathbb{R} : \gamma \mapsto \int_{S^1} \gamma^*(\beta - H\, d\theta).$$

For loops of the form $\gamma(t) = (x(t), t)$ with $x : S^1 \to V$ contractible, this reduces (give or take a sign—see Remark 10.32) to the usual formula for the Floer action functional

$$\mathcal{A}_H(\gamma) = \int_{S^1} x^*\beta - \int_{S^1} H(x(t))\, dt = \int_{\mathbb{D}} \bar{x}^* d\beta - \int_{S^1} H(x(t))\, dt, \tag{10.6}$$

where $\bar{x} : \mathbb{D} \to V$ is any map satisfying $\bar{x}|_{\partial\mathbb{D}} = x$. Stokes' theorem gives an easy relation between the action and the so-called *Ω-energy* if $u : \mathbb{R} \times S^1 \to \mathbb{R} \times (V \times S^1)$

is a J-holomorphic curve for $J \in \mathcal{J}(\mathcal{H}_\epsilon)$ and is positively/negatively asymptotic to orbits $\gamma^\pm : S^1 \to V \times S^1$ at $s = \pm\infty$: we have

$$0 \le \int_{\mathbb{R}\times S^1} u^*\Omega = \mathcal{A}_\epsilon(\gamma^+) - \mathcal{A}_\epsilon(\gamma^-).$$

If $u(s,t) = (s, v(s,t), t)$, then the left hand side is identical to the definition of energy in Floer homology, namely

$$E_H(v) := \int_{\mathbb{R}\times S^1} d\beta(\partial_s v, \partial_t v - X_t(v))\, ds \wedge dt = \int_{\mathbb{R}\times S^1} d\beta(\partial_s v, J_t(v)\partial_s v)\, ds \wedge dt,$$

thus giving the familiar relation

$$E_H(v) = \mathcal{A}_H(\gamma^+) - \mathcal{A}_H(\gamma^-). \tag{10.7}$$

To relate this to the usual notion of energy with respect to a stable Hamiltonian structure, we write the usual formula

$$E_\epsilon(u) := \sup_{\varphi\in\mathcal{T}} \int_{\dot\Sigma} u^* \left[d\big(\varphi(r)\lambda_\epsilon\big) + \Omega \right],$$

with $\mathcal{T} := \left\{ \varphi \in C^\infty(\mathbb{R}, (-\epsilon_0, \epsilon_0)) \;\middle|\; \varphi' > 0 \right\}$ for some constant $\epsilon_0 > 0$ sufficiently small. Notice first that for any fixed ϵ, Stokes' theorem gives a bound for $E_\epsilon(u)$ in terms of the asymptotic orbits of u since Ω is exact. Finally, in the case $\epsilon = 0$ with $u(s,t) = (s, v(s,t), t)$, we find

$$E_0(u) = \sup_{\varphi\in\mathcal{T}} \int_{\mathbb{R}\times S^1} \varphi'(s)\, ds \wedge dt + \int_{\mathbb{R}\times S^1} u^*\Omega = 2\epsilon_0 + E_H(v),$$

so bounds on $E_0(u)$ are equivalent to bounds on the Floer homological energy $E_H(v)$. The basic fact that Floer trajectories $v : \mathbb{R} \times S^1 \to V$ with $E_H(v) < \infty$ are asymptotic to contractible 1-periodic Hamiltonian orbits can now be regarded as a corollary of our Theorem 9.6 in Lecture 9.

The above discussion gives a one-to-one correspondence between a certain moduli space of unparametrized J-holomorphic cylinders in $\mathbb{R} \times (V \times S^1)$ and the moduli space of Floer trajectories between contractible 1-periodic orbits in $(V, d\beta)$ with Hamiltonian function H. If we can adequately understand the moduli space of Floer trajectories—in particular if we can classify them and prove that they are regular—then the idea will be to extend this classification via the implicit function theorem to any $J_\epsilon \in \mathcal{J}(\lambda_\epsilon)$ sufficiently close to J for $\epsilon > 0$ small. As the reader may be aware, classifying Floer trajectories is also not easy in general, but it does become easy under certain conditions. Simple examples of contractible 1-periodic Hamiltonian orbits are furnished by the constant loops $\gamma(t) = x$ at critical points $x \in \mathrm{Crit}(H)$, and for each such orbit, $\gamma^*\Xi_0$ has a canonical homotopy class of unitary trivializations, the so-called **constant trivialization**. The following fundamental result is commonly used in proving the isomorphism from Hamiltonian Floer homology to singular homology.

Theorem 10.30. *Suppose $H : V \to \mathbb{R}$ is a smooth Morse function with no critical points on the boundary, J is a fixed $d\beta$-compatible almost complex structure on V, and the gradient flow of H with respect to the metric $d\beta(\cdot, J\cdot)$ is Morse-Smale* *and transverse to* ∂V*. Given* $\delta > 0$*, let* $H^\delta := \delta H : V \to \mathbb{R}$*, with Hamiltonian vector field* $X_{H^\delta} = \delta X_H$*, and consider the stable Hamiltonian structure*

$$\mathcal{H}_0^\delta := (d\beta + d\theta \wedge dH^\delta, d\theta)$$

on $V \times S^1$ *with induced Reeb vector field* $R_0^\delta = \partial_\theta + X_{H^\delta}$*. Then for all* $\delta > 0$ *sufficiently small, the following statements hold.*

(1) *The* 1*-periodic* R_0^δ*-orbit* $\gamma_x : S^1 \to V \times S^1 : t \mapsto (x,t)$ *arising from any critical point* $x \in \mathrm{Crit}(H)$ *is nondegenerate, and its Conley-Zehnder index relative to the constant trivialization* τ *is related to the Morse index* $\mathrm{ind}(x) \in \{0, \ldots, 2n\}$ *by*

$$\mu_{CZ}^\tau(\gamma_x) = n - \mathrm{ind}(x). \tag{10.8}$$

(2) *Any trajectory* $\gamma : \mathbb{R} \to V$ *satisfying the negative gradient flow question* $\dot\gamma = -\nabla H^\delta(\gamma)$ *gives rise to a Fredholm regular solution* $v : \mathbb{R} \times S^1 \to V : (s,t) \mapsto \gamma(s)$ *of the time-independent Floer equation*

$$\partial_s v + J(v)(\partial_t v - X_{H^\delta}(v)) = 0, \tag{10.9}$$

and the virtual dimensions of the spaces of Floer trajectories near v *and gradient flow trajectories near* γ *are the same.*

(3) *Every* 1*-periodic orbit of* X_{H^δ} *in* $\mathring{V}$ *is a constant loop at a critical point of* H*.*

(4) *Every finite-energy solution* $v : \mathbb{R} \times S^1 \to \mathring{V}$ *of* (10.9) *is of the form* $v(s,t) = \gamma(s)$ *for some negative gradient flow trajectory* $\gamma : \mathbb{R} \to V$*.*

PROOF. The following proof is based on arguments in [SZ92], see in particular Theorem 7.3.

For the first statement, let $\gamma(t) = (x,t)$ for $x \in \mathrm{Crit}(H)$ and recall from Lecture 3 the formula for the asymptotic operator of a 1-periodic orbit,

$$\mathbf{A}_\gamma : \Gamma(\gamma^*\Xi_0) \to \Gamma(\gamma^*\Xi_0) : \eta \mapsto -J\left(\nabla_t \eta - \nabla_\eta R_0^\delta\right),$$

where ∇ is any symmetric connection on $V \times S^1$. Identifying $\Gamma(\gamma^*\Xi_0)$ in the natural way with $C^\infty(S^1, T_x V)$, using the trivial connection and writing $R_0^\delta(z,\theta) = \partial_\theta + X_{H^\delta}(z) = \partial_\theta + \delta J(z)\nabla H(z)$, $\mathbf{A}_\gamma$ becomes the operator

$$\mathbf{A}_\gamma = -J\partial_t - \delta\nabla^2 H(x)$$

on $C^\infty(S^1, T_x V)$, where $\nabla^2 H(x) : T_x V \to T_x V$ denotes the Hessian of H at x. Choosing a unitary basis for $T_x V$ identifies this with $-J_0\partial_t - \delta S$ for some symmetric $2n$-by-$2n$ matrix S and the standard complex structure $J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, so $\ker \mathbf{A}_\gamma$ corresponds to the space of 1-periodic solutions to $\dot\eta = \delta J_0 S\eta$. The Morse condition implies that S is nonsingular, so the eigenvalues of $\delta J_0 S$ are all nonzero, but they are also small since δ is small. It follows that nontrivial solutions of $\dot\eta = \delta J_0 S\eta$ cannot be 1-periodic if S is nonsingular and δ is sufficiently small, thus proving that $\ker \mathbf{A}_\gamma$ is trivial, hence γ is nondegenerate.

To calculate $\mu_{\mathrm{CZ}}^{\tau}(\gamma)$, note that $\lambda \in \sigma(\mathbf{A}_\gamma)$ if and only if there exists a nontrivial 1-periodic solution η to the equation

$$\dot{\eta} = J_0(\delta S + \lambda)\eta.$$

If δ and λ are both close to 0, then the same argument again implies that no such solution exists unless $\delta S + \lambda$ is singular, meaning $\lambda \in \sigma(-\delta S)$. On the other hand, any constant loop $\eta(t) \in \ker(\lambda + \delta S)$ furnishes an element of the λ-eigenspace of $\mathbf{A}_\gamma$, so we obtain a bijection between the spectra of $\mathbf{A}_\gamma$ and $-\delta S$ in some neighborhood of 0. It follows that if $S_\pm$ denotes a pair of nonsingular symmetric matrices defining asymptotic operators $\mathbf{A}_\pm = -J_0\partial_t - \delta S_\pm$, then the spectral flows are related by

$$\mu^{\mathrm{spec}}(\mathbf{A}_-, \mathbf{A}_+) = -\mu^{\mathrm{spec}}(S_-, S_+)$$

when $\delta > 0$ is sufficiently small. Denoting the maximal negative-definite subspace of $S_\pm$ by $E^-(S_\pm)$, this relation implies

$$\dim E^-(S_+) - \dim E^-(S_-) = \mu_{\mathrm{CZ}}(\mathbf{A}_-) - \mu_{\mathrm{CZ}}(\mathbf{A}_+).$$

Now suppose S_+ is a coordinate expression for the Hessian $\nabla^2 H(x)$, hence $\dim E^-(S_+) = \mathrm{ind}(x)$ and $\mu_{\mathrm{CZ}}(\mathbf{A}_+) = \mu_{\mathrm{CZ}}^{\tau}(\gamma)$. Choosing $S_- = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ then gives $\dim E^-(S_-) = n$ and $\mu_{\mathrm{CZ}}(\mathbf{A}_-) = 0$ by definition, so $\mu_{\mathrm{CZ}}^{\tau}(\gamma) = n - \mathrm{ind}(x)$ follows.

The second statement follows in a similar manner by writing down and comparing the linearized operators for the Floer equation and the negative gradient flow equation. Let's leave this as an exercise.

For the third statement, suppose we have a sequence $\delta_k \to 0$ and a sequence of loops $x_k : S^1 \to \mathring{V}$ satisfying $\dot{x}_k = X_{H^{\delta_k}}(x_k) = \delta_k X_H(x_k)$. Pick a number $c > 0$ small enough for part (1) of the theorem to hold with $\delta = c$, choose a sequence of integers $N_k \in \mathbb{N}$ such that

$$N_k \delta_k \to c,$$

and consider the loops $y_k : S^1 \to \mathring{V} : t \mapsto x_k(N_k t)$. These satisfy

$$\dot{y}_k = N_k \delta_k X_H(y_k),$$

and since X_H is C^∞-bounded on V and $N_k \delta_k$ is also bounded, the Arzelà-Ascoli theorem provides a subsequence with

$$y_k \to y_\infty \quad \text{in} \quad C^\infty(S^1, V),$$

where $y_\infty : S^1 \to V$ satisfies $\dot{y}_\infty = X_{H^c}(y_\infty)$ for $H^c := cH : V \to \mathbb{R}$. But y_∞ is also constant: indeed, since $y_k(t + 1/N_k) = y_k(t)$ and $N_k \to \infty$, we can find for any $t \in S^1$ a sequence $q_k \in \mathbb{Z}$ satisfying $q_k / N_k \to t$, so

$$y_\infty(t) = \lim_{k\to\infty} y_k(q_k/N_k) = \lim_{k\to\infty} y_k(0) = y_\infty(0). \tag{10.10}$$

Since the constant orbit y_∞ is nondegenerate by part (1) of the theorem, there can only be one sequence of solutions to $\dot{y}_k = X_{H^{N_k\delta_k}}(y_k)$ converging to y_∞, and we conclude that y_k is also constant for all k sufficiently large.

We will now use a similar trick to prove the fourth statement in the theorem. We shall work under the additional assumption that

$$|\,\mathrm{ind}(x) - \mathrm{ind}(y)| \le 1 \quad \text{for all pairs} \quad x, y \in \mathrm{Crit}(H), \tag{10.11}$$

which suffices for the application in §3.3 below.[5]

Suppose to the contrary that there exists a sequence of positive numbers $\delta_k \to 0$ with finite-energy solutions $v_k : \mathbb{R} \times S^1 \to \mathring{V}$ of the equation $\partial_s v_k + J(v_k)(\partial_t v_k - X_{H^{\delta_k}}(v_k)) = 0$, where each $v_k(s,t)$ is not t-independent. By part (3) of the theorem, we can restrict to a subsequence and assume each v_k for large k is asymptotic to a fixed pair of critical points $x_\pm = \lim_{s\to\pm\infty} v_k(s,\cdot) \in \mathrm{Crit}(H)$, and $x_+ \neq x_-$ since v_k would otherwise by constant and therefore t-independent. Choose a sequence $N_k \in \mathbb{N}$ with

$$N_k \to \infty \quad \text{and} \quad N_k \delta_k \to c,$$

where $c > 0$ is chosen sufficiently small for the first three statements in the theorem to hold with $\delta = c$. Define $w_k : \mathbb{R} \times S^1 \to V$ by

$$w_k(s,t) = v_k(N_k s, N_k t).$$

Then w_k satisfies another time-independent Floer equation,

$$\partial_s w_k + J(w_k)\left(\partial_t w_k - X_{H^{N_k\delta_k}}(w_k)\right) = 0, \tag{10.12}$$

where the Hamiltonian functions $H^{N_k\delta_k}$ converge to H^c. The standard compactness theorem for Floer trajectories should now imply that a subsequence of w_k converges to a broken Floer trajectory whose levels will be t-independent. Since the setting may seem a bit nonstandard, here are some details.

The sequence w_k is uniformly C^0-bounded since V is compact. We claim that it is also C^1-bounded. If not, then there is a sequence $z_k = (s_k, t_k) \in \mathbb{R} \times S^1$ with $|dw_k(z_k)| =: R_k \to \infty$, and we can use the usual rescaling trick from Lecture 9 to define a sequence

$$f_k : \mathbb{D}_{\epsilon_k R_k} \to V : z \mapsto w_k(z_k + z/R_k)$$

for a suitable sequence $\epsilon_k \to 0$ with $\epsilon_k R_k \to \infty$ and $|dw_k(z)| \le 2R_k$ for all $z \in \mathbb{D}_{\epsilon_k}(z_k)$. The latter implies that f_k satisfies a local C^1-bound independent of k, and since

$$\partial_s f_k + J(f_k)\left(\partial_t f_k - \frac{1}{R_k} J(f_k) X_{H^{N_k\delta_k}}(f_k)\right),$$

elliptic regularity (see Remark 10.31 below) provides a subsequence for which f_k converges in $C^\infty_{\mathrm{loc}}(\mathbb{C}, V)$ to a J-holomorphic plane $f_\infty : \mathbb{C} \to V$, which is nonconstant since

$$|df_\infty(0)| = \lim_{k\to\infty} |df_k(0)| = 1.$$

Since v_k and therefore w_k are all asymptotic to fixed constant orbits $x_\pm$, we have a uniform bound on the Floer energies of w_k,

$$E_{H^{N_k\delta_k}}(w_k) = \mathcal{A}_{H^{N_k\delta_k}}(x_+) - \mathcal{A}_{H^{N_k\delta_k}}(x_-) = N_k\delta_k\left[H(x_-) - H(x_+)\right], \tag{10.13}$$

[5] Lifting this assumption requires gluing, whereas we shall only need the usual implicit function theorem for Fredholm regular solutions of the Floer equation.

where the right hand side is bounded since $N_k \delta_k \to c$. Using change of variables and the fact that $d\beta(\partial_s f_k, J(f_k)\,\partial_s f_k) \ge 0$, this implies a uniform bound

$$\int_{\mathbb{D}_{\epsilon_k R_k}} d\beta(\partial_s f_k, J(f_k)\,\partial_s f_k)\, ds \wedge dt = \int_{\mathbb{D}_{\epsilon_k}(z_k)} d\beta(\partial_s v_k, J(v_k)\,\partial_s v_k)\, ds \wedge dt$$
$$\le \int_{\mathbb{R}\times S^1} d\beta(\partial_s v_k, J(v_k)\,\partial_s v_k)\, ds \wedge dt = E_{H^{N_k \delta_k}}(w_k) \le C,$$

thus

$$\int_{\mathbb{C}} f_\infty^* d\beta = \int_{\mathbb{C}} d\beta(\partial_s f_\infty, \partial_t f_\infty)\, ds \wedge dt = \int_{\mathbb{C}} d\beta(\partial_s f_\infty, J(f_\infty)\,\partial_s f_\infty)\, ds \wedge dt < \infty.$$

The removable singularity theorem now extends f_∞ to a nonconstant J-holomorphic sphere $f_\infty : S^2 \to V$, but this violates Stokes' theorem since J is tamed by an exact symplectic form.

We've now shown that the sequence $w_k : \mathbb{R} \times S^1 \to V$ is uniformly C^1-bounded, and it has bounded energy due to (10.13). Pick any sequence $s_k \in \mathbb{R}$ and consider the sequence of translated Floer trajectories

$$\tilde{w}_k(s,t) := w_k(s + s_k, t).$$

These are also uniformly C^1-bounded, so by elliptic regularity (see Remark 10.31 again), a subsequence converges in $C^\infty_{\text{loc}}(\mathbb{R}\times S^1)$ to a map $w_\infty : \mathbb{R}\times S^1 \to V$ satisfying

$$\partial_s w_\infty + J(w_\infty)\,(\partial_t w_\infty - X_{H^c}(w_\infty)) = 0,$$

and it has finite energy $E_{H^c}(w_\infty) < \infty$ due to (10.13), implying that w_∞ is asymptotic to a pair of 1-periodic orbits of X_{H^c} as $s \to \pm\infty$. By the same argument used in (10.10) above, w_∞ is also t-independent. It follows that $w_\infty(s,t) = \gamma_\infty(s)$ for some nonconstant gradient flow trajectory $\gamma_\infty : \mathbb{R} \to \mathring{V}$. Depending on the choice of sequence s_k, this trajectory may or may not be constant, but we can always choose s_k to guarantee that γ_∞ is not constant: indeed, since each w_k is asymptotic to two separate critical points at $\pm\infty$, $s_k \in \mathbb{R}$ can be chosen such that $w_k(s_k, 0)$ stays a fixed distance away from every critical point of H, and then

$$w_\infty(0,0) = \lim_{k\to\infty} w_k(s_k, 0) \notin \operatorname{Crit}(H^c).$$

One can now adapt the argument of Proposition 10.19 to find various sequences $s_k \in \mathbb{R}$ that yield potentially separate limiting trajectories forming the levels of a broken trajectory, which is the limit of w_k in the Floer topology. But since all the levels are t-independent and the gradient flow of H^c is Morse-Smale, condition (10.11) implies that the most complicated (and therefore the only) limit possible involves a single level $w_\infty(s,t) = \gamma(s)$, which is a gradient flow trajectory between critical points whose Morse indices differ by 1. This trajectory is Fredholm regular and has index 1 due to part (2) of the theorem, thus by the implicit function theorem, the only solutions to (10.12) that can converge to w_∞ are the obvious reparametrizations of γ, i.e. they are also t-independent. This is a contradiction. □

REMARK 10.31. In previous lectures we've used the theorem that "C^1-bounds imply C^∞-bounds" to prove compactness for J-holomorphic curves, but not for solutions of inhomogeneous Cauchy-Riemann type equations such as the Floer trajectories w_k and rescalings f_k in the above proof. There is an easy trick to reduce these to our standard setup: as we've already seen, solutions of the Floer equation are equivalent to honest pseudoholomorphic curves in the symplectization of a certain stable Hamiltonian structure, which is a manifold of two dimensions higher. A similar trick can be used for any inhomogeneous Cauchy-Riemann type equation $\bar{\partial}_J f = \nu$, reducing it to an honest Cauchy-Riemann type equation at the cost of adding two dimensions. This trick was used already by Gromov, see [**Gro85**, 1.4.C].

REMARK 10.32. You may notice with some horror that (10.8) differs by a sign from what is stated in [**SZ92**]. As far as I can tell, the discrepancy arises from the fact that while Floer homology is traditionally defined in terms of a negative gradient flow for the action functional, SFT is based on a *positive* gradient flow—this is also why the action functional in (10.6) differs by a sign from what we saw in Lecture 1. If one takes as an axiom that the Conley-Zehnder index should serve as a "relative Morse index" for the action functional, then changing the sign of the functional also reverses the signs of Conley-Zehnder indices, so as a result there appear to be two parallel sign conventions for Conley-Zehnder indices in different sectors of the literature. I'm sorry. It's not my fault.

Returning now to the family $\mathcal{H}_\epsilon$, choose $\delta > 0$ sufficiently small for Theorem 10.30 to hold and define a modified family of stable Hamiltonian structures on $V \times S^1$ by
$$\mathcal{H}_\epsilon^\delta = (\Omega^\delta, \lambda_\epsilon^\delta),$$
where
$$\Omega^\delta := d\beta + d\theta \wedge dH^\delta \quad \text{and} \quad \lambda_\epsilon^\delta := d\theta + \epsilon(\beta - H^\delta\, d\theta).$$
Denote the induced hyperplane distributions and Reeb vector fields by Ξ_ϵ^δ and R_ϵ^δ respectively. We have only changed the Hamiltonian H by rescaling, so all previous statements about $\mathcal{H}_\epsilon$ also apply to $\mathcal{H}_\epsilon^\delta$, in particular λ_ϵ^δ is contact and $\mathcal{J}(\mathcal{H}_\epsilon^\delta) = \mathcal{J}(\lambda_\epsilon^\delta)$ for all $\epsilon > 0$ sufficiently small, though the upper bound for the allowed range of ϵ may now depend on δ. Once $\delta > 0$ is fixed by the requirements of Theorem 10.30, we are still free to take $\epsilon > 0$ is small as we like.

THEOREM 10.33. *Assume the same hypotheses as in Theorem 10.30, including (10.11), and denote the unique extension of J to an $\mathbb{R}$-invariant almost complex structure in $\mathcal{J}(\mathcal{H}_0^\delta)$ by J_0. Given δ sufficiently small and any smooth family of compatible $\mathbb{R}$-invariant almost complex structures $J_\epsilon \in \mathcal{J}(\mathcal{H}_\epsilon^\delta)$ matching J_0 at $\epsilon = 0$, there exists $\epsilon_0 > 0$ such that every critical point $x \in \mathrm{Crit}(H)$ gives rise to a smooth family of nondegenerate closed R_ϵ^δ-orbits*
$$x^\epsilon : S^1 \to V \times S^1 \qquad \epsilon \in [0, \epsilon_0]$$
with $x^0(t) = (x, t)$, and every gradient flow trajectory $\gamma : \mathbb{R} \to V$ for H gives rise to a smooth family of Fredholm regular J_ϵ-holomorphic cylinders
$$u_\gamma^\epsilon : \mathbb{R} \times S^1 \to \mathbb{R} \times (V \times S^1) \qquad \epsilon \in [0, \epsilon_0]$$
with $u_\gamma^0(s, t) = (s, \gamma(\delta s), t)$. Moreover, for all $\epsilon \in [0, \epsilon_0]$, every closed R_ϵ^δ-orbit homotopic to $t \mapsto (\mathrm{const}, t)$ belongs to one of the families x^ϵ up to parametrization, and

every J_ϵ-holomorphic cylinder with a positive and a negative end asymptotic to orbits of this type belongs to one of the families u^ϵ_γ, up to biholomorphic parametrization.

PROOF. The first part is immediate from the implicit function theorem since the orbits $x^0(t) = (x, t)$ are nondegenerate and the curves $u^0_\gamma(s,t) = (s, \gamma(\delta s), t)$ are Fredholm regular by Theorem 10.30. For the uniqueness statement, observe that if $\epsilon_k \to 0$ and γ_k is a sequence of $R^\delta_{\epsilon_k}$-orbits in the relevant homotopy class, then their periods are uniformly bounded, so Arzelà-Ascoli gives a subsequence convergent to a closed R^δ_0-orbit, which is a nondegenerate orbit of the form $x^0(t) = (x, t)$ for $x \in \mathrm{Crit}(H)$ by Theorem 10.30, so sequences converging to this orbit are unique by the implicit function theorem. A similar argument proves uniqueness of J_ϵ-holomorphic cylinders: if $\epsilon_k \to 0$ and u_k is a J_{ϵ_k}-holomorphic sequence, then first by the uniqueness of the orbits, we can extract a subsequence for which all u_k are asymptotic at both ends to orbits in fixed families $x^{\epsilon_k}_\pm$ converging to $x^0_\pm(t) = (x_\pm, t)$ as $k \to \infty$. Since Ω is exact, Stokes' theorem then gives a uniform bound on the energies $E_{\epsilon_k}(u_k)$. Since all R^δ_0-orbits in the relevant homotopy class are nondegenerate and none are contractible, one can now prove as in Proposition 10.19 that u_k has a subsequence convergent to a finite-energy stable J_0-holomorphic building $\mathbf{u}_\infty$ consisting only of cylinders. Its levels are asymptotic to orbits of the form $x(t) = (x, t)$ for $x \in \mathrm{Crit}(H)$, thus they can be parametrized as $(s,t) \mapsto (s, v(s,t), t)$ for $v : \mathbb{R} \times S^1 \to V$ satisfying the H^δ-Floer equation, hence $v(s,t) = \gamma(\delta s)$ by Theorem 10.30. Now since ∇H is Morse-Smale and indices of critical points can only differ by at most 1, the building $\mathbf{u}_\infty$ can have at most one nontrivial level $u_\infty(s,t) = (s, \gamma(\delta s), t)$, implying $u_k \to u_\infty$. Since u_∞ is Fredholm regular, the implicit function theorem does the rest. □

3.3. Admissible data for $(\mathbb{T}^3, \xi_k)$. We now complete the computation of the cylindrical contact homology $HC^h_*(\mathbb{T}^3, \xi_k)$. We can assume via Lemma 10.26 that h is the homotopy class of the orbits in the special set of tori

$$T_m = \{m/k\} \times \mathbb{T}^2 \subset \mathbb{T}^3, \qquad m = 0, \dots, k-1.$$

Let's focus for now on the case $k = 1$, as the general case will simply be a k-fold cover of this. Thanks to the Morse-Bott discussion in §3.1, we know what we're looking for: we want an h-admissible contact form α for $(\mathbb{T}^3, \xi_1)$ such that $\mathcal{P}_h(\alpha)$ contains exactly two orbits, both in $T_0 \subset \mathbb{T}^3$, along with an h-regular $J \in \mathcal{J}(\alpha)$ such that the differential on $CC^h_*(\mathbb{T}^3, \alpha)$ counts exactly two J-holomorphic cylinders that connect the two orbits in T_0. Let $\mathbb{A}$ denote the annulus

$$\mathbb{A} = [-1, 1] \times S^1$$

with coordinates (ρ, ϕ). This will play the role of the Liouville manifold $(V, d\beta)$ from the previous section, and we set

$$\beta := \rho \, d\phi.$$

For the Hamiltonian $H : \mathbb{A} \to \mathbb{R}$, choose a Morse function with the following properties:

(1) H has a minimum at $x_0 = (0, 0)$, an index 1 critical point at $x_1 = (0, 1/2)$, and no other critical points;
(2) $H(\rho, \phi) = |\rho|$ for $1/2 \le |\rho| \le 1$;

(3) The gradient flow of H with respect to the standard Euclidean metric on $[-1,1] \times S^1$ is Morse-Smale.

Fix a number $\delta > 0$ sufficiently small so that Theorem 10.30 applies for Floer trajectories of $H^\delta := \delta H$ in $\mathbb{A}$, and since it will turn out to be useful in Lemma 10.34 below, assume without loss of generality

$$\delta \in \mathbb{Q}.$$

Then following the prescription described above, we consider the family of stable Hamiltonian structures $\mathcal{H}^\delta_\epsilon = (\Omega^\delta, \lambda^\delta_\epsilon)$ on $\mathbb{A} \times S^1$ for $\epsilon \geq 0$ small, where

$$\lambda^\delta_\epsilon = (1 - \epsilon\delta H)\, d\theta + \epsilon\rho\, d\phi, \qquad \Omega^\delta = d\rho \wedge d\phi + \delta\, d\theta \wedge dH,$$

with induced Reeb vector fields R^δ_ϵ and hyperplane distributions $\Xi^\delta_\epsilon := \ker \lambda^\delta_\epsilon$. Choose $J_\epsilon \in \mathcal{J}(\mathcal{H}^\delta_\epsilon)$ to be any smooth family such that $J_0|_{\Xi^\delta_0}$ matches the standard complex structure on $\mathbb{A}$ defined by $J_0\partial_\rho = \partial_\phi$. Then for all $\epsilon > 0$ sufficiently small, Theorems 10.30 and 10.33 give a complete classification of all closed R^δ_ϵ-orbits in $\mathbb{A} \times S^1$ homotopic to $t \mapsto (0,0,t)$, as well as a classification of all J_ϵ-holomorphic cylinders asymptotic to them. Up to parametrization, there are exactly two such orbits,

$$\gamma^\epsilon_i : S^1 \to \mathbb{A} \times S^1, \qquad i = 0, 1,$$

which correspond to the Morse critical points x_0 and x_1 and thus by (10.8) have Conley-Zehnder indices

$$\mu^\tau_{\mathrm{CZ}}(\gamma^\epsilon_i) = 1 - \operatorname{ind}(x_i) = 1 - i \in \{0, 1\}$$

relative to the constant trivialization τ. There are also exactly two J_ϵ-holomorphic cylinders

$$u^\epsilon_\pm : \mathbb{R} \times S^1 \to \mathbb{R} \times (\mathbb{A} \times S^1),$$

corresponding to the two negative gradient flow lines that descend from x_1 to x_0, thus the $u^\epsilon_\pm$ are index 1 curves with a negative end approaching γ^ϵ_1 and a positive end approaching γ^ϵ_0. If we can suitably embed this model into $(\mathbb{T}^3, \xi_1)$ and show that all the orbits and curves needing to be counted are contained in the model, then we will have a complete description of $HC^h_*(\mathbb{T}^3, \xi_1)$, with two generators $\langle \gamma^\epsilon_0 \rangle$ and $\langle \gamma^\epsilon_1 \rangle$, of even and odd degree respectively, satisfying

$$\partial \langle \gamma^\epsilon_0 \rangle = 2\langle \gamma^\epsilon_1 \rangle = 0 \quad \text{and} \quad \partial \langle \gamma^\epsilon_1 \rangle = 0$$

since the former counts two curves and the latter counts none.

LEMMA 10.34. *For any $\epsilon > 0$ sufficiently small, there exists a contact embedding of*

$$(\mathbb{A} \times S^1, \ker \lambda^\delta_\epsilon) \hookrightarrow (\mathbb{T}^3, \xi_1)$$

identifying the homotopy class of the loops $t \mapsto (0,0,t)$ in $\mathbb{A} \times S^1$ with h. Moreover, the contact form λ^δ_ϵ and almost complex structure $J_\epsilon \in \mathcal{J}(\mathcal{H}^\delta_\epsilon)$ can then be extended to an h-admissible contact form α on $(\mathbb{T}^3, \xi_1)$ and an h-regular almost complex structure $J \in \mathcal{J}(\alpha)$ such that γ^ϵ_0 and γ^ϵ_1 are the only orbits in $\mathcal{P}_h(\alpha)$, and all J-holomorphic cylinders with a positive and a negative end asymptotic to either of these orbits are contained in the interior of $\mathbb{A} \times S^1$.

PROOF. We've chosen β and H so that in the region $1/2 \le |\rho| \le 1$,

$$\alpha := \lambda_\epsilon^\delta = (1 - \epsilon\delta|\rho|)\, d\theta + \epsilon\rho\, d\phi =: f(\rho)\, d\theta + g(\rho)\, d\phi,$$

so the Reeb vector field on this region has the form $\frac{1}{D(\rho)}(g'(\rho)\,\partial_\theta - f'(\rho)\,\partial_\phi)$. Notice that

$$\frac{f'(\rho)}{g'(\rho)} = \mp\frac{\epsilon\delta}{\epsilon} = \mp\delta,$$

and we assumed $\delta \in \mathbb{Q}$, so the Reeb orbits in this region are all periodic. Next, pick a large number $N \gg 1$ and extend α to a contact form on $[-N, N] \times S^1 \times S^1$ via the same formula. Now extend the path $(f, g) : [-N, N] \to \mathbb{R}^2$ to $\mathbb{R}$ such that it has period $2N + 2$ and winds once around the origin over the interval $[-N - 1, N + 1]$, with positive angular velocity. This produces a contact form α on

$$\mathbb{T}^3_N := \left(\mathbb{R}\middle/(2N + 2)\mathbb{Z}\right) \times S^1 \times S^1$$

which takes the form $f(\rho)\, d\theta + g(\rho)\, d\phi$ outside of $|\rho| \le 1/2$. We claim in fact that α is homotopic through contact forms to one that takes this form globally, where (f, g) may be assumed to be a smooth loop winding once around the origin. To see this, one need only homotop H in the region $|\rho| \le 1/2$ to a Morse-Bott function that depends only on the ρ-coordinate; the contact condition holds for all Hamiltonians in this homotopy as long as $\epsilon > 0$ is sufficiently small. With this understood, the obvious diffeomorphism

$$\mathbb{T}^3_N \to \mathbb{T}^3 : (\rho, \phi, \theta) \mapsto \left(\frac{\rho}{2N + 2}, \phi, \theta\right)$$

pushes $\ker\alpha$ forward to a contact structure isotopic to one of the form $F(\rho)\, d\theta + G(\rho)\, d\phi$ for a loop $(F, G) : S^1 \to \mathbb{R}^2$ winding once around the origin, so taking a homotopy of this loop to $(\cos(2\pi\rho), \sin(2\pi\rho))$ and applying Gray's stability theorem produces a contactomorphism

$$(\mathbb{T}^3_N, \ker\alpha) \to (\mathbb{T}^3, \xi_1)$$

that is isotopic to the above diffeomorphism.

The construction clearly guarantees that no closed Reeb orbit of α outside $\mathbb{A} \times S^1$ is homotopic to the preferred class h, and there are also no contractible orbits, so α is an h-admissible contact form on $\mathbb{T}^3_N$. Choose any extension of J_ϵ to some $J \in \mathcal{J}(\alpha)$ on $\mathbb{T}^3_N$. We claim now that if N is chosen sufficiently large, then no J-holomorphic cylinder in $\mathbb{R} \times \mathbb{T}^3_N$ with one positive end at either of the orbits γ_i^ϵ can ever venture outside the region $\mathbb{R} \times (-1/2, 1/2) \times \mathbb{T}^2$. Suppose in particular that u is such a curve and its image intersects $\mathbb{R} \times \{1/2\} \times \mathbb{T}^2$. Since the entire region $[1/2, N] \times \mathbb{T}^2$ is foliated by closed Reeb orbits, we can define Υ to be the set of Reeb orbits γ in that region for which the image of u intersects $\mathbb{R} \times \gamma$. This is a closed subset of the connected topological space of all Reeb orbits in $[1/2, N] \times \mathbb{T}^2$: indeed, if $\gamma_k \in \Upsilon$ is a sequence converging to some orbit γ_∞, then $u(z_k) \in \mathbb{R} \times \gamma_k$ for some sequence $z_k \in \mathbb{R} \times S^1$, which must be contained in a compact subset since the asymptotic orbits of u lie outside of $[1/2, N] \times \mathbb{T}^2$, hence z_k has a convergent subsequence $z_k \to z_\infty \in \mathbb{R} \times S^1$ with $u(z_\infty) \in \mathbb{R} \times \gamma_\infty$, proving $\gamma_\infty \in \Upsilon$. We claim that Υ is also an open subset of the space of orbits in $[1/2, N] \times \mathbb{T}^2$. This follows from positivity of intersections, as every $\mathbb{R} \times \gamma$ is also a J-holomorphic curve: if $u(z) \in \mathbb{R} \times \gamma$, then for every other closed orbit γ' close enough to γ, there is a point $z' \in \mathbb{R} \times S^1$ near z with $u(z') \in \mathbb{R} \times \gamma'$. This proves that, in fact, u passes through $\mathbb{R} \times \gamma$ for *every* orbit γ in the region $[1/2, N] \times \mathbb{T}^2$. We will now use this to show that if N is sufficiently large, the contact area of u will be larger than is allowed by Stokes' theorem.

Let us write

$$u(s,t) = (r(s,t), \rho(s,t), \phi(s,t), \theta(s,t)) \in \mathbb{R} \times \big(\mathbb{R}/(2N+2)\mathbb{Z}\big) \times S^1 \times S^1$$

and choose two points $\rho_1 \in [1/2, 1]$ and $\rho_2 \in [N-1, N]$ which are both regular values of the function $\rho : \mathbb{R} \times S^1 \to \mathbb{R}/(2N+2)\mathbb{Z}$. The intersections of u with the orbits in $[1/2, N] \times \mathbb{T}^2$ imply that the function $\rho(s,t)$ attains every value in $[1/2, N]$, and since the asymptotic limits of u lie outside this region,

$$\mathcal{U} := \rho^{-1}([\rho_1, \rho_2]) \subset \mathbb{R} \times S^1$$

is then a nonempty and compact smooth submanifold with boundary

$$\partial \mathcal{U} = -C_1 \sqcup C_2,$$

where $C_i := \rho^{-1}(\rho_i)$ for $i = 1, 2$. Restricting u to the multicurves C_i then gives a pair of smooth maps

$$w_i : C_i \to \mathbb{T}^2 : (s,t) \mapsto (\phi(s,t), \theta(s,t)), \qquad i = 1, 2,$$

which are homologous to each other. Denote the generators of $H_1(\mathbb{T}^2)$ corresponding to the ϕ- and θ-coordinates by ℓ_ϕ and ℓ_θ respectively, and suppose $[w_i] = m\ell_\phi + n\ell_\theta$ for $m, n \in \mathbb{Z}$. The key observation now is that the restriction of α to each of the tori $\{\rho_i\} \times \mathbb{T}^2$ is a closed 1-form, thus for each $i = 1, 2$, $\int_{C_i} u^*\alpha$ depends only on the homology class $m\ell_\phi + n\ell_\theta \in H_1(\mathbb{T}^2)$ and not any further on the maps w_i. In particular,

$$\int_{C_i} u^*\alpha = f(\rho_i)n + g(\rho_i)m$$

for $i = 1, 2$. We now compute,

$$\begin{aligned}
\int_{\mathcal{U}} u^*d\alpha = \int_{C_2} u^*\alpha - \int_{C_1} u^*\alpha &= n[f(\rho_2) - f(\rho_1)] + m[g(\rho_2) - g(\rho_1)] \\
&= n[(1 - \epsilon\delta\rho_2) - (1 - \epsilon\delta\rho_1)] + m[\epsilon\rho_2 - \epsilon\rho_1] \\
&= \epsilon(\rho_2 - \rho_1)(m - n\delta)
\end{aligned}$$

This integral has to be positive since $u^*d\alpha \geq 0$ and u is not a trivial cylinder, thus $m - n\delta > 0$. Moreover, δ was assumed rational, so if $\delta = p/q$ for some $p, q \in \mathbb{N}$, we have

$$m - n\delta = \frac{1}{q}(mq - np) \geq \frac{1}{q},$$

implying

$$\int_{\mathbb{R} \times S^1} u^*d\alpha \geq \int_{\mathcal{U}} u^*d\alpha \geq \frac{\epsilon}{q}(\rho_2 - \rho_1) \geq \frac{\epsilon(N-2)}{q}.$$

Having chosen δ (which determines q) and ϵ in advance, we are free to make N as large as we like. But by (10.2), $\int_{\mathbb{R} \times S^1} u^*d\alpha$ cannot be any larger than the period of its positive asymptotic orbit, which does not depend on N. So this gives a contradiction, proving that u cannot touch the region $\{\rho \ge 1/2\}$. The mirror image of this argument shows that u also cannot touch the region $\{\rho \le -1/2\}$. □

With Lemma 10.34 in hand, the calculation of $HC_*^h(\mathbb{T}^3_N, \alpha, J)$ for sufficiently large N is straightforward: there is one odd generator and one even generator, with a trivial differential, giving

$$HC_*^h(\mathbb{T}^3, \xi_1) \cong \begin{cases} \mathbb{Z}_2 & * = \text{odd}, \\ \mathbb{Z}_2 & * = \text{even}. \end{cases}$$

This calculation can now be extended to $(\mathbb{T}^3, \xi_k)$ by a cheap trick: using the contactomorphism $(\mathbb{T}^3_N, \ker \alpha) \to (\mathbb{T}^3, \xi_1)$, let us identify $\mathbb{T}^3_N$ with $\mathbb{T}^3$ and write $\alpha = F\alpha_1$ for some function $F : \mathbb{T}^3 \to (0, \infty)$. Then the k-fold covering map

$$\Phi_k : \mathbb{T}^3 \to \mathbb{T}^3 : (\rho, \phi, \theta) \mapsto (k\rho, \phi, \theta)$$

maps the homotopy class h to itself and pulls back ξ_1 to ξ_k, so $\Phi_k^*\alpha$ is a contact form for ξ_k. It is also h-admissible: indeed, $\Phi_k^*\alpha$ admits no contractible orbits since they would project down to contractible orbits on $(\mathbb{T}^3, \alpha)$, and every orbit in $\mathcal{P}_h(\Phi_k^*\alpha)$ projects to one in $\mathcal{P}_h(\alpha)$, hence they are all nondegenerate. The almost complex structure $\Phi_k^* J \in \mathcal{J}(\Phi_k^*\alpha)$ then makes the map $\mathrm{Id} \times \Phi_k : (\mathbb{R} \times \mathbb{T}^3, \Phi_k^* J) \to (\mathbb{R} \times \mathbb{T}^3, J)$ holomorphic, so every $\Phi_k^* J$-holomorphic cylinder counted by $HC_*^h(\mathbb{T}^3, \Phi_k^*\alpha, \Phi_k^* J)$ projects to a J-holomorphic cylinder counted by $HC_*^h(\mathbb{T}^3, \alpha, J)$, and conversely, each orbit in $\mathcal{P}_h(\alpha)$ and each J-holomorphic cylinder has exactly k lifts to the cover. The generators of $CC_*^h(\mathbb{T}^3, \Phi_k^*\alpha)$ thus consist of $2k$ orbits, k odd and k even, with $2k$ connecting $\Phi_k^* J$-holomorphic cylinders that cancel each other in pairs, giving a trivial differential. In summary:

$$HC_*^h(\mathbb{T}^3, \xi_k) \cong \begin{cases} \mathbb{Z}_2^k & * = \text{odd}, \\ \mathbb{Z}_2^k & * = \text{even}. \end{cases}$$

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