LECTURE 5

The index formula

(edited by Agustín Moreno)

Contents

1. Riemann-Roch with punctures

As in the previous lecture, let $\mathbf{D}$ denote a linear Cauchy-Riemann type operator on an asymptotically Hermitian vector bundle E of complex rank m over a punctured Riemann surface $(\dot{\Sigma} = \Sigma \setminus (\Gamma^+ \cup \Gamma^-), j)$, and assume that $\mathbf{D}$ is asymptotic at each puncture $z \in \Gamma$ to a nondegenerate asymptotic operator $\mathbf{A}_z$ on the asymptotic bundle (E_z, J_z, ω_z) over S^1. Writing

$$F := \overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, E)$$

for the bundle of complex-antilinear homomorphisms $T\dot{\Sigma} \to E$, the main result of the previous lecture was that

$$\mathbf{D} : W^{k,p}(E) \to W^{k-1,p}(F)$$

is Fredholm for any $k \in \mathbb{N}$ and $p \in (1, \infty)$, and its kernel and index do not depend on k or p. The main goal of this lecture is to compute $\mathrm{ind}(\mathbf{D}) \in \mathbb{Z}$.

The index will depend on the Conley-Zehnder indices $\mu^{\tau}_{\mathrm{CZ}}(\mathbf{A}_z) \in \mathbb{Z}$ introduced in Lecture 3, but since these depend on arbitrary choices of unitary trivializations τ, we need a way of selecting preferred trivializations. The most natural condition is to require that every (E_z, J_z, ω_z) be endowed with a unitary trivialization such that the corresponding asymptotic trivializations of (E, J) extend to a global trivialization[1];

[1]Note that (E, J) is always globally trivializable unless $\Gamma = \emptyset$, as a punctured surface can be retracted to its 1-skeleton.

if there is only one puncture z, for instance, then this condition determines $\mu_{\mathrm{CZ}}^{\tau}(\mathbf{A}_z)$ uniquely. This convention has been used to state the formula for $\mathrm{ind}(\mathbf{D})$ in several of the standard references, e.g. in [HWZ99]. We would prefer however to state a formula which is also valid when $\Gamma = \emptyset$ and $E \to \Sigma$ is nontrivial. One way to do this is by allowing completely arbitrary asymptotic trivializations, but introducing a topological invariant to measure their failure to extend globally over E.

DEFINITION 5.1. Fix a compact oriented surface S with boundary. The **relative first Chern number** associates to every complex vector bundle (E, J) over S and trivialization τ of $E|_{\partial S}$ an integer

$$c_1^{\tau}(E) \in \mathbb{Z}$$

satisfying the following properties:

(1) If $(E, J) \to S$ is a line bundle, then $c_1^{\tau}(E)$ is the signed count of zeroes for a generic smooth section $\eta \in \Gamma(E)$ that appears as a nonzero constant at ∂S with respect to τ.
(2) For any two bundles (E_1, J_1) and (E_2, J_2) with trivializations τ_1 and τ_2 respectively over ∂S,

$$c_1^{\tau_1 \oplus \tau_2}(E_1 \oplus E_2) = c_1^{\tau_1}(E_1) + c_1^{\tau_2}(E_2).$$

These two conditions uniquely determine $c_1^{\tau}(E)$ for all complex vector bundles since bundles of higher rank can always be split into direct sums of line bundles. The definition clearly matches the usual first Chern number $c_1(E)$ when $\partial S = \emptyset$, and it extends in an obvious way to the category of asymptotically Hermitian vector bundles with asymptotic trivializations.

EXERCISE 5.2. Given two distinct choices of asymptotic trivializations τ_1 and τ_2 for an asymptotically Hermitian bundle E of rank m, show that

$$c_1^{\tau_2}(E) = c_1^{\tau_1}(E) - \deg(\tau_2 \circ \tau_1^{-1}),$$

where $\deg(\tau_2 \circ \tau_1^{-1}) \in \mathbb{Z}$ denotes the sum over all punctures of the winding numbers of the determinants of the transition maps $S^1 \to \mathrm{U}(m)$.[2]

EXERCISE 5.3. Combining Exercise 5.2 above with Exercise 3.36 from Lecture 3, show that for our asymptotically Hermitian vector bundle E with Cauchy-Riemann type operator $\mathbf{D}$ and asymptotic operators $\mathbf{A}_z$, the number

$$2c_1^{\tau}(E) + \sum_{z \in \Gamma^+} \mu_{\mathrm{CZ}}^{\tau}(\mathbf{A}_z) - \sum_{z \in \Gamma^-} \mu_{\mathrm{CZ}}^{\tau}(\mathbf{A}_z)$$

is independent of the choice of asymptotic trivializations τ.

The above exercise shows that the right hand side of the following index formula is independent of all choices.

[2] Caution: to compute this winding number at a negative puncture using cylindrical coordinates $(s, t) \in (-\infty, 0] \times S^1$, one must traverse $\{-s\} \times S^1$ for $s \gg 1$ in the *wrong direction*, as this is consistent with the orientation induced on $\{-s\} \times S^1$ as a boundary component of a large compact subdomain of $\dot{\Sigma}$.

THEOREM 5.4. *The Fredholm index of* $\mathbf{D}$ *is given by*

$$\operatorname{ind}\mathbf{D} = m\chi(\dot{\Sigma}) + 2c_1^\tau(E) + \sum_{z\in\Gamma^+}\mu_{CZ}^\tau(\mathbf{A}_z) - \sum_{z\in\Gamma^-}\mu_{CZ}^\tau(\mathbf{A}_z),$$

where $m = \operatorname{rank}_{\mathbb{C}} E$ *and* τ *is an arbitrary choice of asymptotic trivializations.*

NOTATION. Throughout this lecture, we shall denote the integer on the right hand side in Theorem 5.4 by

$$I(\mathbf{D}) := m\chi(\dot{\Sigma}) + 2c_1^\tau(E) + \sum_{z\in\Gamma^+}\mu_{\mathrm{CZ}}^\tau(\mathbf{A}_z) - \sum_{z\in\Gamma^-}\mu_{\mathrm{CZ}}^\tau(\mathbf{A}_z) \in \mathbb{Z}.$$

Our goal is thus to prove that $\operatorname{ind}(\mathbf{D}) = I(\mathbf{D})$.

When $\Gamma = \emptyset$, Theorem 5.4 is equivalent to the classical Riemann-Roch formula, which is more often stated for *holomorphic* vector bundles over a closed Riemann surface (Σ, j) with genus g as

$$\operatorname{ind}_{\mathbb{C}}(\mathbf{D}_0) = m(1-g) + c_1(E). \tag{5.1}$$

This formula assumes that the Cauchy-Riemann type operator $\mathbf{D}_0$ is complex linear, but an arbitrary real-linear Cauchy-Riemann operator is then of the form $\mathbf{D} = \mathbf{D}_0 + B$, where the zeroth-order term $B \in \Gamma(\operatorname{Hom}_{\mathbb{R}}(E, F))$ defines a compact perturbation since the inclusion $W^{k,p}(\Sigma) \hookrightarrow W^{k-1,p}(\Sigma)$ is compact. It follows that $\mathbf{D}$ has the same *real* Fredholm index as $\mathbf{D}_0$, namely twice the complex index shown on the right hand side of (5.1), which matches what we see in Theorem 5.4.

REMARK 5.5. Now seems a good moment to clarify explicitly that all dimensions (and therefore also Fredholm indices) in this lecture are *real* dimensions, not complex dimensions, unless otherwise stated.

Reduction to the complex-linear case does not work in general if there are punctures: it remains true that arbitrary Cauchy-Riemann type operators can be written as $\mathbf{D} = \mathbf{D}_0 + B$ where $\mathbf{D}_0$ is complex linear, but the perturbation introduced by the zeroth-order term B is not compact since $W^{k,p}(\dot{\Sigma}) \hookrightarrow W^{k-1,p}(\dot{\Sigma})$ is not compact when $\Gamma \neq \emptyset$. Another indication that this idea cannot work is the fact that while the formula in Theorem 5.4 always gives an *even* integer when $\Gamma = \emptyset$, it can be odd when there are punctures, in which case $\mathbf{D}$ clearly cannot have the same index is any complex-linear operator. Our proof will therefore have to deal with more than just the complex category.

The punctured version of Theorem 5.4 was first proved by Schwarz in his thesis [**Sch95**], its main purpose at the time being to help define algebraic operations (notably the *pair-of-pants product*) in Hamiltonian Floer homology. Schwarz's proof used a "linear gluing" construction that gives a relation between indices of operators on bundles over surfaces obtained by gluing together constituent surfaces along matching cylindrical ends. Since any surface with ends can be "capped off" to form a closed surface, one obtains the general index formula if one already knows how to compute it for closed surfaces and for planes (i.e. caps). For the latter, it is simple enough to write down model Cauchy-Riemann operators on planes and compute their kernels and cokernels explicitly, so in this way the general case is reduced to the classical Riemann-Roch formula. An analogous linear gluing argument for compact surfaces with boundary is used in [**MS04**, Appendix C] to reduce the general Riemann-Roch formula to an explicit computation for Cauchy-Riemann operators on the disk with a totally real boundary condition.

In this lecture, we will follow a different path and use an argument that was first sketched by Taubes for the closed case in [**Tau96a**, §7], with an additional argument for the punctured case suggested by Chris Gerig [**Ger**]. The argument is (in my opinion) analytically somewhat easier than the more standard approaches, and in addition to proving the formula we need for punctured surfaces, it produces a new proof in the closed case without assuming the classical Riemann-Roch formula. It also provides a gentle preview of two analytical phenomena that will later assume prominent roles in our discussion of SFT: *bubbling* and *gluing*.

To see the idea behind Taubes's argument, we can start by noticing an apparent numerical coincidence in the closed case. Assume (E, J) is a complex line bundle over a closed Riemann surface (Σ, j), and $\mathbf{D} : \Gamma(E) \to \Gamma(F) = \Omega^{0,1}(\Sigma, E)$ is a Cauchy-Riemann type operator. We know that $\operatorname{ind}(\mathbf{D}) = \operatorname{ind}(\mathbf{D} + B)$ for any zeroth-order term $B \in \Gamma(\operatorname{Hom}_{\mathbb{R}}(E, F))$. But E and F are both complex vector bundles, so B can always be split uniquely into its complex-linear and complex-antilinear parts, i.e. there is a natural splitting of $\operatorname{Hom}_{\mathbb{R}}(E, F)$ into a direct sum of complex line bundles[3]

$$\operatorname{Hom}_{\mathbb{R}}(E, F) = \operatorname{Hom}_{\mathbb{C}}(E, F) \oplus \overline{\operatorname{Hom}}_{\mathbb{C}}(E, F).$$

Out of curiosity, let's compute the first Chern number of the second factor; this will be the signed count of zeroes of a generic complex-*antilinear* zeroth-order perturbation. To start with, note that

$$\overline{\operatorname{Hom}}_{\mathbb{C}}(E, F) = \overline{\operatorname{Hom}}_{\mathbb{C}}(E, \mathbb{C}) \otimes F,$$

and then observe that $\overline{\operatorname{Hom}}_{\mathbb{C}}(E, \mathbb{C})$ and E are isomorphic: indeed, any Hermitian bundle metric $\langle\ ,\ \rangle_E$ on E gives rise to a bundle isomorphism[4]

$$E \to \overline{\operatorname{Hom}}_{\mathbb{C}}(E, \mathbb{C}) : \eta \mapsto \langle \cdot, \eta \rangle_E.$$

We thus have $\overline{\operatorname{Hom}}_{\mathbb{C}}(E, F) \cong E \otimes F$, so $c_1(\overline{\operatorname{Hom}}_{\mathbb{C}}(E, F)) = c_1(E) + c_1(F)$. We can compute $c_1(F)$ by the same trick since

$$F = \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, E) = \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, \mathbb{C}) \otimes E \cong T\Sigma \otimes E,$$

so $c_1(F) = c_1(T\Sigma) + c_1(E) = \chi(\Sigma) + c_1(E)$, and thus

$$c_1(\overline{\operatorname{Hom}}_{\mathbb{C}}(E, F)) = \chi(\Sigma) + 2c_1(E).$$

Since we're looking at a line bundle over a surface without punctures, this number is the same as $I(\mathbf{D})$. This coincidence is too improbable to ignore, and indeed, it turns out not to be coincidental. Here is an informal statement of a result that we will later prove a more precise version of in order to deduce Theorem 5.4.

[3] Here the complex structure on $\operatorname{Hom}_{\mathbb{R}}(E, F)$ and its subbundles is defined in terms of the complex structure of F, i.e. it sends $B \in \operatorname{Hom}_{\mathbb{R}}(E, F)$ to $J \circ B \in \operatorname{Hom}_{\mathbb{R}}(E, F)$.

[4] We are assuming as usual that Hermitian inner products are complex antilinear in the first argument and linear in the second.

"THEOREM". *Given a Cauchy-Riemann type operator* $\mathbf{D} : H^1(E) \to L^2(F)$ *on a line bundle* (E, J) *over a closed Riemann surface* (Σ, j)*, choose a complex-antilinear zeroth-order perturbation* $B \in \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(E, F))$ *whose zeroes are all nondegenerate. Then for sufficiently large* $\sigma > 0$, $\ker(\mathbf{D} + \sigma B)$ *is approximately spanned by* 1*-dimensional spaces of sections with support localized near the positive zeroes of* B*. In particular,* $\dim \ker(\mathbf{D} + \sigma B)$ *equals the number of positive zeroes of* B*.*

To deduce $\mathrm{ind}(\mathbf{D}) = I(\mathbf{D})$ from this, we need to apply the same trick to the formal adjoint $\mathbf{D}^*$. As we will review in §2, $-\mathbf{D}^*$ can be regarded under certain natural assumptions as a Cauchy-Riemann type operator on the bundle $\bar{F}$ conjugate to F, and the formal adjoint of $\mathbf{D} + \sigma B$ then gives rise to a Cauchy-Riemann type operator of the form

$$-\mathbf{D}^* + \sigma B' : \Gamma(\bar{F}) \to \Gamma(\bar{E}) = \Omega^{0,1}(\Sigma, \bar{F}),$$

where $B' : \bar{F} \to \bar{E}$ is also complex antilinear and has the same zeroes as B, but with opposite signs. Applying the above "theorem" to $-\mathbf{D}^*$ thus identifies $\ker(\mathbf{D} + \sigma B)^*$ for sufficiently large $\sigma > 0$ with a space whose dimension equals the number of *negative* zeroes of B. This gives

$$\begin{aligned} \mathrm{ind}(\mathbf{D}) = \mathrm{ind}(\mathbf{D} + \sigma B) &= \dim \ker(\mathbf{D} + \sigma B) - \dim \ker(\mathbf{D} + \sigma B)^* \\ &= c_1(\overline{\mathrm{Hom}}_{\mathbb{C}}(E, F)) = I(\mathbf{D}). \end{aligned}$$

It's worth mentioning that the "large perturbation" argument we've just sketched is only one simple example of an idea with a long and illustrious history: another simple example is the observation by Witten [**Wit82**] that after choosing a Morse function on a Riemannian manifold, certain large deformations of the de Rham complex lead to an approximation of the Morse complex, with generators of the de Rham complex having support concentrated near the critical points of the Morse function—this yields a somewhat novel proof of de Rham's theorem. A much deeper example is Taubes's isomorphism [**Tau96b**] between the Seiberg-Witten invariants of symplectic 4-manifolds and certain holomorphic curve invariants: here also, the idea is to consider a large compact perturbation of the Seiberg-Witten equations and show that, in the limit where the perturbation becomes infinitely large, solutions of the Seiberg-Witten equations localize near J-holomorphic curves. For a more recent exploration of this idea in the context of Dirac operators, see [**Mar**].

Before proceeding with the details, let us fix two simplifying assumptions that can be imposed without loss of generality:

ASSUMPTION 1. (E, J) has complex rank 1.

Indeed, an asymptotically Hermitian bundle E of complex rank $m \in \mathbb{N}$ always admits a decomposition into asymptotically Hermitian line bundles $E = E_1 \oplus \ldots \oplus E_m$, producing a corresponding splitting of the target bundle $F = F_1 \oplus \ldots \oplus F_m$. The operator $\mathbf{D}$ need not respect these splittings, but it is always *homotopic through Fredholm operators* to one that does: we saw in Theorem 3.33 of Lecture 3 that the asymptotic operators $\mathbf{A}_z$ are homotopic through nondegenerate asymptotic operators to any other operators $\mathbf{A}'_z$ that have the same Conley-Zehnder indices, so one can choose $\mathbf{A}'_z$ to respect the splitting. Any homotopy of Cauchy-Riemann operators following such a homotopy of nondegenerate asymptotic operators then produces a continuous family of Fredholm operators by the main result of Lecture 4, implying that their indices do not change. The general index formula then follows from the line bundle case since any two Cauchy-Riemann type Fredholm operators $\mathbf{D}_1$ and $\mathbf{D}_2$ over the same Riemann surface satisfy

$$\operatorname{ind}(\mathbf{D}_1 \oplus \mathbf{D}_2) = \operatorname{ind}(\mathbf{D}_1) + \operatorname{ind}(\mathbf{D}_2) \quad \text{and} \quad I(\mathbf{D}_1 \oplus \mathbf{D}_2) = I(\mathbf{D}_1) + I(\mathbf{D}_2).$$

ASSUMPTION 2. $k = 1$ and $p = 2$.

This means we will concretely be considering the operator

$$\mathbf{D} : H^1(E) \to L^2(F),$$

where H^1 as usual is an abbreviation for $W^{1,2}$. This assumption is clearly harmless since we know that $\operatorname{ind} \mathbf{D}$ does not depend on the choice of k and p.

2. Some remarks on the formal adjoint

For the beginning of this section we can drop the assumption that (E, J) is a line bundle and assume $\operatorname{rank}_{\mathbb{C}} E = m \in \mathbb{N}$, though later we will again set $m = 1$.

Recall from the end of Lecture 4 that if we fix global Hermitian structures $\langle\ ,\ \rangle_E$ and $\langle\ ,\ \rangle_F$ on (E, J) and (F, J) respectively and an area form $d\,\mathrm{vol}$ on $\dot{\Sigma}$ that matches $ds \wedge dt$ on the cylindrical ends, then $\mathbf{D}$ has a *formal adjoint*

$$\mathbf{D}^* : \Gamma(F) \to \Gamma(E)$$

satisfying

$$\langle \lambda, \mathbf{D}\eta \rangle_{L^2(F)} = \langle \mathbf{D}^*\lambda, \eta \rangle_{L^2(E)} \quad \text{for all} \quad \eta \in H^1(E),\ \lambda \in H^1(F).$$

Here the real-valued L^2 pairings are defined by

$$\langle \eta, \xi \rangle_{L^2(E)} := \operatorname{Re} \int_{\dot{\Sigma}} \langle \eta, \xi \rangle_E \, d\,\mathrm{vol} \quad \text{for} \quad \eta, \xi \in \Gamma(E),$$

and similarly for sections of F. The essential features of the formal adjoint are that $\ker \mathbf{D}^* \cong \operatorname{coker} \mathbf{D}$ and $\operatorname{coker} \mathbf{D}^* \cong \ker \mathbf{D}$, hence $\operatorname{ind}(\mathbf{D}^*) = -\operatorname{ind}(\mathbf{D})$. Recall moreover that $d\,\mathrm{vol}$ induces a natural Hermitian bundle metric on $\dot{\Sigma}$ by

$$\langle \cdot, \cdot \rangle_\Sigma = d\,\mathrm{vol}(\cdot, j\cdot) + i\, d\,\mathrm{vol}(\cdot, \cdot),$$

which determines a bundle isomorphism

$$T\dot{\Sigma} \to \Lambda^{0,1} T^*\dot{\Sigma} : X \mapsto X^{0,1} := \langle \cdot, X \rangle_\Sigma,$$

as well as a complex-*antilinear* isomorphism

$$T\dot{\Sigma} \to \Lambda^{1,0} T^*\dot{\Sigma} : X \mapsto X^{1,0} := \langle X, \cdot \rangle_\Sigma.$$

If $\langle\ ,\ \rangle_F$ is then chosen to be the tensor product metric determined via the natural isomorphism

$$F = \overline{\operatorname{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, E) = \Lambda^{0,1} T^*\dot{\Sigma} \otimes E = T\dot{\Sigma} \otimes E,$$

then E admits a natural isomorphism to $\Lambda^{1,0} T^*\dot{\Sigma} \otimes F$ such that

$$-\mathbf{D}^* : \Gamma(F) \to \Gamma(E) = \Omega^{1,0}(\dot{\Sigma}, F)$$

becomes an *anti-Cauchy-Riemann* type operator, i.e. it satisfies the Leibniz rule

$$-\mathbf{D}^*(f\lambda) = (\partial f)\lambda + f(-\mathbf{D}^*\lambda)$$

for all $f \in C^\infty(\dot{\Sigma}, \mathbb{R})$, with $\partial f := df - i\, df \circ j \in \Omega^{1,0}(\dot{\Sigma})$. Equivalently, $-\mathbf{D}^*$ defines a Cauchy-Riemann type operator on the **conjugate** bundle $\bar{F} \to \dot{\Sigma}$, defined as the real bundle $F \to \dot{\Sigma}$ but with the sign of its complex structure reversed; we shall distinguish this Cauchy-Riemann operator from $-\mathbf{D}^*$ by writing it as

$$-\overline{\mathbf{D}}^* : \Gamma(\bar{F}) \to \Omega^{0,1}(\dot{\Sigma}, \bar{F}),$$

though it is technically the same operator. Recall that the identity map defines a natural complex-antilinear isomorphism between any complex vector bundle and its conjugate bundle; we shall denote this isomorphism generally by

$$E \to \bar{E} : v \mapsto \bar{v},$$

so in particular it satisfies $\overline{cv} = \bar{c}\bar{v}$ for all scalars $c \in \mathbb{C}$, and similarly

$$\overline{\mathbf{D}}^*\bar{\lambda} = \overline{\mathbf{D}^*\lambda}$$

for $\lambda \in \Gamma(F)$. The asymptotic operators for $-\overline{\mathbf{D}}^*$ are

$$\overline{\mathbf{A}}_z = -\mathbf{A}_z : \Gamma(\bar{E}_z) \to \Gamma(\bar{E}_z).$$

LEMMA 5.6. *If τ is a choice of asymptotic trivialization on E and $\bar{\tau}$ denotes the* conjugate *asymptotic trivialization*[5]*, then*

$$c_1^{\bar{\tau}}(\bar{E}) = -c_1^{\tau}(E), \qquad \textit{and} \qquad \mu_{CZ}^{\bar{\tau}}(\overline{\mathbf{A}}_z) = -\mu_{CZ}^{\tau}(\mathbf{A}_z) \textit{ for all } z \in \Gamma.$$

PROOF. Assuming E is a line bundle, suppose η is a generic section of E that matches a nonzero constant with respect to τ on the cylindrical ends, so $c_1^\tau(E)$ is the signed count of zeroes of η. Then $\bar{\eta} \in \Gamma(\bar{E})$ is similarly a nonzero constant on the ends with respect to $\bar{\tau}$, but the signs of its zeroes are opposite those of η because they are defined as winding numbers with respect to *conjugate* local trivializations. This proves $c_1^{\bar{\tau}}(\bar{E}) = -c_1^{\tau}(E)$.

The Conley-Zehnder indices can be computed from the formula

$$\mu_{\mathrm{CZ}}^{\tau}(\mathbf{A}_z) = \alpha_+^\tau(\mathbf{A}_z) + \alpha_-^\tau(\mathbf{A}_z),$$

see Theorem 3.35 in Lecture 3. Here $\alpha_-^\tau(\mathbf{A}_z)$ is the largest possible winding number relative to τ of an eigenfunction for $\mathbf{A}_z$ with negative eigenvalue, and $\alpha_+^\tau(\mathbf{A}_z)$ is the smallest possible winding number with positive eigenvalue. The eigenfunctions of $\overline{\mathbf{A}}_z = -\mathbf{A}_z$ are the same, but the signs of their eigenvalues are reversed, and the signs of their winding numbers are also reversed because they must be measured relative to the conjugate trivialization, thus

$$\alpha_\pm^{\bar{\tau}}(\overline{\mathbf{A}}_z) = -\alpha_\mp^{\tau}(\mathbf{A}_z),$$

implying

$$\mu_{\mathrm{CZ}}^{\bar{\tau}}(\overline{\mathbf{A}}_z) = \alpha_+^{\bar{\tau}}(\overline{\mathbf{A}}_z) + \alpha_-^{\bar{\tau}}(\overline{\mathbf{A}}_z) = -\alpha_-^{\tau}(\mathbf{A}_z) - \alpha_+^{\tau}(\mathbf{A}_z) = -\mu_{\mathrm{CZ}}^{\tau}(\mathbf{A}_z).$$

[5] If $\tau : E|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{C}^m$ is a local trivialization of E with $\tau(v) = (z, w)$, the conjugate trivialization $\bar{\tau} : \bar{E}|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{C}^m$ is defined by $\bar{\tau}(\bar{v}) = (z, \bar{w})$.

The above calculations are all valid for line bundles, but the general case follows by taking direct sums. $\square$

We are now able to show that Theorem 5.4 is consistent with what we already know about the formal adjoint.

PROPOSITION 5.7. $I(-\overline{\mathbf{D}}^*) = -I(\mathbf{D})$.

PROOF. Under the isomorphism $F = \Lambda^{0,1}T^*\dot{\Sigma} \otimes E = T\dot{\Sigma} \otimes E$, an asymptotic trivialization τ on E induces an asymptotic trivialization $\partial_s \otimes \tau$ on F, where ∂_s denotes the asymptotic trivialization of $T\dot{\Sigma}$ defined via an outward pointing vector field on the cylindrical ends. Counting zeroes of vector fields then proves $c_1^{\partial_s}(T\dot{\Sigma}) = \chi(\dot{\Sigma})$, so

$$c_1^{\partial_s \otimes \tau}(F) = c_1^{\partial_s \otimes \tau}(T\dot{\Sigma} \otimes E) = m c_1^{\partial_s}(T\dot{\Sigma}) + c_1^\tau(E) = m\chi(\dot{\Sigma}) + c_1^\tau(E).$$

Applying Lemma 5.6 to the conjugate bundle then gives

$$c_1^{\overline{\partial_s \otimes \tau}}(\bar{F}) = -m\chi(\dot{\Sigma}) - c_1^\tau(E).$$

The unitary trivializations of the asymptotic bundles $\bar{E}_z$ corresponding to $\overline{\partial_s \otimes \tau}$ are simply $\bar{\tau}$, thus using Lemma 5.6 again for the Conley-Zehnder terms,

$$\begin{aligned} I(-\overline{\mathbf{D}}^*) &= m\chi(\dot{\Sigma}) + 2c_1^{\overline{\partial_s \otimes \tau}}(\bar{F}) + \sum_{z \in \Gamma^+} \mu_{\mathrm{CZ}}^{\bar{\tau}}(\overline{\mathbf{A}}_z) - \sum_{z \in \Gamma^-} \mu_{\mathrm{CZ}}^{\bar{\tau}}(\overline{\mathbf{A}}_z) \\ &= -m\chi(\dot{\Sigma}) - 2c_1^\tau(E) - \sum_{z \in \Gamma^+} \mu_{\mathrm{CZ}}^{\tau}(\mathbf{A}_z) + \sum_{z \in \Gamma^-} \mu_{\mathrm{CZ}}^{\tau}(\mathbf{A}_z) \\ &= -I(\mathbf{D}). \end{aligned}$$

$\square$

We next consider the effect of an antilinear zeroth-order perturbation on the formal adjoint. By "antilinear zeroth-order perturbation," we generally mean a smooth section

$$B \in \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(E, F)).$$

It is perhaps easier to understand B in terms of the conjugate bundle $\bar{E}$: indeed, there exists a unique

$$\beta \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(\bar{E}, F))$$

such that

$$B\eta = \beta\bar{\eta},$$

and this correspondence defines a bundle isomorphism $\overline{\mathrm{Hom}}_{\mathbb{C}}(E, F) = \mathrm{Hom}_{\mathbb{C}}(\bar{E}, F)$.

EXERCISE 5.8. Assume X and Y are complex vector bundles over the same base.

(a) Show that $\bar{X} \otimes \bar{Y}$ is canonically isomorphic to the conjugate bundle of $X \otimes Y$.
(b) Show that $\mathrm{Hom}_{\mathbb{C}}(\bar{X}, \bar{Y})$ is canonically isomorphic to the conjugate bundle of $\mathrm{Hom}_{\mathbb{C}}(X, Y)$, and $\overline{\mathrm{Hom}}_{\mathbb{C}}(\bar{X}, \bar{Y})$ is canonically isomorphic to the conjugate bundle of $\overline{\mathrm{Hom}}_{\mathbb{C}}(X, Y)$.
(c) Show that $\Lambda^{0,1}X := \overline{\mathrm{Hom}}_{\mathbb{C}}(X, \mathbb{C})$ is canonically isomorphic to the conjugate bundle of $\Lambda^{1,0}X := \mathrm{Hom}_{\mathbb{C}}(X, \mathbb{C})$.

Define the Cauchy-Riemann type operator

$$\mathbf{D}_B := \mathbf{D} + B : \Gamma(E) \to \Gamma(F) = \Omega^{0,1}(\dot{\Sigma}, E),$$

so $\mathbf{D}_B\eta = \mathbf{D}\eta + \beta\bar{\eta}$. To write down $\mathbf{D}_B^*$, observe that since $\beta : \bar{E} \to F$ is a complex-linear bundle map between Hermitian bundles, it has a complex-linear adjoint

$$\beta^\dagger : F \to \bar{E} \quad \text{such that} \quad \langle \beta^\dagger\lambda, \bar{\eta}\rangle_{\bar{E}} = \langle \lambda, \beta\bar{\eta}\rangle_F \text{ for } \lambda \in F,\ \bar{\eta} \in \bar{E}.$$

Here the bundle metric on $\bar{E}$ is defined by $\langle \bar{\eta}, \bar{\xi}\rangle_{\bar{E}} := \langle \xi, \eta\rangle_E$. We then have

$$\begin{aligned}\operatorname{Re}\langle \lambda, B\eta\rangle_F &= \operatorname{Re}\langle \lambda, \beta\bar{\eta}\rangle_F = \operatorname{Re}\langle \beta^\dagger\lambda, \bar{\eta}\rangle_{\bar{E}} = \operatorname{Re}\langle \eta, \overline{\beta^\dagger\lambda}\rangle_E = \operatorname{Re}\langle \overline{\beta^\dagger\lambda}, \eta\rangle_E \\ &= \operatorname{Re}\langle \overline{\beta^\dagger}\bar{\lambda}, \eta\rangle_E,\end{aligned}$$

where $\overline{\beta^\dagger} \in \Gamma(\operatorname{Hom}_{\mathbb{C}}(\bar{F}, E))$ denotes the image of $\beta^\dagger \in \Gamma(\operatorname{Hom}_{\mathbb{C}}(F, \bar{E}))$ under the complex-antilinear identity map from $\operatorname{Hom}_{\mathbb{C}}(F, \bar{E})$ to its conjugate bundle (see Exercise 5.8). The formal adjoint of $\mathbf{D}_B$ is thus

$$\mathbf{D}_B^* = \mathbf{D}^* + B^* : \Gamma(F) \to \Gamma(E),$$

where $B^* : F \to E$ is defined by

$$B^*\lambda := \overline{\beta^\dagger}\bar{\lambda}.$$

To write down the resulting Cauchy-Riemann type operator on $\bar{F}$, we replace $B^* : F \to E$ with $\overline{B}^* : \bar{F} \to \bar{E}$, defined by

$$\overline{B}^*\bar{\lambda} := \overline{B^*\lambda} = \beta^\dagger\lambda,$$

giving a Cauchy-Riemann operator

$$-\overline{\mathbf{D}}_B^* = -\overline{\mathbf{D}}^* + (-\overline{B}^*) : \Gamma(\bar{F}) \to \Gamma(\bar{E}) = \Omega^{0,1}(\dot{\Sigma}, \bar{F}).$$

The point of writing down this formula is to make the following observations:

LEMMA 5.9. *The zeroth-order perturbation $-\overline{B}^* : \bar{F} \to \bar{E}$ appearing in $-\overline{\mathbf{D}}_B^*$ has the following properties:*

(1) $-\overline{B}^* : \bar{F} \to \bar{E}$ is complex antilinear;
(2) There is a natural complex bundle isomorphism $\overline{\operatorname{Hom}}_{\mathbb{C}}(\bar{F}, \bar{E}) = \operatorname{Hom}_{\mathbb{C}}(F, \bar{E})$ that identifies $-\overline{B}^*$ with $-\beta^\dagger$;
(3) If $m = 1$ and $B \in \Gamma(\overline{\operatorname{Hom}}_{\mathbb{C}}(E, F))$ has only nondegenerate zeroes, then $-\overline{B}^* \in \Gamma(\overline{\operatorname{Hom}}_{\mathbb{C}}(\bar{F}, \bar{E}))$ has the same zeroes but with opposite signs.

PROOF. The first two statements follow immediately from the fact that $-\overline{B}^*$ is the composition of the canonical conjugation map $\bar{F} \to F$ with the complex-linear bundle map $-\beta^\dagger : F \to \bar{E}$. For the third, it suffices to compare what $\beta \in \Gamma(\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F))$ and $-\beta^\dagger : \Gamma(\operatorname{Hom}_{\mathbb{C}}(F, \bar{E}))$ look like in local trivializations near a zero: one is minus the complex conjugate of the other, hence their zeroes count with opposite signs. □

3. The index zero case on a torus

As a warmup for the general case, we now fill in the details of Taubes's proof of Theorem 5.4 in the case

$$\dot{\Sigma} = \mathbb{T}^2 := \mathbb{C} \setminus (\mathbb{Z} \oplus i\mathbb{Z})$$

and $E = \mathbb{T}^2 \times \mathbb{C}$, i.e. a trivial line bundle. In this case $I(\mathbf{D}) = \chi(\mathbb{T}^2) + 2c_1(E) = 0$, so our aim is to prove $\operatorname{ind}(\mathbf{D}) = 0$. What we will show in fact is that $\mathbf{D}$ is homotopic through a continuous family of Fredholm operators to one that is an isomorphism. Since E and F are now both trivial, it will suffice to consider the operator

$$\mathbf{D} := \bar{\partial} = \partial_s + i\partial_t : H^1(\mathbb{T}^2, \mathbb{C}) \to L^2(\mathbb{T}^2, \mathbb{C}),$$

whose formal adjoint is $\mathbf{D}^* := -\partial = -\partial_s + i\partial_t$. An antilinear zeroth-order perturbation is then equivalent to a choice of function $\beta : \mathbb{T}^2 \to \mathbb{C}$, giving rise to a family of operators

$$\mathbf{D}_\sigma \eta := \bar{\partial}\eta + \sigma\beta\bar{\eta}$$

for $\sigma \in \mathbb{R}$, where $\bar{\eta} : \mathbb{T}^2 \to \mathbb{C}$ now denotes the straightforward complex conjugate of η. Let us assume that $\beta : \mathbb{T}^2 \to \mathbb{C}$ is nowhere zero; note that this would not be possible in more general situations, but is possible here because $\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F)$ is a trivial bundle.

LEMMA 5.10. *$\mathbf{D}_\sigma$ is injective for all $\sigma > 0$ sufficiently large.*

PROOF. Elliptic regularity implies any $\eta \in \ker \mathbf{D}_\sigma$ is smooth, so we shall restrict our attention to smooth functions $\eta : \mathbb{T}^2 \to \mathbb{C}$. We start by comparing the two second-order differential operators

$$\mathbf{D}^*\mathbf{D} \text{ and } \mathbf{D}_\sigma^*\mathbf{D}_\sigma : C^\infty(\mathbb{T}^2, \mathbb{C}) \to C^\infty(\mathbb{T}^2, \mathbb{C}).$$

Both are nonnegative L^2-symmetric operators, and in fact the first is simply the Laplacian

$$\mathbf{D}^*\mathbf{D} = -\partial\bar{\partial} = (-\partial_s + i\partial_t)(\partial_s + i\partial_t) = -\partial_s^2 - \partial_t^2 = -\Delta.$$

The formal adjoint of $\mathbf{D}_\sigma$ takes the form

$$\mathbf{D}_\sigma^*\eta = \mathbf{D}^*\eta + \sigma B^*\eta = \mathbf{D}^*\eta + \sigma\beta\bar{\eta},$$

thus for any $\eta \in C^\infty(\mathbb{T}^2, \mathbb{C})$,

$$\begin{aligned} \mathbf{D}_\sigma^*\mathbf{D}_\sigma\eta &= (\mathbf{D}^* + \sigma B^*)(\mathbf{D} + \sigma B)\eta \\ &= \mathbf{D}^*\mathbf{D}\eta + \sigma\left(\beta\overline{\bar{\partial}\eta} - \partial(\beta\bar{\eta})\right) + \sigma^2 B^*B\eta \\ &= \mathbf{D}^*\mathbf{D}\eta + \sigma\left(\beta\partial\bar{\eta} - (\partial\beta)\bar{\eta} - \beta\partial\bar{\eta}\right) + \sigma^2 B^*B\eta \\ &= \mathbf{D}^*\mathbf{D}\eta + \sigma^2 B^*B\eta - \sigma(\partial\beta)\bar{\eta}. \end{aligned} \tag{5.2}$$

This is a *Weitzenböck formula*: its main message is that the Laplacian $\mathbf{D}^*\mathbf{D}$ and the related operator $\mathbf{D}_\sigma^*\mathbf{D}_\sigma$ differ from each other only by a zeroth-order term that will be positive definite if σ is sufficiently large. Indeed, since β is nowhere zero, we have $|B\eta| \geq c|\eta|$ for some constant $c > 0$, thus

$$\begin{aligned}\|\mathbf{D}_\sigma\eta\|_{L^2}^2 &= \langle \eta, \mathbf{D}_\sigma^*\mathbf{D}_\sigma\eta\rangle_{L^2} = \langle \eta, \mathbf{D}^*\mathbf{D}\eta\rangle_{L^2} + \sigma^2\langle \eta, B^*B\eta\rangle_{L^2} - \sigma\langle \eta, (\partial\beta)\bar{\eta}\rangle_{L^2} \\ &= \|\mathbf{D}\eta\|_{L^2}^2 + \sigma^2\|B\eta\|_{L^2}^2 - \sigma\langle \eta, (\partial\beta)\bar{\eta}\rangle_{L^2} \\ &\geq \left(\sigma^2 c^2 - \sigma\|\partial\beta\|_{C^0}\right)\|\eta\|_{L^2}^2.\end{aligned}$$

We conclude that as soon as $\sigma > 0$ is large enough to make the quantity in parentheses positive, $\mathbf{D}_\sigma\eta$ cannot vanish unless $\|\eta\|_{L^2} = 0$. □

PROOF OF THEOREM 5.4 FOR $E = \mathbb{T}^2 \times \mathbb{C}$. The lemma above shows that one can add a large antilinear perturbation to $\mathbf{D} = \bar{\partial}$ making the deformed operator $\mathbf{D}_\sigma$ injective. By Lemma 5.9, the same argument applies to the formal adjoint $\mathbf{D}^*$, implying that for sufficiently large $\sigma > 0$, $\mathbf{D}_\sigma^*$ is injective and thus $\mathbf{D}_\sigma$ is also surjective, and therefore an isomorphism. This proves $\operatorname{ind}(\mathbf{D}) = \operatorname{ind}(\mathbf{D}_\sigma) = 0$. □

Let's consider which particular details of the setup made the proof above possible.

First, the zeroth-order perturbation is complex antilinear. We used this, if only implicitly, in deriving the Weitzenböck formula (5.2): the key step is in the third line, where the two terms involving $\partial\bar{\eta}$ cancel each other out and leave nothing but zeroth-order terms remaining. This would not have happened if e.g. $B : E \to F$ had been complex linear—we would then have seen terms depending on the first derivative of η in $\mathbf{D}_\sigma^*\mathbf{D}_\sigma\eta - \mathbf{D}^*\mathbf{D}\eta$, and this would have killed the whole argument. The fact that this cancellation happens when the perturbation is antilinear probably looks like magic at this point, but there is a principle behind it; we will discuss it further in §4 below, see Remark 5.15.

The second crucial fact we used was that $\beta : \mathbb{T}^2 \to \mathbb{C}$ is nowhere zero, in order to obtain the lower bound on $\|B\eta\|_{L^2}$ in terms of $\|\eta\|_{L^2}$. This cannot always be achieved—it is possible in this special case only because E and F are both trivial bundles and thus so is $\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F)$. On more general bundles, the best we could hope for would be to pick $\beta \in \Gamma(\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F))$ with finitely many zeroes, all nondegenerate. In this case the above argument fails, but it still tells us something. Suppose $\Sigma_\epsilon \subset \mathbb{T}^2$ is a region disjoint from the isolated zeroes of β. Then there exists a constant $c_\epsilon > 0$, dependent on the region Σ_ϵ, such that

$$\|\beta\bar{\eta}\|_{L^2(\mathbb{T}^2)}^2 \geq \|\beta\bar{\eta}\|_{L^2(\Sigma_\epsilon)}^2 \geq c_\epsilon\|\eta\|_{L^2(\Sigma_\epsilon)}^2,$$

so instead of the estimate at the end of the proof above implying $\mathbf{D}_\sigma$ is injective, we obtain one of the form

$$\|\mathbf{D}_\sigma\eta\|_{L^2(\mathbb{T}^2)}^2 \geq c_\epsilon\sigma^2\|\eta\|_{L^2(\Sigma_\epsilon)} - c\sigma\|\eta\|_{L^2(\mathbb{T}^2)}^2.$$

To see what this means, imagine we have sequences $\sigma_\nu \to \infty$ and $\eta_\nu \in \ker \mathbf{D}_{\sigma_\nu}$, normalized so that $\|\eta_\nu\|_{L^2} = 1$ for all ν. The estimate above then implies

$$\|\eta_\nu\|_{L^2(\Sigma_\epsilon)}^2 \leq \frac{c}{c_\epsilon\sigma_\nu} \to 0 \quad \text{as} \quad \nu \to \infty,$$

so while all sections η_ν have the same amount of "energy" (as measured via their L^2-norms), the energy is escaping from Σ_ϵ as σ_ν increases. This is true for *any* domain Σ_ϵ disjoint from the zeroes, so we conclude that in the limit as $\sigma \to \infty$, sections in ker $\mathbf{D}_\sigma$ have their energy concentrated in infinitesimally small neighborhoods of the zeroes of β. We will see in the following how to extract useful information from this concentration of energy.

4. A Weitzenböck formula for Cauchy-Riemann operators

The Weitzenböck formula (5.2) can be generalized to a useful relation between any two Cauchy-Riemann type operators that differ by an *antilinear* zeroth-order term. To see this, we start with a short digression on holomorphic and antiholomorphic vector bundles.

A smooth function $f : \mathbb{C} \supset \mathcal{U} \to \mathbb{C}$ is called **antiholomorphic** if it satisfies $(\partial_s - i\partial_t)f = 0$, which means its differential anticommutes with the complex structure on $\mathbb{C}$. The class of antiholomorphic functions is not closed under composition, but it is closed under products, hence one can define an **antiholomorphic structure** on a complex vector bundle to be a system of local trivializations for which all transition maps are antiholomorphic. Given the standard correspondence between holomorphic structures and Cauchy-Riemann type operators, it is easy to establish a similar correspondence between aniholomorphic structures and (complex-linear) **anti-Cauchy-Riemann type** operators, i.e. those which satisfy

$$\mathbf{D}(f\eta) = (\partial f)\eta + f\mathbf{D}\eta$$

for all $f \in C^\infty(\dot{\Sigma}, \mathbb{C})$, where $\partial f := df - i\, df \circ j \in \Omega^{1,0}(\dot{\Sigma})$. We've seen one important example of such an operator already: if $\mathbf{D} : \Gamma(E) \to \Gamma(F)$ is complex linear, then $-\mathbf{D}^*$ is a complex-linear anti-Cauchy-Riemann operator on F and thus endows F with an antiholomorphic structure. Another natural example occurs naturally on conjugate bundles: if E has a holomorphic structure, then $\bar{E}$ inherits from this an antiholomorphic structure. This is immediate from the fact that $f : \mathbb{C} \supset \mathcal{U} \to \mathbb{C}$ is holomorphic if and only if $\bar{f} : \mathcal{U} \to \mathbb{C}$ is antiholomorphic. If $\mathbf{D} : \Gamma(E) \to \Gamma(F) = \Omega^{0,1}(\dot{\Sigma}, E)$ is the corresponding complex-linear Cauchy-Riemann type operator on E, we shall denote the resulting anti-Cauchy-Riemann operator by

$$\overline{\mathbf{D}} : \Gamma(\bar{E}) \to \Gamma(\bar{F}) = \Omega^{1,0}(\dot{\Sigma}, \bar{E}),$$

where by definition $\overline{\mathbf{D}}\bar{\eta} = \overline{\mathbf{D}\eta}$.

EXERCISE 5.11. Show that if X and Y are antiholomorphic vector bundles over the same base, then $X \otimes Y$ and $\mathrm{Hom}_{\mathbb{C}}(X, Y)$ both naturally inherit antiholomorphic bundle structures such that the obvious Leibniz rules are satisfied. *Remark: the proof of this is exactly the same as for holomorphic bundles, one only needs to change some signs.*

EXERCISE 5.12. Suppose X and Y are complex vector bundles over the same base, carrying real-linear anti-Cauchy-Riemann operators ∂_X and ∂_Y respectively. Show that $H := \mathrm{Hom}_{\mathbb{R}}(X, Y)$ then admits a real-linear anti-Cauchy-Riemann operator ∂_H such that for all $\Phi \in \Gamma(H)$ and $\eta \in \Gamma(X)$,

$$\partial_Y(\Phi\eta) = (\partial_H\Phi)\eta + \Phi(\partial_X\eta).$$

Hint: write ∂_X and ∂_Y as complex-linear operators with real-linear zeroth-order perturbations, and apply Exercise 5.11. Show moreover that any C^k-bounds satisfied by the zeroth-order terms in ∂_X and ∂_Y are inherited by the zeroth-order term in ∂_H.

The setup for the next result is as follows. We assume again $m = 1$, so E and F are line bundles. Fix $\beta \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(\bar{E}, F))$, define $B \in \Gamma(\overline{\mathrm{Hom}_{\mathbb{C}}}(E, F))$ by $B\eta := \beta\bar{\eta}$, and use this to define the perturbed Cauchy-Riemann type operator

$$\mathbf{D}_B := \mathbf{D} + B : \Gamma(E) \to \Gamma(F),$$

whose formal adjoint is $\mathbf{D}_B^* = \mathbf{D}^* + B^*$ with $B^*\lambda := \overline{\beta^\dagger}\bar{\lambda}$.

PROPOSITION 5.13. *The second-order differential operators $\mathbf{D}^*\mathbf{D}$ and $\mathbf{D}_B^*\mathbf{D}_B$ on E are related by*

$$\mathbf{D}_B^*\mathbf{D}_B\eta = \mathbf{D}^*\mathbf{D}^*\eta + B^*B\eta - (\partial_H\beta)\bar{\eta},$$

where ∂_H is a real-linear anti-Cauchy-Riemann type operator on $\mathrm{Hom}_{\mathbb{C}}(\bar{E}, F)$. Moreover, if β is C^1-bounded on $\dot{\Sigma}$, then $\partial_H\beta$ is C^0-bounded.

PROOF. We have real-linear anti-Cauchy-Riemann operators $\overline{\mathbf{D}}$ and $-\mathbf{D}^*$ on $\bar{E}$ and F respectively, so Exercise 5.12 produces an operator ∂_H on $\mathrm{Hom}_{\mathbb{C}}(\bar{E}, F)$ for which the Leibniz rule is satisfied. We can then write

$$\begin{aligned}
\mathbf{D}_B^*\mathbf{D}_B\eta &= (\mathbf{D}^* + B^*)(\mathbf{D} + B)\eta \\
&= \mathbf{D}^*\mathbf{D}\eta + \overline{\beta^\dagger}\overline{\mathbf{D}\eta} - (-\mathbf{D}^*)(\beta\bar{\eta}) + B^*B\eta \\
&= \mathbf{D}^*\mathbf{D}\eta + \overline{\beta^\dagger}\overline{\mathbf{D}}\bar{\eta} - (\partial_H\beta)\bar{\eta} - \beta\overline{\mathbf{D}}\bar{\eta} + B^*B\eta \\
&= \mathbf{D}^*\mathbf{D}\eta + B^*B\eta - (\partial_H\beta)\bar{\eta} + \left(\overline{\beta^\dagger} - \beta\right)\overline{\mathbf{D}}\bar{\eta}.
\end{aligned}$$

Here β and $\overline{\beta^\dagger}$ are both viewed as complex-linear bundle maps $\bar{F} \to E$, the latter in the obvious way, and the former acting as $\mathbb{1} \otimes \beta$ on $\bar{F} = \Lambda^{1,0}T^*\dot{\Sigma} \otimes \bar{E}$ with target $\Lambda^{1,0}T^*\dot{\Sigma} \otimes F = \Lambda^{1,0}T^*\dot{\Sigma} \otimes \Lambda^{0,1}T^*\dot{\Sigma} \otimes E = E$. Choosing unitary local trivializations, β and $\overline{\beta^\dagger}$ are represented by the same complex-valued function: indeed, the latter is the transpose of the former as m-by-m complex matrices, but since $m = 1$, this means they are identical.

Finally, we observe that the asymptotic convergence conditions satisfied by $\mathbf{D}$ on the cylindrical ends imply similar conditions for all other Cauchy-Riemann and anti-Cauchy-Riemann operators in this picture, yielding an estimate of the form $\|\partial_H\beta\|_{C^0} \le c\|\beta\|_{C^1}$ globally on $\dot{\Sigma}$. □

REMARK 5.14. The above proof used the assumption $m = 1$ in order to conclude $\overline{\beta^\dagger} - \beta \equiv 0$. For higher rank bundles, this imposes a nontrivial condition that must be satisfied in order for the Weitzenböck formula to hold, cf. [GW].

REMARK 5.15. We can now pick out a geometric reason for the miraculous cancellation in the Weitzenböck formula: the perturbation B is described by a complex bundle map $\bar{E} \to F$, where $\bar{E}$ and F both have natural antiholomorphic bundle structures defined via the complex-linear parts of $\overline{\mathbf{D}}$ and $-\mathbf{D}^*$ respectively. A complex-linear perturbation $B : E \to F$ would not work because E is holomorphic rather than antiholomorphic: while $\overline{\mathbf{D}}$ can be fit into the same Leibniz rule with $-\mathbf{D}^*$, the same is not true of $\mathbf{D}$.

5. Large antilinear perturbations and energy concentration

We continue in the setting of Proposition 5.13 and set

$$\mathbf{D}_\sigma := \mathbf{D} + \sigma B : \Gamma(E) \to \Gamma(F)$$

for $\sigma > 0$. After a compact perturbation of $\mathbf{D}$, we can without loss of generality also impose the following assumptions on $\mathbf{D}$, $\beta \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(\bar{E}, F))$ and the area form $d\,\mathrm{vol}$:

(i) All zeroes of β are nondegenerate.
(ii) Both $|\beta|$ and $1/|\beta|$ are bounded outside of a compact subset of $\dot{\Sigma}$.
(iii) Near each point $\zeta \in \dot{\Sigma}$ with $\beta(\zeta) = 0$, there exists a neighborhood $\mathcal{D}(\zeta) \subset \dot{\Sigma}$ of ζ, a holomorphic coordinate chart identifying $(\mathcal{D}(\zeta), j, \zeta)$ with the unit disk $(\mathbb{D}, i, 0)$, and a local trivialization of E over $\mathcal{D}(\zeta)$ that identifies $\mathbf{D}$ with $\bar{\partial} = \partial_s + i\partial_t : C^\infty(\mathbb{D}, \mathbb{C}) \to C^\infty(\mathbb{D}, \mathbb{C})$ and β with one of the functions

$$\beta(z) = z \qquad \text{or} \qquad \beta(z) = \bar{z},$$

the former if ζ is a positive zero and the latter if it is negative.
(iv) In the holomorphic coordinate on $\mathcal{D}(\zeta)$ described above, $d\,\mathrm{vol}$ is the standard Lebesgue measure.

As in the torus case discussed in §3, we will see that the Weitzenböck formula implies a concentration of energy near the zeroes of β for sections $\eta \in \ker \mathbf{D}_\sigma$ as $\sigma \to \infty$. To understand what really happens in this limit, we will use a rescaling trick. Denote the zero set of β by

$$Z(\beta) = Z^+(\beta) \cup Z^-(\beta) \subset \dot{\Sigma},$$

partitioned into the positive and negative zeroes. For any $\eta \in \Gamma(E)$, $\zeta \in Z^\pm(\beta)$ and $\sigma > 0$, we then define a rescaled function

$$\eta^{(\zeta,\sigma)} : \mathbb{D}_{\sqrt{\sigma}} \to \mathbb{C} : z \mapsto \frac{1}{\sqrt{\sigma}} \eta(z/\sqrt{\sigma}),$$

where the right hand side denotes the local representation of η on $\mathcal{D}(\zeta)$ in the chosen coordinate and trivialization. Notice that the equation $\mathbf{D}_\sigma \eta = 0$ appears in this local representation as either $\bar{\partial}\eta + \sigma z \bar{\eta} = 0$ or $\bar{\partial}\eta + \sigma \bar{z} \bar{\eta} = 0$ depending on the sign of ζ, and the function $f := \eta^{(\zeta,\sigma)}$ then satisfies

$$\bar{\partial} f + z \bar{f} = 0 \qquad \text{or} \qquad \bar{\partial} f + \bar{z} \bar{f} = 0 \qquad \text{on } \mathbb{D}_{\sqrt{\sigma}}.$$

We will take a closer look at these two PDEs in §6 below. But first, observe that by change of variables,

$$\left\| \eta^{(\zeta,\sigma)} \right\|_{L^2(\mathbb{D}_{\sqrt{\sigma}})} = \|\eta\|_{L^2(\mathcal{D}(\zeta))}.$$

LEMMA 5.16. *Assume $\sigma_\nu \to \infty$, and $\eta_\nu \in \ker \mathbf{D}_{\sigma_\nu}$ is a sequence satisfying a uniform L^2-bound. Then after passing to a subsequence, the rescaled functions $\eta_\nu^\zeta := \eta_\nu^{(\zeta,\sigma_\nu)} : \mathbb{D}_{\sqrt{\sigma_\nu}} \to \mathbb{C}$ for each $\zeta \in Z^\pm(\beta)$ converge in $C^\infty_{\mathrm{loc}}(\mathbb{C})$ to smooth functions*

$\eta_\infty^\zeta \in L^2(\mathbb{C})$ *satisfying*

$$\begin{aligned} \bar\partial \eta_\infty^\zeta + z\overline{\eta_\infty^\zeta} &= 0 \qquad \text{if } \zeta \in Z^+(\beta), \\ \bar\partial \eta_\infty^\zeta + \bar z\overline{\eta_\infty^\zeta} &= 0 \qquad \text{if } \zeta \in Z^-(\beta). \end{aligned}$$

Moreover, if $\xi_\nu \in \ker \mathbf{D}_{\sigma_\nu}$ *is another sequence with these same properties and convergence* $\xi_\nu^\zeta \to \xi_\infty^\zeta$*, then*

$$\lim_{\nu\to\infty} \langle \eta_\nu, \xi_\nu \rangle_{L^2(E)} = \sum_{\zeta\in Z(\beta)} \langle \eta_\infty^\zeta, \xi_\infty^\zeta \rangle_{L^2(\mathbb{C})}.$$

Proof. The uniform L^2-bound implies uniform bounds on $\|\eta_\nu^\zeta\|_{L^2(\mathbb{D}_R)}$ for every $R > 0$, where ν here is assumed sufficiently large so that $R < \sqrt{\sigma_\nu}$. Since η_ν^ζ satisfies a Cauchy-Riemann type equation on $\mathbb{D}_R$, the usual elliptic estimates (see Lecture 2) then imply uniform H^k-bounds for every $k \in \mathbb{N}$ on every compact subset in the interior of $\mathbb{D}_R$, hence η_ν^ζ has a C^∞_{loc}-convergent subsequence on $\mathbb{C}$, and the limit η_∞^ζ clearly satisfies the stated PDE. The uniform L^2-bound also implies a uniform bound on $\|\eta_\nu^\zeta\|_{L^2(\mathbb{D}_{\sqrt{\sigma_\nu}})}$ and thus an R-independent uniform bound on $\|\eta_\nu^\zeta\|_{L^2(\mathbb{D}_R)}$ as $\nu \to \infty$, implying that η_∞^ζ is in $L^2(\mathbb{C})$.

The limit of $\langle \eta_\nu, \xi_\nu \rangle_{L^2(E)}$ is now proved using the Weitzenböck formula. Let

$$\dot\Sigma_\epsilon := \dot\Sigma \setminus \bigcup_{\zeta\in Z(\beta)} \mathcal{D}(\zeta),$$

so there exists a constant $c > 0$ such that β satisfies $|\beta(z)\bar v| \ge c|v|$ for all $v \in E_z$, $z \in \dot\Sigma_\epsilon$. (Note that this depends on the assumption of $1/|\beta|$ being bounded outside of a compact subset.) Now by Proposition 5.13,

$$\begin{aligned} 0 = \|\mathbf{D}_{\sigma_\nu}\eta_\nu\|^2_{L^2(\dot\Sigma)} &= \langle \eta_\nu, \mathbf{D}^*_{\sigma_\nu}\mathbf{D}_{\sigma_\nu}\eta_\nu\rangle_{L^2(\dot\Sigma)} \\ &= \langle \eta_\nu, \mathbf{D}^*\mathbf{D}\eta_\nu\rangle_{L^2(\dot\Sigma)} + \sigma_\nu^2 \langle \eta_\nu, B^*B\eta_\nu\rangle_{L^2(\dot\Sigma)} - \sigma_\nu \langle \eta_\nu, (\partial_H\beta)\bar\eta_\nu\rangle_{L^2(\dot\Sigma)} \\ &\ge \|\mathbf{D}\eta_\nu\|^2_{L^2(\dot\Sigma)} + \sigma_\nu^2 c^2 \|\eta_\nu\|^2_{L^2(\dot\Sigma_\epsilon)} - \sigma_\nu c' \|\eta_\nu\|^2_{L^2(\dot\Sigma)} \\ &\ge \sigma_\nu^2 c^2 \|\eta_\nu\|^2_{L^2(\dot\Sigma_\epsilon)} - \sigma_\nu c' \|\eta_\nu\|^2_{L^2(\dot\Sigma)} \end{aligned}$$

for some constant $c' > 0$ independent of ν. This implies

$$\|\eta_\nu\|^2_{L^2(\dot\Sigma_\epsilon)} \le \frac{c'}{c^2\sigma_\nu}\|\eta_\nu\|^2_{L^2(\dot\Sigma)} \to 0 \qquad \text{as} \qquad \nu \to \infty$$

since $\|\eta_\nu\|_{L^2(\dot\Sigma)}$ is uniformly bounded. The same estimate applies to ξ_ν, so that $\langle \eta_\nu, \xi_\nu\rangle_{L^2(\dot\Sigma_\epsilon)} \to 0$ and thus by change of variables,

$$\begin{aligned} \lim_{\nu\to\infty} \langle \eta_\nu, \xi_\nu\rangle_{L^2(\dot\Sigma)} &= \lim_{\nu\to\infty} \sum_{\zeta\in Z(\beta)} \langle \eta_\nu, \xi_\nu\rangle_{L^2(\mathcal{D}(\zeta))} = \lim_{\nu\to\infty} \sum_{\zeta\in Z(\beta)} \langle \eta_\nu^\zeta, \xi_\nu^\zeta\rangle_{L^2(\mathbb{D}_{\sqrt{\sigma_\nu}})} \\ &= \sum_{\zeta\in Z(\beta)} \langle \eta_\infty^\zeta, \xi_\infty^\zeta\rangle_{L^2(\mathbb{C})}. \end{aligned}$$

□

6. Two Cauchy-Riemann type problems on the plane

The rescaling trick in the previous section produced smooth solutions $f : \mathbb{C} \to \mathbb{C}$ of class $L^2(\mathbb{C})$ to the two equations

$$\bar{\partial} f + z\bar{f} = 0, \qquad \bar{\partial} f + \bar{z}\bar{f} = 0.$$

It turns out that we can say precisely what all such solutions are. Write $\mathbf{D}_+ f := \bar{\partial} f + z\bar{f}$ and $\mathbf{D}_- f := \bar{\partial} f + \bar{z}\bar{f}$. Both operators differ from $\bar{\partial}$ by antilinear perturbations, so they satisfy Weitzenböck formulas relating $\mathbf{D}_\pm^* \mathbf{D}_\pm$ to the Laplacian $-\Delta = \bar{\partial}^* \bar{\partial} = -\partial_s^2 - \partial_t^2$. Indeed, repeating Proposition 5.13 in these special cases gives

$$\mathbf{D}_+^* \mathbf{D}_+ f = -\Delta f + |z|^2 f - 2\bar{f} \qquad \text{and} \qquad \mathbf{D}_-^* \mathbf{D}_- f = -\Delta f + |z|^2 f.$$

To make use of this, recall that a smooth function $u : \mathcal{U} \to \mathbb{R}$ on an open subset $\mathcal{U} \subset \mathbb{C}$ is called **subharmonic** if it satisfies

$$-\Delta u \le 0.$$

Subharmonic functions satisfy a **mean value property**:

$$-\Delta u \le 0 \text{ on } \mathcal{U} \qquad \Rightarrow \qquad u(z_0) \le \frac{1}{\pi r^2} \int_{\mathbb{D}_r(z_0)} u(z)\, d\mu(z) \quad \text{for all} \quad \mathbb{D}_r(z_0) \subset \mathcal{U},$$

where $\mathbb{D}_r(z_0) \subset \mathbb{C}$ denotes the disk of radius $r > 0$ about a point $z_0 \in \mathcal{U}$, and $d\mu(z)$ is the Lebesgue measure on $\mathbb{C}$; see e.g. [**Eva98**, p. 85].

EXERCISE 5.17. Show that for any smooth complex-valued function f on an open subset of $\mathbb{C}$,

$$\Delta |f|^2 = 2\,\mathrm{Re}\langle f, \Delta f\rangle + 2|\nabla f|^2,$$

where $\langle\ ,\ \rangle$ denotes the standard Hermitian inner product on $\mathbb{C}$ and $|\nabla f|^2 := |\partial_s f|^2 + |\partial_t f|^2$.

PROPOSITION 5.18. *The equation $\bar{\partial} f + \bar{z}\bar{f} = 0$ does not admit any nontrivial smooth solutions $f \in L^2(\mathbb{C}, \mathbb{C})$.*

PROOF. If $f : \mathbb{C} \to \mathbb{C}$ is smooth with $\mathbf{D}_- f = 0$, then the Weitzenböck formula for $\mathbf{D}_-$ implies $\Delta f = |z|^2 f$. Then by Exercise 5.17,

$$\Delta |f|^2 = 2\,\mathrm{Re}\langle f, |z|^2 f\rangle + 2|\nabla f|^2 = 2|z|^2 |f|^2 + 2|\nabla f|^2,$$

implying that $|f|^2 : \mathbb{C} \to \mathbb{R}$ is subharmonic. Now if $f(z_0) \ne 0$ for some $z_0 \in \mathbb{C}$, the mean value property implies

$$\int_{\mathbb{D}_r(z_0)} |f(z)|^2\, d\mu(z) \ge \pi r^2 |f(z_0)|^2 \to \infty \quad \text{as} \quad r \to \infty,$$

so $f \notin L^2(\mathbb{C})$. □

PROPOSITION 5.19. *Every smooth solution $f \in L^2(\mathbb{C}, \mathbb{C})$ to the equation $\bar{\partial} f + z\bar{f} = 0$ is a constant real multiple of $f_0(z) := e^{-\frac{1}{2}|z|^2}$.*

Proof. We claim first that every smooth solution in $L^2(\mathbb{C},\mathbb{C})$ of $\mathbf{D}_+ f = 0$ is purely real valued. The Weitzenböck formula for this case gives $\Delta f = |z|^2 f - 2\bar{f}$, and taking the difference between this equation and its complex conjugate then implies that $u := \operatorname{Im} f : \mathbb{C} \to \mathbb{R}$ satisfies

$$\Delta u = (|z|^2 + 2)u.$$

Now by Exercise 5.17,

$$\Delta(u^2) = 2|\nabla u|^2 + 2(|z|^2 + 2)u^2 \ge 0,$$

so $u^2 : \mathbb{C} \to \mathbb{R}$ is subharmonic, and the mean value property implies as in the proof of Prop. 5.18 that $u \notin L^2(\mathbb{C})$ and hence $f \notin L^2(\mathbb{C})$ unless $u \equiv 0$. This proves the claim.

It is easy to check however that f_0 is a solution and is in $L^2(\mathbb{C})$. Since it is also nowhere zero, every other solution f must then take the form $f(z) = v(z) f_0(z)$ for some *real-valued* function $v : \mathbb{C} \to \mathbb{R}$. Since $\mathbf{D}_+$ is a Cauchy-Riemann type operator, the Leibniz rule then implies $\bar{\partial} v \equiv 0$. But the only globally holomorphic functions with trivial imaginary parts are constant. □

7. A linear gluing argument

Now we're getting somewhere.

Lemma 5.20. *Suppose the assumptions of §5 hold and $\beta \in \Gamma(\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F))$ has $I_+ \ge 0$ positive and $I_- \ge 0$ negative zeroes. Then for all $\sigma > 0$ sufficiently large,*

$$\dim \ker \mathbf{D}_\sigma \le I_+ \quad \text{and} \quad \dim \operatorname{coker} \mathbf{D}_\sigma \le I_-.$$

In particular, for sufficiently large σ, $\mathbf{D}_\sigma$ is injective if all zeroes of β are negative and surjective if all zeroes are positive.

Proof. Arguing by contradiction, suppose there exists a sequence $\sigma_\nu \to \infty$ such that $\dim \ker \mathbf{D}_{\sigma_\nu} > I_+$, and pick $(I_+ + 1)$ sequences of sections $\eta^1_\nu, \ldots, \eta^{I_+ + 1}_\nu \in \ker \mathbf{D}_{\sigma_\nu}$ which form L^2-orthonormal sets for each ν. By Lemma 5.16, we can then extract a subsequence such that rescaling near the zeroes of β produces C^∞_{loc}-convergent sequences whose limits form an $(I_+ + 1)$-dimensional orthonormal set in

$$\bigoplus_{\zeta \in Z(\beta)} L^2(\mathbb{C},\mathbb{C}),$$

where the component functions $f \in L^2(\mathbb{C},\mathbb{C})$ for $\zeta \in Z^+(\zeta)$ satisfy $\bar{\partial} f + z\bar{f} = 0$, while those for $\zeta \in Z^-(\zeta)$ satisfy $\bar{\partial} f + \bar{z}\bar{f} = 0$. Proposition 5.18 now implies that the component functions for $\zeta \in Z^-(\zeta)$ are all trivial, and by Proposition 5.19, the components for $\zeta \in Z^+(\zeta)$ belong to 1-dimensional subspaces $\ker \mathbf{D}_+ \subset L^2(\mathbb{C})$ generated by the function $e^{-\frac{1}{2}|z|^2}$. We conclude that the limiting orthonormal set lives in a precisely I_+-dimensional subspace

$$\bigoplus_{\zeta \in Z^+(\beta)} \ker \mathbf{D}_+ \subset \bigoplus_{\zeta \in Z(\beta)} L^2(\mathbb{C},\mathbb{C}),$$

and this is a contradiction since there are $I_+ + 1$ elements in the set.

Applying the same argument to the formal adjoint implies similarly $\dim \ker \mathbf{D}_\sigma^* \leq I_-$ for σ sufficiently large. □

We would next like to turn the two inequalities in the above lemma into equalities, which means showing that the I_+-dimensional subspace of $\bigoplus_{\zeta \in Z^+(\beta)} L^2(\mathbb{C}, \mathbb{C})$ generated by solutions of $\bar{\partial} f + z\bar{f} = 0$ is isomorphic to $\ker \mathbf{D}_\sigma$ for σ sufficiently large. This requires a simple example of a *linear gluing* argument, the point of which is to reverse the "convergence after rescaling" process that we saw in Lemma 5.16. The first step is a **pregluing** construction which turns elements of $\bigoplus_{\zeta \in Z^+(\beta)} \ker \mathbf{D}_+$ into *approximate* solutions to $\mathbf{D}_\sigma \eta = 0$ for large σ. To this end, fix a smooth bump function

$$\rho \in C_0^\infty(\mathring{\mathbb{D}}, [0,1]), \qquad \rho|_{\mathbb{D}_{1/2}} \equiv 1$$

and define for each $\zeta \in Z^+(\beta)$ and $\sigma > 0$ a linear map

$$\Phi_\sigma^\zeta : \ker \mathbf{D}_+ \to \Gamma(E)$$

such that $\Phi_\sigma^\zeta(f)$ is a section with support in $\mathcal{D}(\zeta)$ whose expression in our fixed coordinate and trivialization on that neighborhood is the function

$$f_\sigma^\zeta(z) = \rho(z)\sqrt{\sigma} f(\sqrt{\sigma} z).$$

Adding up the Φ_σ^ζ for all $\zeta \in Z^+(\beta)$ then produces a linear map

$$\Phi_\sigma : \bigoplus_{\zeta \in Z^+(\beta)} \ker \mathbf{D}_+ \to \Gamma(E)$$

whose image consists of sections supported near $Z^+(\beta)$, each a linear combination of cut-off Gaussians with energy concentrated in smaller neighborhoods of $Z^+(\beta)$ for larger σ. These sections are manifestly not in $\ker \mathbf{D}_\sigma$ since they vanish on open subsets and thus violate unique continuation, but they are close, in a quantitative sense:

LEMMA 5.21. *For each $\sigma > 0$, there exists a constant $c_\sigma > 0$ such that*

$$\|\mathbf{D}_\sigma \Phi_\sigma(f)\|_{L^2} \leq c_\sigma \|f\|_{L^2} \quad \text{for all} \quad f \in \bigoplus_{\zeta \in Z^+(\beta)} \ker \mathbf{D}_+,$$

and $c_\sigma \to 0$ as $\sigma \to \infty$. Moreover, for every pair $f, g \in \bigoplus_{\zeta \in Z^+(\beta)} \ker \mathbf{D}_+$,

$$\langle \Phi_\sigma(f), \Phi_\sigma(g) \rangle_{L^2} \to \langle f, g \rangle_{L^2}$$

as $\sigma \to \infty$.

PROOF. First, observe that any $f \in \bigoplus_{\zeta \in Z^+(\beta)} \ker \mathbf{D}_+$ is described by a collection of functions $\{f_\zeta \in L^2(\mathbb{C})\}_{\zeta \in \beta^+(Z)}$ which take the form

$$f_\zeta(z) = K_\zeta e^{-\frac{1}{2}|z|^2},$$

for some constants $K_\zeta \in \mathbb{R}$. Since each f_ζ is in $\ker \mathbf{D}_+$, we find

$$
\begin{aligned}
\mathbf{D}_\sigma \left(\Phi_\sigma(f)|_{\mathcal{D}(\zeta)}\right)(z) &= \overline{\partial}\rho(z)\sqrt{\sigma} f_\zeta(\sqrt{\sigma}z) + \rho(z)\sigma\overline{\partial} f_\zeta(\sqrt{\sigma}z) \\
&\quad + \sigma z \rho(z)\sqrt{\sigma} f_\zeta(\sqrt{\sigma}z) \\
&= \overline{\partial}\rho(z)\sqrt{\sigma} f_\zeta(\sqrt{\sigma}z) + \rho(z)\sigma(\mathbf{D}_+ f_\zeta)(\sqrt{\sigma}z) \\
&= \overline{\partial}\rho(z)\sqrt{\sigma} K_\zeta e^{-\frac{1}{2}\sigma|z|^2}.
\end{aligned}
\tag{5.3}
$$

Now since $\overline{\partial}\rho = 0$ in $\mathbb{D}_{1/2}$, we obtain

$$
\begin{aligned}
\|\mathbf{D}_\sigma \Phi_\sigma(f)\|_{L^2}^2 &= \sum_{\zeta \in Z^+(\beta)} \int_{\mathcal{D}(\zeta)} |\mathbf{D}_\sigma \Phi_\sigma(f)(z)|^2 \, d\mu(z) \\
&= \sum_{\zeta \in Z^+(\beta)} \int_{\mathbb{D}\setminus\mathbb{D}_{1/2}} |\overline{\partial}\rho(z)|^2 \sigma K_\zeta^2 e^{-\sigma|z|^2} \, d\mu(z) \\
&\le I\sigma e^{-\sigma/4} \sum_{\zeta \in Z^+(\beta)} K_\zeta^2,
\end{aligned}
$$

where we abbreviate $I := \int_{\mathbb{D}\setminus\mathbb{D}_{1/2}} \left|\bar{\partial}\rho(z)\right|^2 \, d\mu(z)$. The norm of f is given by

$$
\|f\|_{L^2}^2 = \sum_{\zeta \in Z^+(\beta)} \int_{\mathbb{C}} K_\zeta^2 e^{-|z|^2} \, d\mu(z) = \left(\int_{\mathbb{C}} e^{-|z|^2} \, d\mu(z)\right) \sum_{\zeta \in Z^+(\beta)} K_\zeta^2.
$$

We conclude that there is a bound of the form

$$
\|\mathbf{D}_\sigma \Phi_\sigma(f)\|_{L^2} \le C\sqrt{\sigma} e^{-\sigma/2} \|f\|_{L^2},
$$

which proves the first statement since $\sqrt{\sigma}e^{-\sigma/2} \to 0$ as $\sigma \to \infty$.

The second statement follows by a change of variable, since

$$
\begin{aligned}
\langle \Phi_\sigma(f), \Phi_\sigma(g)\rangle_{L^2} &= \sum_{\zeta \in Z^+(\beta)} \langle \Phi_\sigma(f)|_{\mathcal{D}(\zeta)}, \Phi_\sigma(g)|_{\mathcal{D}(\zeta)}\rangle_{L^2(\mathcal{D}(\zeta))} \\
&= \sum_{\zeta \in Z^+(\beta)} \int_{\mathbb{D}} \rho^2(z)\sigma f_\zeta(\sqrt{\sigma}z) g_\zeta(\sqrt{\sigma}z) \, d\mu(z) \\
&= \sum_{\zeta \in Z^+(\beta)} \int_{\mathbb{D}_{\sqrt{\sigma}}} \rho^2\left(\frac{z}{\sqrt{\sigma}}\right) f_\zeta(z) g_\zeta(z) \, d\mu(z)
\end{aligned}
$$

The functions f_ζ and g_ζ are both real multiples of $e^{-\frac{1}{2}|z|^2}$, so this last integral for each $\zeta \in Z^+(\beta)$ is bounded between $\int_{\mathbb{D}_{\sqrt{\sigma}/2}} f_\zeta(z) g_\zeta(z) \, d\mu(z)$ and $\int_{\mathbb{D}_{\sqrt{\sigma}}} f_\zeta(z) g_\zeta(z) \, d\mu(z)$, both of which converge to $\int_{\mathbb{C}} f_\zeta(z) g_\zeta(z) \, d\mu(z)$ as $\sigma \to \infty$, thus

$$
\lim_{\sigma\to\infty} \langle \Phi_\sigma(f), \Phi_\sigma(g)\rangle_{L^2} = \langle f, g\rangle_{L^2}.
$$

□

To turn approximate solutions into actual solutions, let

$$
\Pi_\sigma : L^2(E) \to \ker \mathbf{D}_\sigma
$$

denote the orthogonal projection. We will prove:

PROPOSITION 5.22. *If all zeroes of β are positive, then the linear map*

$$\Pi_\sigma \circ \Phi_\sigma : \bigoplus_{\zeta \in Z^+(\beta)} \ker \mathbf{D}_+ \to \ker \mathbf{D}_\sigma$$

is injective for all $\sigma > 0$ sufficiently large.

This statement says in effect that whenever $\sigma > 0$ is large enough and $\eta := \Phi_\sigma(f) \in \Gamma(E)$ is in the image of the pregluing map, with f normalized by $\|f\|_{L^2} = 1$, we can find a "correction" $\xi \in (\ker \mathbf{D}_\sigma)^\perp$ such that

$$\eta + \xi \neq 0 \quad \text{but} \quad \mathbf{D}_\sigma(\eta + \xi) = 0.$$

An element $\xi \in (\ker \mathbf{D}_\sigma)^\perp$ with the second property certainly exists, and in fact it's unique: indeed, the assumption $Z^-(\beta) = \emptyset$ implies via Lemma 5.20 that $\mathbf{D}_\sigma$ is surjective and thus restricts to an isomorphism from $(\ker \mathbf{D})^\perp \cap H^1(E)$ to $L^2(F)$, with a bounded right inverse

$$\mathbf{Q}_\sigma : L^2(F) \to H^1(E) \cap (\ker \mathbf{D})^\perp,$$

hence $\xi := -\mathbf{Q}_\sigma(\mathbf{D}_\sigma \eta)$. We know moreover from Lemma 5.21 that $\|\eta\|_{L^2}$ is close to $\|f\|_{L^2} = 1$, so to prove $\eta + \xi \neq 0$, it would suffice to show $\|\xi\|_{L^2}$ is small, which sounds likely since we also know $\|\mathbf{D}_\sigma \eta\|_{L^2}$ is small and $\mathbf{Q}_\sigma$ is a bounded operator. To make this reasoning precise, we just need to have some control over $\|\mathbf{Q}_\sigma\|$ as $\sigma \to \infty$, or equivalently, a quantitative measure of the injectivity of $\mathbf{D}_\sigma|_{(\ker \mathbf{D}_\sigma)^\perp \cap H^1(E)}$. This requires one last appeal to the Weitzenböck formula.

LEMMA 5.23. *Assume all zeroes of β are positive. Then there exist constants $c > 0$ and σ_0 such that for all $\sigma > \sigma_0$,*

$$\|\eta\|_{L^2} \le c\|\mathbf{D}_\sigma \eta\|_{L^2} \quad \textit{for all} \quad \eta \in H^1(E) \cap (\ker \mathbf{D}_\sigma)^\perp.$$

PROOF. Let us instead prove that if zeroes of β are all *negative*, then the same bound holds for all $\eta \in H^1(E)$. The stated result follows from this by considering the formal adjoint and using Exercise 5.24 below. Note that by density, it suffices to prove the estimate holds for all $\eta \in C_0^\infty(E)$.

Assume therefore that $Z^+(\beta) = \emptyset$ and, arguing by contradiction, suppose there exist sequences $\sigma_\nu \to \infty$ and $\eta_\nu \in C_0^\infty(E)$ with $\|\eta_\nu\|_{L^2} = 1$ and

$$\|\mathbf{D}_{\sigma_\nu} \eta_\nu\|_{L^2} \to 0.$$

The usual rescaling trick and application of the Weitzenböck formula then produces for each $\zeta \in Z^-(\beta)$ a sequence of functions $\eta_\nu^\zeta := \eta_\nu^{(\zeta,\sigma_\nu)} : \mathbb{D}_{\sqrt{\sigma_\nu}} \to \mathbb{C}$ which satisfy

$$\sum_{\zeta \in Z^-(\beta)} \|\eta_\nu^\zeta\|^2_{L^2(\mathbb{D}_{\sqrt{\sigma_\nu}})} \to 1 \quad \text{and} \quad \|\mathbf{D}_- \eta_\nu^\zeta\|_{L^2(\mathbb{D}_{\sqrt{\sigma_\nu}})} \to 0$$

as $\nu \to \infty$. Indeed, defining $\dot{\Sigma}_\epsilon$ as in the proof of Lemma 5.16, a similar application of the Weitzenböck formula yields

$$\|\mathbf{D}_{\sigma_\nu} \eta_\nu\|^2_{L^2(\dot{\Sigma})} \ge \sigma_\nu^2 c^2 \|\eta_\nu\|^2_{L^2(\dot{\Sigma}_\epsilon)} - \sigma_\nu c' \|\eta_\nu\|^2_{L^2(\dot{\Sigma})} = \sigma_\nu^2 c^2 \|\eta_\nu\|^2_{L^2(\dot{\Sigma}_\epsilon)} - \sigma_\nu c',$$

for some $c' > 0$. Thus we obtain

$$\|\eta_\nu\|^2_{L^2(\dot{\Sigma}_\epsilon)} \le \frac{\|\mathbf{D}_{\sigma_\nu}\eta_\nu\|^2_{L^2(\dot{\Sigma})}}{c^2\sigma_\nu^2} + \frac{c'}{\sigma_\nu c^2} \to 0 \quad \text{as} \quad \nu \to \infty,$$

so there is again concentration of energy near the zeroes of the antilinear perturbation: in particular,

$$\begin{aligned} 1 &= \lim_{\nu\to\infty} \|\eta_\nu\|^2_{L^2(\dot{\Sigma})} \\ &= \lim_{\nu\to\infty} \|\eta_\nu\|^2_{L^2(\dot{\Sigma}_\epsilon)} + \lim_{\nu\to\infty} \sum_{\zeta\in Z^-(\beta)} \|\eta_\nu\|^2_{L^2(\mathcal{D}(\zeta))} \\ &= \lim_{\nu\to\infty} \sum_{\zeta\in Z^-(\beta)} \|\eta_\nu^\zeta\|^2_{L^2(\mathbb{D}_{\sqrt{\sigma_\nu}})}. \end{aligned}$$

Moreover, we have

$$\mathbf{D}_-\eta_\nu^\zeta(z) = \frac{1}{\sigma_\nu}\bar{\partial}\eta_\nu\left(\frac{z}{\sqrt{\sigma_\nu}}\right) + \frac{\bar{z}}{\sqrt{\sigma_\nu}}\bar{\eta}_\nu\left(\frac{z}{\sqrt{\sigma_\nu}}\right) = \frac{1}{\sigma_\nu}\mathbf{D}_{\sigma_\nu}\eta_\nu\left(\frac{z}{\sqrt{\sigma_\nu}}\right).$$

Taking the square of the norms on each side, we may integrate and use change of variables to obtain

$$\|\mathbf{D}_-\eta_\nu^\zeta\|_{L^2(\mathbb{D}_{\sqrt{\sigma_\nu}})} = \frac{1}{\sqrt{\sigma_\nu}}\|\mathbf{D}_{\sigma_\nu}\eta_\nu\|_{L^2(\mathcal{D}(\zeta))} \to 0 \quad \text{as} \quad \nu \to \infty.$$

The elliptic estimates from Lecture 2 now provide uniform H^k-bounds for each η_ν^ζ on compact subsets of $\mathbb{C}$ for every $k \in \mathbb{N}$, so that a subsequence converges in $C^\infty_{\text{loc}}(\mathbb{C})$ to a smooth map $\eta_\infty^\zeta \in L^2(\mathbb{C},\mathbb{C})$ satisfying $\mathbf{D}_-\eta_\infty^\zeta = 0$. But $\sum_{\zeta\in Z^-(\beta)} \|\eta_\infty^\zeta\|^2_{L^2(\mathbb{C})} = 1$, so at least one of these solutions is nontrivial and thus contradicts Proposition 5.18. □

EXERCISE 5.24. Show that for any Fredholm Cauchy-Riemann type operator $\mathbf{D}$ on E, the following two estimates are equivalent, with the same constant $c > 0$ in both:

(i) $\|\eta\|_{L^2(E)} \le c\|\mathbf{D}\eta\|_{L^2(F)}$ for all $\eta \in H^1(E) \cap (\ker \mathbf{D})^\perp$;
(ii) $\|\lambda\|_{L^2(F)} \le c\|\mathbf{D}^*\lambda\|_{L^2(E)}$ for all $\lambda \in H^1(F) \cap (\ker \mathbf{D}^*)^\perp$.

Hint: elliptic regularity implies that for $\mathbf{D}$ *and* $\mathbf{D}^*$ *as bounded linear operators* $H^1 \to L^2$, $(\ker \mathbf{D})^\perp = \operatorname{im} \mathbf{D}^*$ *and* $(\ker \mathbf{D}^*)^\perp = \operatorname{im} \mathbf{D}$.

PROOF OF PROPOSITION 5.22. If the statement is not true, then there exist sequences $\sigma_\nu \to \infty$ and

$$f_\nu \in \bigoplus_{\zeta\in Z^+(\beta)} \ker \mathbf{D}_+$$

such that $\|f_\nu\|_{L^2} = 1$ and $\eta_\nu := \Phi_{\sigma_\nu}(f_\nu) \in (\ker \mathbf{D}_{\sigma_\nu})^\perp$ for all ν. Lemmas 5.21 and 5.23 then provide estimates of the form

- $\|\eta_\nu\|_{L^2} \to 1$,
- $\|\mathbf{D}_{\sigma_\nu}\eta_\nu\|_{L^2} \to 0$, and
- $\|\eta_\nu\|_{L^2} \le c\|\mathbf{D}_{\sigma_\nu}\eta_\nu\|_{L^2}$

as $\nu \to \infty$, with $c > 0$ independent of ν. These imply:

$$1 = \lim_{\nu\to\infty} \|\eta_\nu\|_{L^2} \leq \lim_{\nu\to\infty} c\|\mathbf{D}_{\sigma_\nu}\eta_\nu\|_{L^2} = 0.$$

□

We've proved:

PROPOSITION 5.25. *Suppose the assumptions of §5 hold and that the section $\beta \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(\bar{E}, F))$ has $I_+ \geq 0$ positive and $I_- \geq 0$ negative zeroes. If $I_- = 0$, then $\mathbf{D}_\sigma$ is surjective with $\dim\ker\mathbf{D}_\sigma = I_+$ for all $\sigma > 0$ sufficiently large. If $I_+ = 0$, then $\mathbf{D}_\sigma$ is injective with $\dim\operatorname{coker}\mathbf{D}_\sigma = I_-$ for all $\sigma > 0$ sufficiently large. In either case,*

$$\mathrm{ind}(\mathbf{D}_\sigma) = I_+ - I_-$$

for all $\sigma > 0$ sufficiently large. □

8. Antilinear deformations of asymptotic operators

Proposition 5.25 suffices to prove the index formula in the closed case, but there is an additional snag if $\Gamma \neq \emptyset$: since $H^1(\dot{\Sigma}) \hookrightarrow L^2(\dot{\Sigma})$ is not a compact inclusion, we have no guarantee that $\mathbf{D}$ and $\mathbf{D}_\sigma := \mathbf{D} + \sigma B$ will have the same index, and generally they will not. A solution to this problem has been pointed out by Chris Gerig [**Ger**], using a special class of asymptotic operators that also originate in the work of Taubes (see [**Tau10**, Lemma 2.3]).

In general, the only obvious way to guarantee $\mathrm{ind}(\mathbf{D}) = \mathrm{ind}(\mathbf{D}_\sigma)$ for large $\sigma > 0$ is if we can arrange for *every* operator in the family $\{\mathbf{D}_\sigma\}_{\sigma\geq 0}$ to be Fredholm, which is not automatic since the zeroth-order perturbation $B : E \to F$ is required to be bounded away from zero near ∞ and must therefore change the asymptotic operators at the punctures. We are therefore led to ask:

QUESTION. *For what nondegenerate asymptotic operators $\mathbf{A} : H^1(E) \to L^2(E)$ on a Hermitian line bundle $(E, J, \omega) \to S^1$ can one find complex-antilinear bundle maps $B : E \to E$ such that*

$$\mathbf{A}_\sigma := \mathbf{A} - \sigma B : H^1(E) \to L^2(E)$$

is an isomorphism for every $\sigma \geq 0$?

It turns out that it will suffice to find, for each unitary trivialization σ and every $k \in \mathbb{Z}$, a particular pair $(\mathbf{A}_k, B_k)$ such that $\mathbf{A}_k - \sigma B_k$ is nondegenerate for all $\sigma \geq 0$ and $\mu^\tau_{\mathrm{CZ}}(\mathbf{A}_k) = k$. To see why, let us proceed under the assumption that such pairs can be found, and use them to compute the index:

LEMMA 5.26. *Given $\mathbf{D}$ as in Theorem 5.4, fix asymptotic trivializations τ and suppose that for each puncture $z \in \Gamma$ there exists an asymptotic operator $\mathbf{A}'_z$ on (E_z, J_z, ω_z) with $\mu^\tau_{CZ}(\mathbf{A}'_z) = \mu^\tau_{CZ}(\mathbf{A}_z)$, such that if $\mathbf{A}'_z$ is written with respect to τ as $-J_0\partial_t - S_z(t)$, then the deformed asymptotic operator*

$$C^\infty(S^1, \mathbb{R}^2) \to C^\infty(S^1, \mathbb{R}^2) : \eta \mapsto -J_0\partial_t\eta - S_z(t)\eta - \sigma\beta_z(t)\bar{\eta} \tag{5.4}$$

is nondegenerate for some loop $\beta_z : S^1 \to \mathbb{C} \setminus \{0\}$ and every $\sigma \geq 0$. Then

$$\operatorname{ind}(\mathbf{D}) = \chi(\dot{\Sigma}) + 2c_1^\tau(E) + \sum_{z\in\Gamma^+} \operatorname{wind}(\beta_z) - \sum_{z\in\Gamma^-} \operatorname{wind}(\beta_z).$$

PROOF. Since $\mu^\tau_{\mathrm{CZ}}(\mathbf{A}_z) = \mu^\tau_{\mathrm{CZ}}(\mathbf{A}'_z)$, we can deform $\mathbf{A}_z$ to $\mathbf{A}'_z$ continuously through a family of nondegenerate asymptotic operators. It follows that we can deform $\mathbf{D}$ through a continuous family of Fredholm Cauchy-Riemann type operators to a new operator $\mathbf{D}'$ whose asymptotic operators are $\mathbf{A}'_z$ for $z \in \Gamma$, and $\operatorname{ind}(\mathbf{D}') = \operatorname{ind}(\mathbf{D})$. We are free to assume in fact that $\mathbf{D}'$ is written with respect to the trivialization τ on the cylindrical end near $z \in \Gamma^\pm$ as

$$\partial_s + J_0\partial_t + S_z(t).$$

Now choose $\beta \in \Gamma(\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F))$ with nondegenerate zeroes such that the deformed operators $\mathbf{D}_\sigma\eta := \mathbf{D}'\eta + \sigma\beta\bar{\eta}$ appear in trivialized form on the cylindrical end near $z \in \Gamma^\pm$ as

$$\mathbf{D}_\sigma\eta = \partial_s\eta + J_0\partial_t\eta + S_z(t)\eta + \sigma\beta_z(t)\bar{\eta}.$$

This means $\mathbf{D}_\sigma$ is asymptotic at z to (5.4), which is nondegenerate for every $\sigma \geq 0$, implying $\mathbf{D}_\sigma$ is Fredholm for every $\sigma \geq 0$ and thus

$$\operatorname{ind}(\mathbf{D}) = \operatorname{ind}(\mathbf{D}_\sigma).$$

The trivializations τ induce trivializations over the cylindrical ends for $\bar{E}$ and $F = \Lambda^{0,1}T^*\dot{\Sigma}\otimes E$, and the expression for β in the resulting asymptotic trivialization of $\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F)$ near $z \in \Gamma$ is $\beta_z(t)$. It follows that the signed count of zeroes of β is

$$\begin{aligned} i(\mathbf{D}) &:= c_1^\tau(\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F)) + \sum_{z\in\Gamma^+} \operatorname{wind}(\beta_z) - \sum_{z\in\Gamma^-} \operatorname{wind}(\beta_z) \\ &= \chi(\dot{\Sigma}) + 2c_1^\tau(E) + \sum_{z\in\Gamma^+} \operatorname{wind}(\beta_z) - \sum_{z\in\Gamma^-} \operatorname{wind}(\beta_z), \end{aligned}$$

where the computation $c_1^\tau(\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F)) = \chi(\dot{\Sigma}) + 2c_1^\tau(E)$ follows from the natural isomorphism

$$\begin{aligned} \operatorname{Hom}_{\mathbb{C}}(\bar{E}, F) &= \bar{E}^* \otimes F = E \otimes F = E \otimes \Lambda^{0,1}T^*\dot{\Sigma} \otimes E = \Lambda^{0,1}T^*\dot{\Sigma} \otimes E \otimes E \\ &= T\dot{\Sigma} \otimes E \otimes E. \end{aligned}$$

We are free to assume that all zeroes of β are either positive or negative, depending on the sign of $i(\mathbf{D})$. Proposition 5.25 then implies $\operatorname{ind}(\mathbf{D}_\sigma) = i(\mathbf{D})$ for large σ. □

Notice that instead of nondegenerate families $\mathbf{A} - \sigma B$ parametrized by $\sigma \in [0, \infty)$, it is just as well to find such families which are nondegenerate and have the right Conley-Zehnder index for all $\sigma > 0$, as the $\sigma \geq 1$ portion of this family can be rewritten as $(\mathbf{A} - B) - \sigma B$ for $\sigma \geq 0$. The following lemma thus completes the proof of Theorem 5.4.

LEMMA 5.27. *For every $k \in \mathbb{Z}$, the trivial Hermitian line bundle over S^1 admits an asymptotic operator $\mathbf{A}_k$ and a loop $\beta_k : S^1 \to \mathbb{C} \setminus \{0\}$ such that the deformed asymptotic operators*

$$\mathbf{A}_{k,\sigma}\eta := \mathbf{A}_k\eta - \sigma\beta_k\bar{\eta}$$

are nondegenerate for every $\sigma > 0$ *and satisfy*

$$\mu_{CZ}(\mathbf{A}_{k,\sigma}) = \mathrm{wind}(\beta_k) = k.$$

PROOF. We claim that the choices

$$\mathbf{A}_k\eta := -J_0\partial_t\eta - \pi k\eta \quad \text{and} \quad \beta_k(t) := e^{2\pi ikt}$$

do the trick. We prove this in three steps.

Step 1: $k = 0$. The above formula gives $\mathbf{A}_{0,\sigma} = -J_0\partial_t\eta - \sigma\bar{\eta}$, in which the $\sigma = 1$ case is precisely the operator that we used in Lecture 3 to normalize the Conley-Zehnder index, hence $\mu_{\mathrm{CZ}}(\mathbf{A}_{0,1}) = 0$ by definition. More generally, all of these operators can be expressed in the form $\mathbf{A} := -J_0\partial_t - S$ where $S \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^2)$ is a constant nonsingular 2-by-2 symmetric matrix that anticommutes with J_0. We claim that *all* asymptotic operators of this form are nondegenerate. Indeed, the conditions $S^{\mathrm{T}} = S$ and $SJ_0 = -J_0S$ for $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ imply that S takes the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ with $\det S = -a^2 - b^2 \neq 0$, and moreover S is of this form if and only if J_0S also is. In particular, J_0S is traceless, symmetric, and nonsingular. Solutions of $\mathbf{A}\eta = 0$ then satisfy $\dot{\eta} = J_0S\eta$, which has no periodic solutions since J_0S has one positive and one negative eigenvalue, hence $\ker \mathbf{A} = \{0\}$.

Step 2: even k. There is a cheap trick to deduce the case $k = 2m$ for any $m \in \mathbb{N}$ from the $k = 0$ case. Recall that by Exercise 3.36 in Lecture 3, conjugating $\mathbf{A}_{0,\sigma}$ by a change of trivialization changes its Conley-Zehnder index by twice the degree of that change. In particular, the operator

$$\tilde{\mathbf{A}}_{0,\sigma}\eta := e^{2\pi imt}\mathbf{A}_{0,\sigma}(e^{-2\pi imt}\eta)$$

is also a nondegenerate asymptotic operator, but with $\mu_{\mathrm{CZ}}(\tilde{\mathbf{A}}_{0,\sigma}) = \mu_{\mathrm{CZ}}(\mathbf{A}_{0,\sigma}) + 2m = k$. Explicitly, we compute

$$\tilde{\mathbf{A}}_{0,\sigma}\eta = -J_0\partial_t\eta - \pi k\eta - \sigma k e^{2\pi ikt}\bar{\eta},$$

so $\mathbf{A}_{k,\sigma} = \tilde{\mathbf{A}}_{0,\sigma/k}$ is also nondegenerate for every $\sigma > 0$.

Step 3: odd k. Another cheap trick relates each $\mathbf{A}_{k,\sigma}$ to $\mathbf{A}_{2k,\sigma}$ after an adjustment in σ. Given an arbitrary asymptotic operator $\mathbf{A} = -J_0\partial_t - S(t)$ and $m \in \mathbb{N}$, define

$$\mathbf{A}^m := -J_0\partial_t - mS(mt).$$

Geometrically, if $\mathbf{A}$ is a trivialized representation for the asymptotic operator of a Reeb orbit $\gamma : S^1 \to M$, then $\mathbf{A}^m$ is the operator for the m-fold covered orbit $\gamma^m : S^1 \to M : t \mapsto \gamma(mt)$. It is easy to check in particular that if we define $\eta^m(t) := \eta(mt)$ for any given loop $\eta : S^1 \to \mathbb{R}^2$, then

$$\mathbf{A}^m\eta^m = m(\mathbf{A}\eta)^m,$$

so this gives an embedding of $\ker \mathbf{A}$ into $\ker \mathbf{A}^m$, implying that whenever $\mathbf{A}^m$ is nondegenerate for some $m \in \mathbb{N}$, so is $\mathbf{A}$. To make use of this, observe that

$$\mathbf{A}^2_{k,\sigma}\eta = -J_0\partial_t\eta - \pi 2k\eta - 2\sigma e^{4\pi ikt}\bar{\eta} = \mathbf{A}_{2k,2\sigma}\eta,$$

so $\mathbf{A}^2_{k,\sigma}$ is nondegenerate for all $\sigma > 0$ by Step 2, and therefore so is $\mathbf{A}_{k,\sigma}$. $\square$

The proof of Theorem 5.4 is now complete.

EXERCISE 5.28. Derive a Weitzenböck formula for asymptotic operators and use it to show that for any asymptotic operator $\mathbf{A}$ on the trivial Hermitian line bundle and any smooth $\beta : S^1 \to \mathbb{C} \setminus \{0\}$, the deformed operators $\mathbf{A}_\sigma \eta := \mathbf{A}\eta - \sigma\beta\bar{\eta}$ are all nondegenerate for $\sigma > 0$ sufficiently large. Deduce from this that $\mu_{\mathrm{CZ}}(\mathbf{A}_\sigma) = \mathrm{wind}(\beta)$ for large $\sigma > 0$.

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