Handle attaching in symplectic topology - a second glance

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Abstract

We give a corrected proof of Cieliebaks important result on the invariance of symplectic homology under handle attachment. This paper is partly based on the authors PhD-thesis, during which he was supported by the Studienstiftung des deutschen Volkes, the graduate school of the SFB 647, "Raum, Zeit, Materie", and the Berlin Mathematical School.

1 Introduction

1.1 Symplectic topology and handle attaching

Symplectic geometry studies the topological, geometrical, dynamical structures of symplectic manifolds, that is of even dimensional manifolds V, $\dim V = 2n$, admitting a 2-form ω such that $d\omega = 0$ and ω^n is a volume form. Such an ω is called a symplectic form or structure.

In this program, symplectic topology studies symplectic manifolds with the help of techniques that are similar to whose of algebraic topology in the study of general manifolds. In particular, it constructs (co)homology theories in order to define invariants of symplectic manifolds. One of these theories is symplectic homology SH – a Floer-type homology for compact symplectic manifolds with contact type boundary (see 1.2).

On the other hand, there are topological techniques to construct new symplectic manifolds from existing ones – in particular the attachment of a symplectic handle to a symplectic manifold along a contact type boundary (see section 2 for a precise definition). In 2001 Kai Cieliebak, [3], first presented the following theorem which relates this construction with the invariants defined by SH.

Theorem 1 (Invariance of SH under subcritical surgery).
Let W and V be compact symplectic manifolds with contact type boundary and assume that the Conley-Zehnder index is well-defined on W. If V is obtained from W by attaching to $\partial W \times [0,1]$ a subcritical symplectic handle H_k^{2n}, $k < n$, then it holds that

$$SH_*(V) \cong SH_*(W) \qquad \textit{and} \qquad SH^*(V) \cong SH^*(W) \qquad \forall * \in \mathbb{Z}.$$

Applications of this theorem include the vanishing of symplectic homology of subcritical Stein manifolds, the proof of certain cases of the Cord conjecture (see [3]) and the distinction of exotic contact structures obtained by handle attachment (see [9]). Unfortunately, Cieliebaks original proof of this theorem has two flaws:

a) His version of the maximum principle is not strong enough for his purposes.

b) The statement about the existence of only one closed 1-periodic Hamiltonian orbit on the handle is not true (see 2.3, Discussion 8).

The purpose of this paper is to fix these problems, thus giving a clean and self-contained proof of this important theorem in symplectic topology. It is organized as follows: First, we give a short introduction to symplectic homology, prove a strong version of the maximum principle and construct the transfer maps due to Viterbo. Then, we study symplectic handle bodies, describe the attaching process and construct specific Hamiltonians on them. Finally, we prove Theorem 1 and transfer it to Rabinowitz-Floer homology with the help of recent results by Cieliebak and Oancea, [7].

1.2 Setup

Let (V,ω) be a compact symplectic manifold with boundary $\partial V = \Sigma$. For the sake of simplicity, we assume that the symplectic manifold (V,ω) is a Liouville domain[1], i.e. we assume that ω is exact with $\omega = d\lambda$ such that the Liouville vector field Y defined by $\omega(Y,\cdot) = \lambda$ points out of V along Σ. Note that any hypersurface M in V transverse to Y is a contact manifold. That is to say that the 1-form $\alpha := \lambda|_{TM}$ is contact, i.e. satisfies $\alpha \wedge (d\alpha)^{n-1} \neq 0$ pointwise. We write $\xi := \ker \alpha$ for the contact structure and R for the Reeb vector field defined by $d\alpha(R,\cdot) = 0$ and $\alpha(R) = 1$. The spectrum $spec(\Sigma,\alpha)$ of a contact form α on Σ is then defined by

$$spec(\Sigma,\alpha) = \{\eta \in \mathbb{R} \,|\, \exists \, \text{closed orbit of } R \text{ with period } \eta\}.$$

A symplectization of a contact manifold Σ with contact form α is a manifold $N = \Sigma \times I$, where $I \subset \mathbb{R}$ is an interval, together with the symplectic form $\omega := d(e^r\alpha)$, $r \in I$. The flow φ_Y of the Liouville vector field on V allows us to identify a collar neighborhood of $\Sigma = \partial V$ with the symplectization[2] $\big(\Sigma \times (-\delta, 0], d(e^r\alpha)\big)$, $r \in (-\delta, 0]$, for δ small enough.

[1] The weaker conditions in [3], namely that $\int_{T^2} f^*\omega = 0 \; \forall f : T^2 \to V$ and that ∂V is a convex contact boundary would also suffice.

[2] In fact, as Y points out of V along Σ and as V is compact, the negative flow of Y stays in V for all time, thus defining an embedding $\Sigma \times (-\infty, 0] \hookrightarrow V$.

This is in particular possible since

$$\begin{aligned}\mathcal{L}_Y\omega &= \iota_Y d\omega + d(\iota_Y\omega) = 0 + d\lambda = \omega\\ \mathcal{L}_Y\lambda &= \iota_Y d\lambda + d(\iota_Y\lambda) = \iota_Y\omega + d\omega(Y,Y) = \lambda,\end{aligned}$$

so that φ_Y preserves ω and expands $\alpha = \lambda|_{T\Sigma}$ exponentially overtime.
The collar neighborhood allows us to define the completion $(\widehat{V},\widehat{\omega})$ of (V,ω) by

$$\widehat{V} := V \cup_{\varphi_Y} \big(\Sigma\times(-\delta,\infty]\big) \qquad \widehat{\omega} := \begin{cases}\omega & \text{on } V\\ d(e^r\alpha) & \text{on } \Sigma\times(-\delta,0].\end{cases}$$

A Hamiltonian on $\widehat{V}$ is a smooth S^1-family of functions $H_t : \widehat{V}\to\mathbb{R}$ with Hamiltonian vector field X_H^t defined by

$$\omega(\cdot, X_H^t) = dH_t. \qquad \text{(for } t\in S^1 \text{ fixed)}$$

The Hamiltonian action of a loop $x : S^1\to\widehat{V}$ with respect to H is defined by

$$\mathcal{A}^H(x) = \int_0^1 x^*\lambda - \int_0^1 H_t(x(t))dt.$$

The critical points of the functional $\mathcal{A}^H$ are exactly the closed 1-periodic orbits of X_H^t. We denote the set of these solutions by $\mathcal{P}(H)$. Let J_t denote an S^1-family of ω-compatible almost complex structures. As usual, ω-compatible means that $\omega(\cdot, J_t\cdot)$ defines a Riemannian metric for every t. The L^2-gradient of $\mathcal{A}^H$ with respect to this metric is then given by

$$\nabla\mathcal{A}^H(x) = -J(\partial_t x - X_H^t).$$

An $\mathcal{A}^H$-gradient trajectory $u : \mathbb{R}\times S^1\to\widehat{V}$ is hence a solution of the following partial differential equation:

$$\partial_s u - \nabla\mathcal{A}^H = 0 \qquad \Leftrightarrow \qquad \partial_s u + J(\partial_t u - X_H^t) \;= 0. \tag{1}$$

In the course of this article, we will be also interested in homotopies H_s of Hamiltonians. In this case, we call solutions of (1) still $\mathcal{A}^{H_s}$-gradient trajectories, where X_H^t is then depending on s.
For the construction of symplectic (co)homology we look at solutions u of (1) satisfying $\lim_{s\to\pm\infty} = x_\pm(t)\in\mathcal{P}(H)$. In general, these solutions might not stay in a compact subset of $\widehat{V}$, even for $x_\pm$ fixed. Hence, it could be that the moduli space of these solutions has no suitable compactification. To avoid this problem, we make the following restrictions, which will by Lemma 2 ensure that all solutions of (1) with asymptotics in $\mathcal{P}(H)$ stay in a compact subset of $\widehat{V}$:

- We call a Hamiltonian H *(strongly) admissible*, writing $H\in Ad(V)$, if all 1-periodic orbits of X_H are *non-degenerate*, i.e. if for the flow $\varphi^t_{X_H}$ of X_H holds $\det(D\varphi^1_{X_H} - Id)\neq 0$ along each 1-periodic orbit $x\in\mathcal{P}(H)$, and if H is *linear at infinity*, that is if there exist $\alpha,\beta,R\in\mathbb{R}$ with $\alpha\notin Spec(\Sigma,\lambda)$ such that H is on $\Sigma\times[R,\infty)\subset\widehat{V}$ of the form

$$H = \alpha\cdot e^r + \beta \quad \text{ or more general } \quad H = h(e^r), \quad h:\mathbb{R}\to\mathbb{R}.$$

- We call a homotopy H_s between admissible Hamiltonians $H_\pm$ admissible if there exist $S, R > 0$ such that $H_s = H_\pm$ for $\pm s \geq S$ and on $\Sigma \times [R, \infty)$ the homotopy has the form

$$H_s = h_s(e^r) \qquad \text{with} \qquad \partial_s \partial_r H_s \leq 0 \qquad \text{on} \qquad \Sigma \times [R, \infty).$$

- We call a Hamiltonian/homotopy H *weakly admissible*, writing $H \in Ad^w(V)$, if there exist $S, R > 0$ such that $H_s = H_\pm$ for $\pm s \geq S$ and on $\Sigma \times [R, \infty)$ it has the form

$$H = \alpha \cdot e^{r - f(y)} + \beta \quad \text{or} \quad H = h\big(e^{r-f(y)}\big) \quad \text{resp.} \quad H_s = h_s\big(e^{r - f_s(y)}\big)$$

for a function $f : \Sigma \to \mathbb{R}$. In the homotopy case we require that

$$(\partial_s \partial_r h_s)\big(e^{r-f_s(y)}\big) - (\partial_r h_s)\big(e^{r-f_s(y)}\big) \cdot \partial_s f_s(y) \leq 0, \quad \text{with } < 0 \quad \text{on} \quad supp\, \partial_s f.$$

If $\partial_r^2 h = 0$ (e.g. if h is linear), then this is equivalent to $\partial_s \partial_r H_s \leq 0$.

- We call a possibly s-dependent almost complex structure J (weakly) admissible, if it is cylindrical and time independent at infinity, that is if

$$d\big(e^{r-f_s}\big) \circ J_s = -\lambda \qquad \text{on} \qquad \Sigma \times [R, \infty)$$

for an $R \in \mathbb{R}$. We may write this shorter as $d(e^{r_s}) \circ J = -\lambda$ for $r_s := r - f_s$.

Lemma 2 (**Maximum Principle**).
Let H be a (weakly) admissible Hamiltonian/homotopy and J an admissible almost complex structure. Let $x_\pm \in P(H_\pm)$, where $H_\pm$ are the ends of the possibly constant homotopy H_s. Then there exists a constant $\sigma \leq 1$ such that for $H_{\sigma \cdot s}$ and $J_{\sigma \cdot s}$ any solution $\boldsymbol{u}$ of (1) with $\lim\limits_{s \to \pm\infty} u(s) = x_\pm$ satisfies

$$e^r \circ u(s,t) \leq e^C \qquad \forall (s,t) \in \mathbb{R} \times S^1$$

for some constant $C \geq R$ not depending on u. If H is a (weakly) admissible Hamiltonian or a strongly admissible homotopy, then we may choose $\sigma = 1$, i.e. the Maximum principle holds already for H and J.

Proof: Our proof is a generalization of similar proofs by A.Oancea,[11], and P.Seidel, [14]. We give the proof only for homotopies H_s, which includes the Hamiltonian case by constant $H_s = H$. Let us consider the function $\rho : \mathbb{R} \times S^1 \to \mathbb{R}$ given by

$$\rho := e^{r - f_s} \circ u = e^{r_s} \circ u, \qquad \text{where } r_s := r - f_s.$$

To ease the notation, we will drop the index s, writing only H, h, f and J. Moreover, we write h' instead of $\partial_r h$. However, we keep r_s and we write u_s, u_t for $\partial_s u$ and $\partial_t u$.

Calculation of $\Delta\rho$

$$\begin{aligned}\partial_s\rho = d\big(e^{r_s}\big)(u_s) + \big(\partial_s e^{r-f_s}\big)(u) &= d\big(e^{r_s}\big)\big(-J(u_t - X_H)\big) + e^{r-f_s}(u)\cdot(-\partial_s f)(u)\\ &= \lambda(u_t) + \lambda(X_H) - \rho\cdot(\partial_s f)(u)\\ &= \lambda(u_t) - \rho\cdot h'(\rho) - \rho\cdot(\partial_s f)(u),\end{aligned}$$

as $\lambda(X_H) = \omega(Y, X_H) = dH(\partial_r) = \partial_r H = \rho\cdot h'(\rho)$. Moreover, we have

$$\partial_t\rho = d\big(e^{r_s}\big)(u_t) = d\big(e^{r_s}\big)(Ju_s - X_H) = -\lambda(u_s),$$

as the orbits of X_{H_s} stay in the level sets of e^{r_s} and hence $d\big(e^{r_s}\big)(X_{H_s}) = 0$. Therefore, we obtain for the Lapacian of ρ

$$\begin{aligned}\Delta\rho &= \partial_s\Big(\lambda(u_t) - \rho\cdot\big[h'(\rho) + (\partial_s f)(u)\big]\Big) - \partial_t\lambda(u_s)\\ &= d\lambda(u_s,u_t) - \lambda(\underbrace{[u_s,u_t]}_{=0}) - \partial_s\rho\cdot(\partial_s f)(u) - \Big(-\rho(\partial_s f)(u) + \underbrace{d\big(e^{r_s}\big)(u_s)\Big)\cdot h'(\rho)}_{=dH(u_s)=d\lambda(u_s,X_H)}\\ &\quad - \rho\cdot\Big[(\partial_s h')(\rho) + h''(\rho)\cdot\partial_s\rho + (\partial_s^2 f)(u) + d(\partial_s f)(u_s)\Big]\\ &= \omega(u_s, \underbrace{u_t - X_H}_{=Ju_s}) - \partial_s\rho\cdot\big((\partial_s f)(u) + \rho\cdot h''(\rho)\big) - \rho\cdot d(\partial_s f)(u_s)\\ &\quad - \rho\cdot\Big[(\partial_s h')(\rho) - h'(\rho)(\partial_s f)(u) + (\partial_s^2 f)(u)\Big]\\ &= |u_s|^2 - \partial_s\rho\cdot\big((\partial_s f)(u) + \rho\cdot h''(\rho)\big) - \rho\cdot d(\partial_s f)(u_s)\\ &\quad - \rho\cdot\Big[(\partial_s h')(\rho) - h'(\rho)(\partial_s f)(u) + (\partial_s^2 f)(u)\Big].\end{aligned}$$

Abbreviating $g(u) := (\partial_s f)(u) + \rho\cdot h''(\rho)$, we find that this is equivalent to

$$\Delta\rho + \partial_s\rho\cdot g(u) = |u_s|^2 - \rho\cdot d(\partial_s f)(u_s) - \rho\cdot\Big[(\partial_s h')(\rho) - h'(\rho)(\partial_s f)(u) + (\partial_s^2 f)(u)\Big]. \quad (*)$$

Now if for $C > R$ holds on $[C,\infty)\times\Sigma$ that the right-hand side of $(*)$ is non-negative, then ρ satisfies on $[C,\infty)\times\Sigma$ a maximum principle and cannot have a local maximum at an interior point of $u^{-1}\big([C,\infty)\times\Sigma\big)$. As the asymptotics of u lie outside of $[C,\infty)\times\Sigma$, it follows that $\rho = e^{r-f_s}\circ u \le e^C$ everywhere.

Estimate of $\kappa := |u_s|^2 - \rho\cdot d(\partial_s f)(u_s)$

At first glance, this term might be unbounded from below. However, as the Liouville form $\lambda = e^r\cdot\lambda_0$ grows exponentially in r, we will see that κ is in fact bounded by a constant, independent of u. Indeed, as $d(\partial_s f)$ is an r-invariant 1-form, there exists a vector field ξ_s on Σ, such that

$$d(\partial_s f)(\cdot) = d\lambda(\tfrac{1}{e^r}\xi_s,\cdot) \;\Rightarrow\; \rho\cdot d(\partial_s f)(u_s) = d\lambda(\xi_s, u_s).$$

For $c := \sup_s|J\xi_s|$, we find that this last expression is bounded by $c\cdot|u_s|$. It will be usefull to introduce σ at this point. Note that if we replace f_s by $f_{\sigma\cdot s}$, then κ becomes $|u_s|^2 - \sigma\cdot\rho\cdot d(\partial_s f)(u_s)$. Then, we have

$$\kappa = |u_s|^2 - \sigma\cdot\rho\cdot d(\partial_s f)(u_s) \ge |u_s|^2 - \sigma\cdot c\cdot|u_s| \ge -\tfrac{1}{4}c^2\cdot\sigma^2. \quad (**)$$

Here, the last estimate is the minimum of the parabola $x^2 - c\sigma x$.

Finally note that outside the s-support of $\partial_s f$, we have $\kappa = |u_s|^2 \geq 0$.

Estimate of the whole right-hand side of $(*)$

Let us introduce σ everywhere in $(*)$. Then, we get the following

$$\begin{aligned}\Delta\rho + \partial_s\rho\, g(u) &= |u_s|^2 - \sigma\rho\, d(\partial_s f)(u_s) - \rho\Big[\sigma(\partial_s h')(\rho) - \sigma\, h'(\rho)(\partial_s f)(u) + \sigma^2(\partial_s^2 f)(u)\Big] \\ &\overset{(**)}{\geq} -\rho\Big[\sigma\big((\partial_s h')(\rho) - h'(\rho)(\partial_s f)(u)\big) + \sigma^2(\partial_s^2 f)(u)\Big] - \tfrac{1}{4}c^2 \cdot \sigma^2. \qquad (***)\end{aligned}$$

For weakly admissable, we assumed that $(\partial_s h') - h'(\rho)(\partial_s f)(u) \leq 0$ with < 0 on the s-support of $\partial_s f$. As this support is bounded, we find for σ sufficiently small that the expression in the brackets is non-positive. Fixing such a σ, we find that for $\rho > R$ sufficiently large that the right-hand side is in fact non-negative. This proves the lemma. □

Remark.
- By decreasing σ, we can in fact achieve that $C = R$.
- If H is a Hamiltonian or a strongly admissible homotopy, then the term $(\partial_s^2 f)(u)$ is zero and there is no need for a reparametrization by σ, i.e. we can choose $\sigma = 1$.

1.3 Symplectic homology

For a (weakly) admissible Hamiltonian H, we define the Floer homology $FH_*(H)$ as follows: The chain groups $FC_*(H)$ are the $\mathbb{Z}_2$-vector space generated by $\mathcal{P}(H)$. Note that due to $h' \notin Spec(\Sigma, \alpha)$ and the non-degeneracy of the 1-periodic orbits, we find that $\mathcal{P}(H)$ is in fact a finite set. Thus, $FC_*(H)$ is a finite vector space of dimension $|\mathcal{P}(H)|$.
For $x_\pm \in \mathcal{P}(H)$ let $\widehat{\mathcal{M}}(x_-, x_+)$ denote the space of solutions u of (1) with $\lim\limits_{s\to\pm\infty} u = x_\pm$. There is an $\mathbb{R}$-action on this space given by time shift. The quotient under this action is called the moduli space of $\mathcal{A}^H$-gradient trajectories between x_- and x_+ and denoted by $\mathcal{M}(x_-, x_+) := \widehat{\mathcal{M}}(x_-, x_+)/\mathbb{R}$.
For a generic J, the space $\mathcal{M}(x_-, x_+)$ is a manifold. Its zero-dimensional component $\mathcal{M}^0(x_-, x_+)$ is compact and hence a finite set. Let $\#_2\mathcal{M}^0(x_-, x_+)$ denote its cardinality modulo 2. We define the operator $\partial : FC_*(H) \to FC_*(H)$ as the linear extension of

$$\partial x := \sum_{y \in \mathcal{P}(H)} \#_2\mathcal{M}^0(y, x) \cdot y.$$

A standard argument in Floer theory, involving the compactification of $\mathcal{M}^1(y, x)$, shows that $\partial^2 = 0$, so that ∂ is a boundary operator. We set as usual

$$FH_*(H) := \frac{\ker \partial}{\operatorname{im} \partial}.$$

To a (weakly) admissible homotopy H_s between admissible Hamiltonians $H_\pm$ we consider for $x_\pm \in \mathcal{P}(H_\pm)$ the moduli space of s-dependent $\mathcal{A}^{H_s}$-gradient trajectories $\mathcal{M}_s(x_-, x_+)$. Note that we have no time shift on this space, as equation (1) now depends on s.

We define the continuation map $\sigma_*(H_-, H_+) : FC_*(H_+) \to FC_*(H_-)$ as the linear extension of

$$\sigma_*(H_-, H_+)x_+ = \sum_{x_- \in \mathcal{P}(H_-)} \#_2\mathcal{M}_s^0(x_-, x_+) \cdot x_-.$$

By considering homotopies of homotopies, one sees that $\sigma_*(H_-, H_+)$ is independent of the chosen homotopy. By considering the compactification of $\mathcal{M}_s^1(x_-, x_+)$, we obtain from Floer theory that $\partial \circ \sigma_* = \sigma_* \circ \partial$, so that $\sigma_*(H_-, H_+)$ is a chain map, which descends to a map $\sigma_*(H_-, H_+) : FH(H_+) \to FH(H_-)$. For three admissible Hamiltonians H_1, H_2 and H_3, the maps σ_* obey the composition rule

$$\sigma_*(H_1, H_3) = \sigma_*(H_1, H_2) \circ \sigma_*(H_2, H_3).$$

We introduce a partial ordering $\prec$ on $Ad^w(V)$ by saying $H_+ \prec H_-$ if and only if $H_+ < H_-$ on $\Sigma \times [R, \infty)$ for some R. Observe that admissibility of a homotopy H_s between H_- and H_+ implies that $H_+ \prec H_-$. It follows from the above that the groups $FH(H)$ together with the maps $\sigma_*(H_-, H_+)$ for $H_+ \prec H_-$ define a direct system over the directed set $(Ad^w(V), \prec)$. The symplectic homology groups $SH_*(V)$ are then defined to be the direct limit of this system:

$$SH_*(V) := \varinjlim FH_*(H).$$

A *cofinal sequence* $(H_n) \subset Ad^w(V)$ is a sequence of Hamiltonians such that $H_n \prec H_{n+1}$ and for any $H \in Ad^w(V)$ there exists $n \in \mathbb{N}$ such that $H \prec H_n$. It follows from the definition of direct limits that it can be computed from any cofinal sequence, i.e. that

$$SH_*(V) = \lim_{n \to \infty} FH_*(H_n).$$

More general, a set $\mathcal{F} \subset Ad^w(V)$ is cofinal if for any $H \in Ad^w(V)$ there exists $F \in \mathcal{F}$ such that $H \prec F$. For cofinal $\mathcal{F}$ also holds $SH_*(V) = \varinjlim FH_*(F), F \in \mathcal{F}$.

Symplectic (co)homology can be given a $\mathbb{Z}$-grading by the Conley-Zehnder index μ_{CZ}. For that, we restrict ourself to contractible 1-periodic orbits of X_H, which is no restriction if V is simply connected. Moreover, we have to assume that $\int_{S^2} s^* c_1(TW) = 0$ for every continuous map $s : S^2 \to V$.
To compute $\mu_{CZ}(v)$ for a closed contractible 1-periodic Hamiltonian orbit v choose a map u from the unit disc $D \subset \mathbb{C}$ to V such that $u(e^{2\pi i t}) = v(t)$. Then choose a symplectic trivialization $\Phi : D \times \mathbb{R}^{2n} \to u^*TV$ of the pullback bundle $(u^*TV, u^*\omega)$. Such trivializations exist and are homotopically unique as D is contractible. The linearization of the Hamiltonian flow $\varphi^t_{X_H}$ along v with respect to Φ defines a path Ψ in the group $Sp(2n)$ staring at $\mathbb{1}$ by

$$\Psi(t) := \Phi(v(t))^{-1} \circ d\varphi^t_{X_H}(v(0)) \circ \Phi(v(0)).$$

The Conley-Zehnder index of this path is the index $\mu_{CZ}(v)$ (see [13] or 2.4 for μ_{CZ} of paths in $Sp(2n)$). The assumption on the first Chern class $c_1(TV)$ guarantees that this definition does not depend on the choice of u.

1.4 Action filtration

The action functional $\mathcal{A}^H$ provides filtrations of $SH(V)$ as follows: For a (weakly) admissible Hamiltonian H and $b \in \mathbb{R}$ consider the subchain groups

$$FC_*^{<b}(H) \subset FC_*(H),$$

which are generated by whose $x \in \mathcal{P}(H)$ with $\mathcal{A}^H(x) < b$. For $a < b$, we set

$$FC_*^{[a,b)}(H) := FC_*^{<b}(H) \Big/ FC_*^{<a}(H)\,.$$

We call $FC_*^{[a,b)}(H)$ truncated chain groups in the action window $[a,b)$. By setting $a = -\infty$, they include the cases $FC_*^{[-\infty,b)}(H) = FC_*^{<b}(H)$. Analogously one defines

$$FC_*^{\leq b}(B),\ FC_*^{>b}(H) := FC_*(H) \Big/ FC_*^{\leq b}(H),\ \ FC_*^{\geq b}(H),$$
$$FC_*^{(a,b]}(H),\ FC_*^{(a,b)}(H) \text{ and } FC_*^{[a,b]}(H).$$

Note that $FC_*^{[a,b)}(H) = FC_*^{(a,b)}(H)$ if $a \notin \mathcal{A}^H(\mathcal{P}(H))$. In the following, we restrict ourself for simplicity to $FC_*^{(a,b)}(H)$. However, most of the subsequent results hold also for all other versions of action windows.

Lemma 3 below shows that the boundary operator ∂ reduces the action. It induces therefore a boundary operator ∂ on the truncated chain groups and for this ∂ we define

$$FH_*^{(a,b)}(H) := \frac{\ker \partial}{\operatorname{im} \partial}.$$

Lemma 3. *If H is a Hamiltonian or a (everywhere) monotone decreasing homotopy and $\boldsymbol{u}$ a solution of (1) with $\lim\limits_{s\to\pm\infty} u = x_\pm \in \mathcal{P}(H)$, then $\mathcal{A}^H(x_+) \geq \mathcal{A}^H(x_-)$.*

Proof:

$$\begin{aligned}\mathcal{A}^H(x_+) - \mathcal{A}^H(x_-) &= \int_{-\infty}^{\infty} \frac{d}{ds}\mathcal{A}^H(u(s))ds \\ &= \int_{-\infty}^{\infty} ||\nabla \mathcal{A}^H||^2 ds - \int_{-\infty}^{\infty}\int_0^1 \left(\frac{d}{ds}H\right)(u(s))dt\,ds \geq 0.\end{aligned}$$

Note that the second term is zero, if H does not depend on s, i.e. if H is a Hamiltonian. This shows that the monotone decreasing condition is only needed for homotopies. □

Let H_-, H_+ be two (weakly) admissible Hamiltonians such that $H_- > H_+$ everywhere. Then we may choose a monotone decreasing (weakly) admissible homotopy H_s between them and it follows from Lemma 3 that the associated continuation map $\sigma_*(H_-, H_+)$ also decreases action. We obtain hence a well-defined map

$$\sigma_*(H_-, H_+) : FH_*^{(a,b)}(H_+) \to FH_*^{(a,b)}(H_-).$$

The truncated symplectic homology in the action window (a,b) is then defined as the direct limit under these maps:

$$SH_*^{(a,b)}(V) := \varinjlim FH_*^{(a,b)}(H).$$

Attention: Without further restrictions, we have always

$$SH^{(a,b)}(V) = 0, \quad \text{for} \quad a > -\infty \qquad \text{and} \qquad SH^{(-\infty,b)}(V) = SH(V) \quad \text{for} \quad b < \infty.$$

To see this, take any cofinal sequence of Hamiltonians (H_n) and take an increasing sequence $(\beta_n) \subset \mathbb{R}$ such that $\beta_n > \max\limits_{x \in \mathcal{P}(H_n)} \mathcal{A}^{H_n}(x)$. Define $K_n := H_n + \beta_n - a$ and $L_n := H_n + \beta_n - b$, which yield also cofinal sequence satisfying

$$\max_{x \in \mathcal{P}(K_n)} \mathcal{A}^{K_n}(x) = \max_{x \in \mathcal{P}(H_n)} \mathcal{A}^{H_n}(x) - \beta_n + a < a \quad \text{and} \quad \max_{x \in \mathcal{P}(L_n)} \mathcal{A}^{L_n}(x) < b.$$

It follows that $FC_*^{(a,b)}(K_n) = FH_*^{(a,b)}(K_n) = 0$ for all n and hence $SH^{(a,b)}(V) = 0$, while $FC_*^{(-\infty,b)}(L_n) = FC_*(L_n)$ for all n and hence $SH^{(-\infty,b)}(V) = SH(V)$.
To obtain a meaningful action filtered version of SH, we hence have to restrict further the set of admissible Hamiltonians. For us, it will be enough to require that all Hamiltonians H are smaller then 0 inside a fixed Liouville subdomain[3] $W \subset V$ bounded by a contact hypersurface ∂W. We write $SH^{(a,b)}(W{\subset}V)$ for the direct limit of these Hamiltonians[4], as this filtration of $SH_*(V)$ gives informations about the embedded subdomain W. Note that different choices of $W \subset V$ give different filtrations of $SH(V)$!
We remark that for the definition of $FH_*^{(a,b)}(H)$ it suffices that only the 1-periodic orbits x of X_H with $\mathcal{A}^H(x) \in (a,b)$ are non-degenerate, as the others are discarded. Therefore, we call a Hamiltonian H admissible for $SH_*^{(a,b)}(W{\subset}V)$, writing $H \in Ad^{(a,b)}(W{\subset}V)$, if it satisfies

- $H|_W < 0$
- $H|_{\Sigma \times [R,\infty)} = h(e^r)$ for R large
- all $x \in \mathcal{P}^{(a,b)}(H) = \{x \in \mathcal{P}(H) \mid \mathcal{A}^H(x) \in (a,b)\}$ are non-degenerate.

The partial ordering on $Ad^{(a,b)}(W \subset V)$ is given by $H \prec K$ if $H < K$ everywhere. Similar, one defines weakly admissible Hamiltonians. Note that we are free to choose for the computation of $SH^{(a,b)}(W{\subset}V)$ cofinal sequences (H_n) which are also admissible for the whole symplectic homology or cofinal sequences, where the 1-periodic orbits of X_{H_n} are only non-degenerate in the action window (a,b).
Note that $Ad(W{\subset}V) := Ad^{(-\infty,\infty)}(W{\subset}V)$ is cofinal in $Ad^w(V)$, so that

$$SH_*^{(-\infty,\infty)}(W{\subset}V) = SH_*(W{\subset}V) = SH_*(V).$$

When taking a cofinal sequence $(H_n) \subset Ad(W{\subset}V)$, we find that the projection

$$FC_*(H) \to FC_*^{>b}(H) = FC_*(H) \Big/ FC_*^{\leq b}(H)$$

[3]Liouville subdomain means that $W \subset V$ is a codimension 0 submanifold and that the Liouville structure of W is the restriction of the Liouville structure of V.

[4]These Hamiltonians coincide with whose defining $SH(W)$ in the sense of [7]. However, $SH(W) \neq SH(W \subset V)$ in general, as $SH(W)$ involves also limits over the action window and is hence an invariant of W, while $SH_*(W{\subset}V)$ is an invariant of V.

or the short exact sequence

$$0 \to FC_*^{(a,b)}(H) \to FC_*^{(a,c)}(H) \to FC_*^{(b,c)}(H) \to 0$$

induce in homology the map

$$FH_*(H) \to FH_*^{\geq b}(H)$$

respectively the long exact sequence

$$\cdots \to FH_*^{(a,b)}(H) \to FH_*^{(a,c)}(H) \to FH_*^{(b,c)}(H) \to \dots$$

Applying the direct limit then yields the map

$$SH_*(V) = SH_*(W \subset V) \to SH_*^{>b}(W \subset V)$$

and (as $\varinjlim$ is an exact functor) the long exact sequence

$$\cdots \to SH_*^{(a,b)}(W \subset V) \to SH_*^{(a,c)}(W \subset V) \to SH_*^{(b,c)}(W \subset V) \to \dots$$

1.5 Symplectic cohomology

By dualizing the constructions from the previous section, we obtain the symplectic cohomology. Explicitly, we define for a (weakly) admissible Hamiltonian H the cochain groups $FC^*(H)$ again as the $\mathbb{Z}_2$-vector space generated by $\mathcal{P}(H)$. The coboundary operator δ is then defined as the linear extension of

$$\delta x := \sum_{y \in \mathcal{P}(H)} \#_2 \mathcal{M}^0(x, y) \cdot y.$$

Note that the operator δ increases action. The analogue construction of chain maps $\sigma^*(H_-, H_+)$ associated to an admissible homotopy H_s between Hamiltonians H_- and H_+ yields hence a map in the opposite direction (compared to $\sigma_*(H_-, H_+)$)

$$\sigma^*(H_-, H_+) : FH^*(H_-) \to FH^*(H_+),$$

where $H_- > H_+$ on $\Sigma \times [R, \infty)$ for R sufficiently large. It obeys the composition rule

$$\sigma^*(H_1, H_3) = \sigma^*(H_2, H_3) \circ \sigma^*(H_1, H_2).$$

By taking the same partial ordering on $Ad^w(V)$ as for homology, we obtain hence an inverse system. The symplectic cohomology $SH^*(V)$ is then defined to be the inverse limit of this system

$$SH^*(V) := \varprojlim FH^*(H).$$

Again, it can be calculated using cofinal sequences (H_n) of admissible Hamiltonians. For the truncated version of symplectic cohomology, we now have to consider

$$FC^*_{>a}(H) \subset FC^*(H)$$

generated by those 1-periodic orbits with action greater then a.

Then, we define

$$FC^*_{(a,b]}(H) := FC^*_{>a}(H) \Big/ FC^*_{>b}(H)$$

and all other truncated groups accordingly. As δ increases action, it is well-defined on the truncated chain groups and yields analogously $FH^*_{>a}(H)$ and $FH^*_{(a,b)}(H)$ as cohomology groups. When considering only (globally) monotone decreasing homotopies, the chain maps σ^* are also well-defined on truncated groups and we obtain as inverse limits

$$SH^*_{>a}(W\subset V) = \lim_{\longleftarrow} FH^*_{>a}(H), \qquad SH^*_{(a,b)}(W\subset V) = \lim_{\longleftarrow} FH^*_{(a,b)}(H),$$

where we restricted again to $H \in Ad^w(W\subset V)$.
Unlike to the homology case, the long exact sequence

$$\dots \to FH^*_{(b,c)}(H) \to FH^*_{(a,c)}(H) \to FH^*_{(a,b)}(H) \to \dots$$

induces in general not a long exact sequence in symplectic cohomology. This is due to the fact that, in general, the inverse limit is not an exact functor, but only left exact (see [1] or [8]). However, the inclusion $FC^*_{>a}(H) \to FC^*(H)$ still induces a map

$$SH^*_{>a}(W\subset V) \to SH^*(W\subset V) = SH^*(V).$$

1.6 The transfer morphisms

In the following, we construct a map $\pi_*(W,V) : SH_*(V) \to SH_*(W)$ for a Liouville subdomain $W \subset V$, as first suggested by Viterbo in [16]. Analogously, we construct a map $\pi^*(W,V) : SH^*(W) \to SH^*(V)$ in cohomology. The maps $\pi_*(W,V)$ and $\pi^*(W,V)$ are called transfer maps and they will provide the isomorphisms in Theorem 1.
As shown above, we have always maps $SH_*(V) \to SH_*^{>0}(W\subset V)$ and $SH^*_{>0}(W\subset V) \to SH^*(V)$. The maps $\pi_*(W,V)$ and $\pi^*(W,V)$ are obtained by showing the identities $SH_*^{>0}(W\subset V) = SH_*(W)$ and $SH^*_{>0}(W\subset V) = SH^*(W)$. This is done in Corollary 5 by giving an explicit cofinal sequence $(H_n) \subset Ad(W\subset V)$.
The following proposition is based on ideas by Viterbo, [16]. Its proof is taken from McLean, [10]. We include it here for completeness and to add a missing argument for the homotopy case. See also Cieliebak, [3], for a slightly different approach.

Proposition 4 (McLean,[10])**.** *There exists a cofinal sequence $(H_n) \subset Ad(W\subset V)$ and a sequence of monotone decreasing admissible homotopies $(H_{n,n+1})$ between them such that*

1. *$K_n := H_n|_W$, $K_{n,n+1} := H_{n,n+1}|_W$ are sequences of admissible Hamiltonians / homotopies on (W,ω).*
2. *all 1-periodic orbits of X_{H_n} in W have positive action and all 1-periodic orbits of X_{H_n} in $V \setminus W$ have negative action.*
3. *all $\mathcal{A}^H$-gradient trajectories of H_n or $H_{n,n+1}$ connecting 1-periodic orbits in W are entirely contained in W.*

Proof: It will be convenient to use $z = e^r$ rather than r for the second coordinate in the completions $(\widehat{W}, \widehat{\omega})$ and $(\widehat{V}, \widehat{\omega})$. Note that we can embed $\widehat{W}$ into $\widehat{V}$ using the flow of the Liouville vector field Y. The cylindrical end $\partial W \times [1, \infty)$ is then a subset of $\widehat{V}$. The second coordinates will be denoted z_W for $\partial W \times (0, \infty)$ and z_V for $\partial V \times (0, \infty)$. Let $\alpha_W := \lambda|_{T\partial W}$, $\alpha_V := \lambda|_{T\partial V}$ and assume that $Spec(\partial W, \alpha_W)$ and $Spec(\partial V, \alpha_V)$ are discrete. Now let

$$k : \mathbb{N} \to \mathbb{R}^+ \setminus \Big(Spec(\partial W, \alpha_W) \cup 4 \cdot Spec(\partial V, \alpha_V) \Big)$$

be an increasing function such that $k(n) \to \infty$. Let $\mu : \mathbb{N} \to \mathbb{R}^+$ be defined by

$$\mu(n) = dist\big(k(n), Spec(\partial W, \alpha_W)\big) = \min_{a \in Spec(\partial W, \alpha_W)} |k(n) - a|.$$

Choose an increasing sequence $A = A(n)$ with $\quad A > \dfrac{2k}{\mu} > 1 \quad$ and $\quad A(n+1) > 2A(n)$

which satisfies additionally the conditions $(\oplus)$ and $(\oplus\oplus)$ below. Note that we can always achieve $\frac{2k}{\mu} > 1$, as we may choose k arbitrarily large whilst making μ arbitrarily small. Let also $\varepsilon(n) > 0$ be a sequence tending to zero.

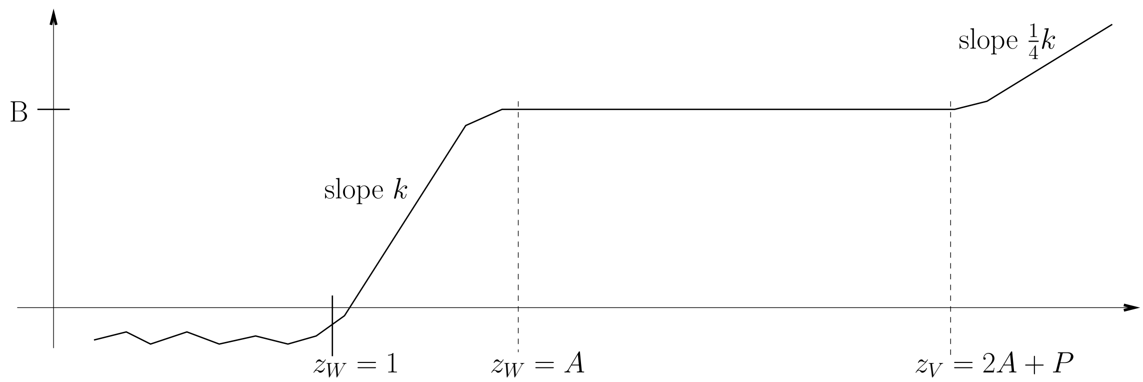

Fig. 1: The Hamiltonian H_n

Next, we describe the Hamiltonian H_n (see Figure 1 for a schematic illustration). Let $H_n|_W$ be a C^2-small negative Morse function inside $W \setminus \big(\partial W \times [1-\varepsilon, 1)\big)$ and for $1 - \varepsilon \le z_W \le A$ of the form $H_n = g(z)$ with $g(1) = -\varepsilon$, $g' \ge 0$ and $g' \equiv k(n)$ for $1 \le z_W \le A - \varepsilon$. For $A \le z_W \le 2A$ let $H_n \equiv B$ be constant with B being arbitrarily close to $k \cdot (A - 1)$.
On $\partial V \times [1, \infty)$ the Hamiltonain H_n should satisfy the following: Coming from ∂W we keep H_n constant until we reach $z_V = 2A + P$, where P is some constant such that $\{z_W \le 1\} \subset \{z_V \le P\}$. Note that this implies $\{z_W \le 2A\} \subset \{z_V \le 2A + P\}$. Then let $H_n = f(z_V)$ for $z_V \ge 2A + P$ with $0 \le f' \le \frac{1}{4}k(n)$ and $f' \equiv \frac{1}{4}k(n)$ for $z_V \ge 2A + P + \varepsilon$.

As the action of an X_H-orbit on a fixed z-level is $h'(z) \cdot z - h(z)$, we distinguish five types of 1-periodic orbits of X_H:

- critical points inside W of action > 0 (as H is negative and C^2-small inside W)
- non-constant orbits near $z_W = 1$ of action $\approx g'(z) \cdot 1 > 0$
- non-constant orbits on $z_W = a$ for a near A of action $\approx g'(a) \cdot a - B < (k - \mu) \cdot A - B \approx -\mu \cdot A + k < -k < 0$
- critical points in $A < z_W$; $z_V < 2A + P$ of action $-B < 0$
- non-constant orbits on $z_V = a$ for a near $2A + P$ of action $\approx f'(a) \cdot a - B \leq \frac{1}{4}k \cdot (2A + P + \varepsilon) - B \approx -\frac{1}{2}kA + k \cdot (\frac{1}{4}P + \frac{1}{4}\varepsilon + 1) < 0$ for A sufficiently large (this is condition $(\oplus)$).

Obviously, (H_n) satisfies 1. and 2. of the proposition's claims. It only remains to show that $\mathcal{A}^H$-gradient trajectories connecting two orbits of non-negative action are contained entirely inside $z_W \leq 1$. By Gromov's Monotonicity Lemma (see [15], Prop. 4.3.1 and [12], Lem. 1) there exists a $K > 0$ such that any J-holomorphic curve which intersects $z_W = A$ and $z_W = 2A$ has area greater than KA. Note that inside $A \leq z_W \leq 2A$ the equation (1) reduces to an ordinary J-holomorphic curve equation, as $X_H \equiv 0$ there. Any $\mathcal{A}^H$-gradient trajectory connecting two orbits of non-negative action which intersects $z_W = A$ and $z_W = 2A$ has therefore area greater than KA – in other words the action difference between its ends is greater than KA.
For $k(n)$ fixed, the maximal action difference of two 1-periodic orbits in W is bounded from above. So for $A(n)$ sufficiently large (this is condition $(\oplus\oplus)$) no such $\mathcal{A}^H$-gradient trajectory can touch $z_W = 2A$. It follows then from the Maximum Principle that in fact all these $\mathcal{A}^H$-gradient trajectories have to remain inside $z_W \leq 1$.
For the construction of the homotopies $H_{n,n+1}$ between H_n and H_{n+1} we have to sharpen this argument. As $A(n+1) > 2A(n)$, we can take for $H_{n,n+1}$ the following interpolations: At first, in time $s \in [0, 1/2]$, decrease H_{n+1} in the area $z_W \leq 2A(n)$ to H_n and keep it unchanged in $z_W \geq A(n + 1)$. Then decrease in time $s \in [1/2, 1]$ the remaining part to H_n (see Figures 2 and 3).
For $s \in [-\infty, 1/2]$ the homotopy $H_{n,n+1}$ is then constant $B(n+1)$ in the area $A(n+1) \leq z_W \leq 2A(n+1)$ so that no $\mathcal{A}^H$-gradient trajectory can leave $z_W \leq 1$ in this time interval. For $s \in [1/2, \infty]$ the homotopy $H_{n,n+1}$ is constant $B(n)$ in the area $A(n) \leq z_W \leq 2A(n)$ so that again no $\mathcal{A}^H$-gradient trajectory can leave $z_W \leq 1$ in this time interval. □

Corollary 5. $SH_*^{>0}(W \subset V) \simeq SH_*(W)$ *and* $SH^*_{>0}(W \subset V) \simeq SH^*(W)$.

Proof: We only prove the corollary for homology, cohomology being completely analog. Take the sequence of Hamiltonians (H_n) constructed in Proposition 4. Clearly it is cofinal and $(H_n) \subset Ad^{>0}(W \subset V)$, as 1-periodic orbits with positive action are either isolated critical points inside W (as H is Morse and C^2-small there) or isolated Reeb-orbits near $z_W = 1$ – in both cases non-degenerate. Hence we have

$$SH_*^{>0}(W \subset V) = \lim_{\longrightarrow} FH_*^{>0}(H_n).$$

Let $\tilde{H}_n \in Ad(W)$ be the linear extension of $H_n|_W$ with slope $k(n)$. Due to $k(n) \notin Spec(\partial W, \lambda)$, we have obviously $FC_*^{>0}(H_n) = FC_*(\tilde{H}_n)$. As any $\mathcal{A}^H$-gradient trajectory connecting 1-periodic orbits in W stays in W, the two boundary operators ∂^{H_n} and $\partial^{\tilde{H}_n}$ coincide and we have $FH_*^{>0}(H_n) = FH_*(\tilde{H}_n)$. As the $\mathcal{A}^H$-gradient trajectories for the homotopies $H_{n,n+1}$ stay inside W, the continuation maps

$$\sigma(H_{n+1}, H_n) : FH_*^{>0}(H_n) \to FH_*^{>0}(H_{n+1})$$

coincide with the continuation maps

$$\sigma(\tilde{H}_{n+1}, \tilde{H}_n) : FH_*(\tilde{H}_n) \to FH_*(\tilde{H}_{n+1}).$$

Hence we have $\quad SH_*^{>0}(W \subset V) = \varinjlim FH_*^{>0}(H_n) = \varinjlim FH_*(\tilde{H}_n) = SH_*(W).$ $\square$

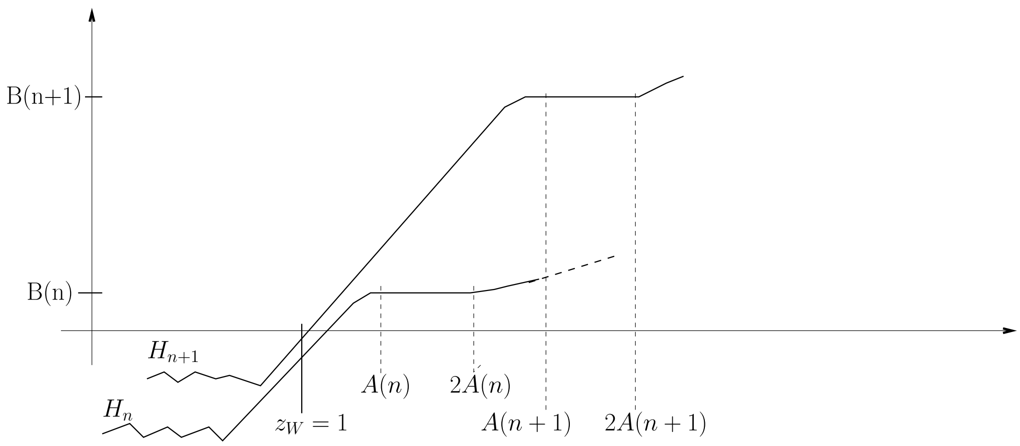

Fig. 2: Two Hamiltonian H_n and H_{n+1}

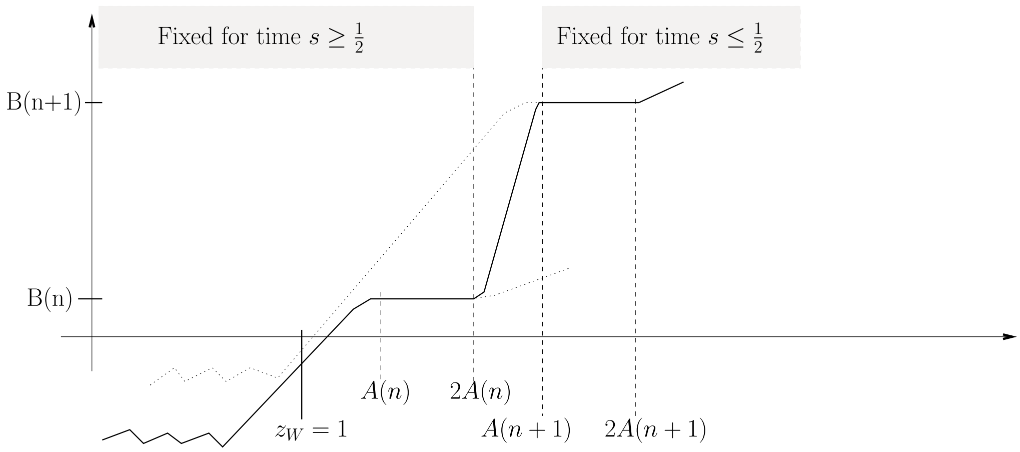

Fig. 3: The homotopy $H_{n,n+1}$ at time $s = \frac{1}{2}$

2 Contact surgery and handle attaching

In this section, we describe the general construction for contact surgery, which is done by attaching a symplectic handle H_k^{2n} to the symplectization of a contact manifold. Then, we describe the symplectic standard handle, which is a subset of $\mathbb{R}^{2n}$ defined as the intersection of two sublevel sets $\{\psi < -1\} \cap \{\phi > -1\}$, where ϕ and ψ are functions on $\mathbb{R}^{2n}$. While ϕ is explicitly given, we describe the construction of a suitable ψ in 2.3. Simultaneously, we describe how to extend an admissible Hamiltonian over the handle to a new admissible Hamiltonian with only few new 1-periodic Hamiltonian orbits. The calculation of Conley-Zehnder indices for these orbits on H_k^{2n} and the proof of the main theorem conclude the section.

2.1 Surgery along isotropic spheres

Let us briefly recall the contact surgery construction due to Weinstein, [17]. Consider an isotropic sphere S^{k-1} in a contact manifold (N^{2n-1}, ξ). The 2-form $\omega = d\lambda$ for a contact form λ (with $\xi = \ker \lambda$) defines a natural conformal symplectic structure on ξ. Denote the ω-orthogonal on ξ by $\perp_\omega$. Since S is isotropic, it holds that $TS \subset TS^{\perp_\omega}$. So, the normal bundle of S in N is given by

$$TN/TS = TN/\xi \,\oplus\, \xi/(TS)^{\perp_\omega} \oplus (TS)^{\perp_\omega}/TS.$$

The Reeb field R_λ trivializes TN/ξ. The bundle $\xi/(TS)^{\perp_\omega}$ is canonically isomorphic to T^*S via $v \mapsto \iota_v\omega$. The conformal symplectic normal bundle $CSN(S) := (TS)^{\perp_\omega}/TS$ carries a natural conformal symplectic structure induced by ω.
Since S is a sphere, the embedding $S^{k-1} \subset \mathbb{R}^k$ provides a natural trivialization of the bundle $\mathbb{R}R_\lambda \oplus T^*S$. This trivialization together with a conformally symplectic trivialization of $CNS(S)$ specifies a standard framing for S in N. Note that we have to assume that $CNS(S)$ is trivializable. This holds certainly true for $S = S^0 = \{N, S\}$ (two points) or $S = S^{n-1}$. In the latter case we have $(TS)^{\perp_\omega} = TS$ and hence $CNS(S) = (0)$. Therefore, taking connected sums and surgery along Legendrian spheres is always possible.
Following Weinstein, we define an isotropic setup as a quintuple $(P, \omega, Y, \Sigma, S)$, where (P, ω) is a symplectic manifold, Y a Liouville vector field for ω, Σ a hypersurface transverse to Y (so Σ is contact) and S an isotropic submanifold of Σ. In [17], Weinstein proves the following variant of his famous neighborhood theorem for isotropic manifolds:

Proposition 6 (Weinstein). *Let $(P_0, \omega_0, Y_0, \Sigma_0, S_0)$ and $(P_1, \omega_1, Y_1, \Sigma_1, S_1)$ be two isotropic setups. Given a diffeomorphism from S_0 to S_1 covered by an isomorphism of their symplectic subnormal bundles, there exist neighborhoods U_j of S_j in P_j and an isomorphism of isotropic setups*

$$\phi : (U_0, \omega_0, Y_0, \Sigma_0 \cap U, S_0) \to (U_1, \omega_1, Y_1, \Sigma_1 \cap U_1, S_1)$$

which restricts to the given mappings on S_0.

We may now define contact surgery along an isotropic sphere as follows:

Let $H_k^{2n} \approx D^k \times D^{2n-k}$ be a symplectic standard handle (see 2.2) and let S^{k-1} be an isotropic sphere in a contact manifold (N^{2n-1}, ξ). Then, Proposition 6 allows us to glue the (lower) boundary $S^k \times D^{2n-k}$ of H_k^{2n} to the symplectization $N \times [0,1]$ along the boundary part $U_1 \cap N \times [0,1]$ of a tubular neighborhood U_1 of $S \times \{1\}$ (see Figure 4). We obtain an exact symplectic manifold $P := N \times [0,1] \cup_S H_k^{2n}$ with a Liouville vector field Y which is on $N \times [0,1]$ simply $\frac{\partial}{\partial t}$, where t denotes the coordinate on $[0,1]$. Note that Y points inwards along $\partial^- P := N \times \{0\}$ and outwards along the other boundary component $\partial^+ P$. Both manifolds are hence contact and $\partial^+ P$ is obtained from N by surgery along S. Moreover, P is an exact symplectic cobordism between $\partial^- P$ and $\partial^+ P$.

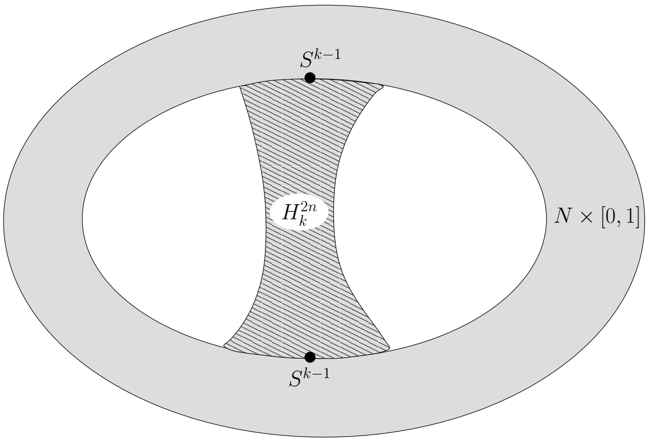

Fig. 4: $N \times [0,1]$ with handle attached

2.2 A standard handle

In order to specify a standard handle H_k^{2n}, we consider $\mathbb{R}^{2n}$ with symplectic coordinates $(q,p) = (q_1, p_1, \dots, q_n, p_n)$ and the following Weinstein structure (cf. [17]):

$$
\begin{aligned}
\lambda &:= \sum_{j=1}^{k} (2q_j dp_j + p_j dq_j) + \sum_{j=k+1}^{n} \frac{1}{2} (q_j dp_j - p_j dq_j) \\
d\lambda = \omega &:= \sum_{j=1}^{n} dq_j \wedge dp_j \\
Y &:= \sum_{j=1}^{k} \left(2q_j \frac{\partial}{\partial q_j} - p_j \frac{\partial}{\partial p_j} \right) + \sum_{j=k+1}^{n} \frac{1}{2} \left(q_j \frac{\partial}{\partial q_j} + p_j \frac{\partial}{\partial p_j} \right) \\
\phi &:= \sum_{j=1}^{k} \left(q_j^2 - \frac{1}{2} p_j^2 \right) + \sum_{j=k+1}^{n} \frac{A_j}{2} \left(q_j^2 + p_j^2 \right), \qquad A_j > 0 \text{ const.}
\end{aligned}
$$

Note that Y is in fact the Liouville vector field for λ, as $\iota_Y \omega = \lambda$. Moreover, observe that $(Y \cdot \phi)(q,p) > 0$ for $(q,p) \neq (0,0)$.

Let us introduce furthermore the following three functions:

$$x := \sum_{j=1}^{k} q_j^2 \qquad y := \sum_{j=1}^{k} \frac{1}{2} p_j^2 \qquad z := \sum_{j=k+1}^{n} \frac{A_j}{2} \left(q_j^2 + p_j^2 \right),$$

whose Hamiltonian vector fields are given by

$$X_x = \sum_{j=1}^{k} 2q_j \frac{\partial}{\partial p_j}, \qquad X_y = \sum_{j=1}^{k} -p_j \frac{\partial}{\partial q_j}, \qquad X_z = \sum_{j=k+1}^{n} A_j \left(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j} \right).$$

This convention allows us to write $\phi = x - y + z$ and $X_\phi = X_x - X_y + X_z$.
Now, consider the level surface $\Sigma^- := \{\phi = -1\}$ and note that Y is transverse to Σ^-, as $Y \cdot \phi|_{\Sigma^-} > 0$. Hence, $\lambda|_{T\Sigma^-}$ is a contact form. The set $S := \{x = z = 0,\ y = +1\}$ is an isotropic sphere in Σ^- and the quintuple $(\mathbb{R}^{2n}, \omega, Y, \Sigma^-, S)$ will be the isotropic setup where we glue H_k^{2n} to a contact manifold. To specify a handle H_k^{2n}, we choose a different Weinstein function ψ on $\mathbb{R}^{2n}$ such that the following holds:

(ψ1) $X_\psi = C_x \cdot X_x + C_y \cdot X_y + C_z \cdot X_z$,

where C_x, C_Y, C_z are smooth functions on $\mathbb{R}^{2n}$ such that $C_x, C_z > 0,\ C_y < 0$.

(ψ2) $\psi = \phi$ for $y > 1 + \varepsilon$ with ε arbitrarily small.

(ψ3) The closure $\overline{\{\psi < -1\} \cap \{\phi > -1\}}$ is diffeomorphic to $\overline{D^k \times D^{2n-k}}$.

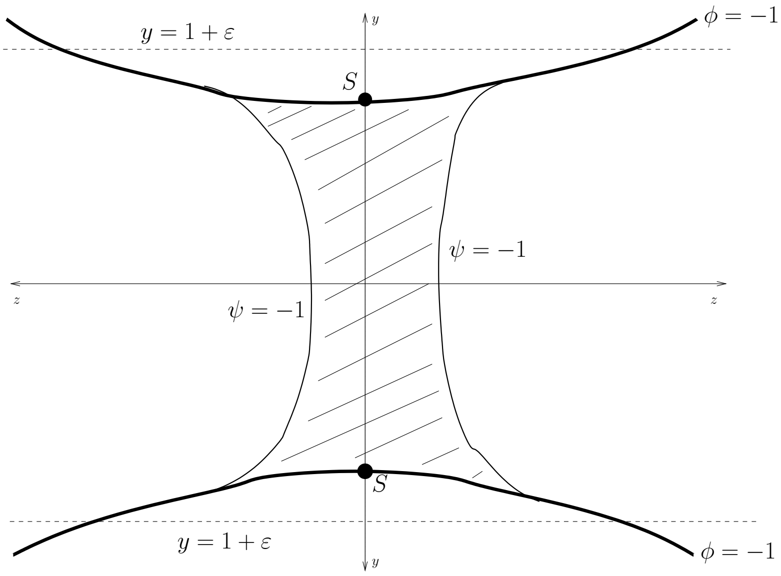

Fig. 5: The handle H_k^{2n}

The handle is then defined as $H_k^{2n} := \overline{\{\psi < -1\} \cap \{\phi > -1\}}$ (see Fig. 5).

Remark.

- If $\psi(0) \neq -1$, it follows from (ψ1) that the level sets $\Sigma^+ := \{\psi = -1\}$ and $\Sigma^- = \{\phi = -1\}$ are both contact hypersurfaces, as $Y \cdot \psi > 0$ away from 0. They coincide for $y \geq 1+\varepsilon$ due to (ψ2) and they contain the boundary of H_k^{2n}. Condition (ψ3) on the other hand assures that Σ^+ is obtained from Σ^- by surgery along S.

- Condition (ψ1) is automatically satisfied if $\psi(p,q) = \psi(x,y,z)$ is given as a function on x, y, z such that $\partial_x\psi, \partial_z\psi > 0$ and $\partial_y\psi < 0$. The Hamiltonian vector field X_ψ of ψ is then given by
$$X_\psi = \left(\frac{\partial\psi}{\partial x}\cdot X_x + \frac{\partial\psi}{\partial y}\cdot X_y + \frac{\partial\psi}{\partial z}\cdot X_z\right).$$

- It follows from (ψ2) that reducing ε makes the handle thinner. Note that one can always choose ε so small, that the handle becomes so thin that its "lower" boundary $\{\phi = -1\} \cap H_k^{2n}$ lies inside any prescribed neighborhood of S.

- The handle stays unchanged if we take $\phi' = \alpha\cdot\phi + \beta$ and $\psi' = \alpha\cdot\psi + \beta$, $\alpha > 0$, provided that we set $\quad H_k^{2n} = \overline{\{\psi' < -\alpha+\beta\} \cap \{\phi' > -\alpha+\beta\}}$.

Discussion 7. Consider the Lyapunov function $f := \sum_{j=1}^k q_j p_j$. Note that ($\psi$1) implies that $X_\psi \cdot f > 0$ away from the critical points of f, which shows that all periodic orbits of X_ψ are contained in the set $\{x = y = 0\}$. The same holds true if we consider $\psi' = \alpha\cdot\psi + \beta, \alpha > 0$, instead.

2.3 An explicit Hamiltonian on and near the handle

It is not difficult to find a Weinstein function $\psi : \mathbb{R}^{2n} \to \mathbb{R}$ which satisfies (ψ1)–(ψ3). Fix $\varepsilon > 0$ and choose a smooth, monotone function $g : \mathbb{R} \to [0,1]$ such that

$$g(t) = \begin{cases} 0 & \text{for} \quad t \leq 0 \\ 1 & \text{for} \quad t \geq 1+\varepsilon \end{cases} \qquad \text{and} \qquad 0 \leq g'(t) < \frac{1}{1+\varepsilon/2}.$$

Then set $\quad \psi := x - y + z - (1+\varepsilon/2) + (1+\varepsilon/2)\cdot g(y). \qquad (2)$

In Section 1.3, we want to use ψ as an admissible Hamiltonian which allows us to compare the symplectic homologies of a Liouville domain W bounded by Σ^- and $W \cup H_k^{2n}$ bounded by Σ^+. We do this by extending an admissible Hamiltonian on W over H_k^{2n} via ψ, where we take great care not to create new 1-periodic orbits away from H_k^{2n}. Hence, we need that ψ is linear, i.e. of the form $\psi = \alpha\cdot e^r + \beta$, on regions which are identified with the symplectizations of $\Sigma^\pm$.

To be more precise, let $\Sigma^- = \{\phi = -1\}$ and $\Sigma^+ = \{\psi = -1\}$ be as above. As both are hypersurfaces transversal to the Liouville vector field Y, the flow φ of Y provides symplectic embeddings $\Phi^\pm$ of the symplectizations of $\Sigma^\pm$ into $\mathbb{R}^{2n}$:

$$\Phi^\pm : \Sigma^\pm \times \mathbb{R} \to \mathbb{R}^{2n}, \qquad \Phi^\pm(y,r) = \varphi^r(y).$$

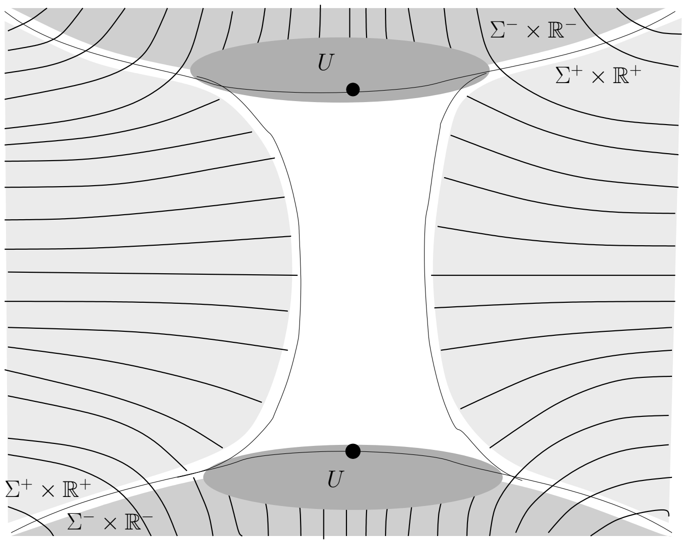

Fig. 6: The symplectizations of $\Sigma^\pm$ and areas where ψ is linear

On a symplectization $(\Sigma \times \mathbb{R}, d(e^r\lambda))$ of a contact manifold we define a function $\tilde{h}_\Sigma$ by

$$\tilde{h}_\Sigma(y,r) := \alpha \cdot e^r + \beta, \quad \alpha, \beta \in \mathbb{R}.$$

We call such a function linear (in fact it is linear when using the coordinate $z := e^r$). Observe that the Hamiltonian vector field of $\tilde{h}_\Sigma$ is given by $X_{\tilde{h}_\Sigma}(s,t) = \alpha \cdot R_\lambda(s)$, where R_λ is the Reeb vector field of λ, the contact form on Σ.
For $\Sigma^\pm$ we choose explicitly $\alpha = 1, \beta = -2$, such that $\tilde{h}^\pm_\Sigma(\Sigma^\pm \times \{0\}) = -1$, and let

$$h^\pm_\Sigma : \Phi^\pm(\Sigma^\pm \times \mathbb{R}) \to \mathbb{R}, \qquad h^\pm_\Sigma := \tilde{h}_{\Sigma^\pm} \circ (\Phi^\pm)^{-1}$$

be their pushforward onto $\Phi^\pm(\Sigma^\pm \times \mathbb{R}) \subset \mathbb{R}^{2n}$. Note that h^+_Σ and h^-_Σ coincide on $\Phi^\pm((\Sigma^+ \cap \Sigma^-) \times \mathbb{R})$, as $\Phi^- = \Phi^+$ on $(\Sigma^- \cap \Sigma^+) \times \mathbb{R}$. For the comparison of the symplectic (co)homologies of W and $W \cup H^{2n}_k$, we need a Hamiltonian that is linear on the negative symplectization of Σ^- and the positive symplectization of Σ^+. As ψ will serve as such a Hamiltonian, we require that $\psi = h^+_\Sigma$ on $\{\psi \geq -1\}$ and $\psi = h^-_\Sigma$ on $\{\phi \leq -1\} \setminus U$, where U is a compact neighborhood of $S = \{x = z = 0,\ y = 1\}$ (see Figure 6).

Discussion 8. It is the extension of ψ beyond the handle, that is not quite correct in [3]: It is stated there that one can extend ψ on the positive symplectization of Σ^+ such that the only 1-periodic orbit of X_ψ on the handle is the constant orbit at the origin. It is suggested that this can be done such that:

(A1) $\psi = \alpha \cdot e^r + \beta,\ \alpha \notin spec(\Sigma^\pm)$ on $\Phi^+\big(\Sigma^+ \times [0,\infty)\big)$ and $\Phi^-\big(\Sigma^- \times (-\infty, 0]\big)$ except a small neighborhood of S,

(A2) ψ is increasing for $y \to 0$ on $\{x = z = 0\}$,

(A3) $\psi = \alpha \cdot e^r + \beta$ on $\{x = z = 0\} \subset \Phi^+\big(\Sigma^+ \times [0,\infty)\big)$.

Assumption (A2) is a consequence of (ψ1), while (A3) guarantees that X_ψ has only the origin as 1-periodic orbit (as $\alpha \notin spec(\Sigma^\pm)$ and as all 1-periodic orbits lie in $\{x = y = 0\}$). However, all 3 assumptions cannot be satisfied simultaneously. To see this consider a path in $\mathbb{R}^{2n}$ as depicted in Figure 7.

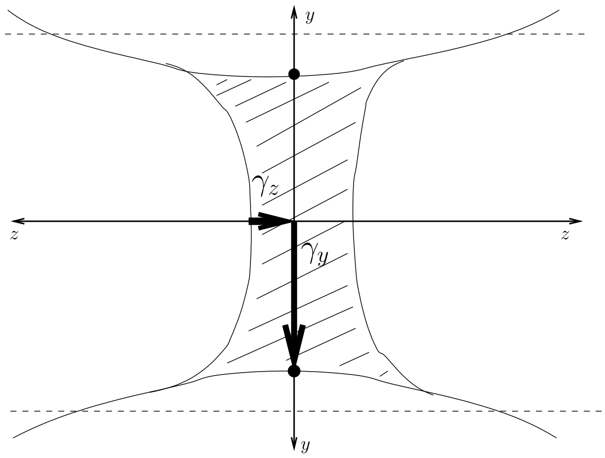

Fig. 7: The problematic path

Here, path γ_z is an integral curve of Y in $\{x = y = 0\}$. As the origin is the only zero of Y, it follows that $\lim_{t\to-\infty} \gamma_z(t) = 0$. By (A3) and the continuity of ψ, we find

$$\psi(0) = \lim_{t\to-\infty} \alpha \cdot e^{r(\gamma_z(t))} + \beta = \beta.$$

With (A2), we find then that $\psi \leq \beta$ on $\{x = z = 0\}$ (on the path γ_y). For $x = z = 0$ and $y \gg 1$ however, we are on $\Phi^-\big(\Sigma^- \times (-\infty, 0]\big)$ and require by (A1) that $\psi = \alpha \cdot e^r + \beta$, which gives on this region the contradiction $\psi = \alpha \cdot e^r + \beta > \beta \geq \psi$.
Our solution to this dilemma is to omit assumption (A3) and to allow ψ to have varying slope α and constant β on $\{x = y = 0\}$, first letting it grow very slowly coming from the origin and increasing the slope sharply near Σ^+.
This creates more than one 1-periodic Hamiltonian orbit on the handle. However, using property (ψ1) and the Lyapunov function f as in Discussion 7, we can show that these 1-periodic X_ψ-orbits stay all in the set $\{x = y = 0\}$. Moreover, they can be described explicitly and are hence manageable.

For the construction of such a ψ, we need the following two technical lemma:

Lemma 9. *Consider $\mathbb{R}^{2n}$ with the standard symplectic structure, the Liouville vector field Y and the functions x, y, z as given in 2.2. Let $\Sigma \subset \mathbb{R}^{2n}$ be a smooth hypersurface transverse to the Liouville vector field Y (i.e. Σ is contact) such that its Reeb vector field R is of the form*

$$R = c_x \cdot X_x + c_y \cdot X_y + c_z \cdot X_z,$$

where $c_x, c_y, c_z : \Sigma \to \mathbb{R}$ are smooth functions with $c_x, c_z > 0, c_y < 0$ and X_x, X_y, X_z are the Hamiltonian vector fields of x, y, z. Let $\tilde{h}_\Sigma$ denote the function $\tilde{h}_\Sigma(y, r) = \alpha \cdot e^r + \beta$

on the symplectization of Σ and let $h_\Sigma := \tilde{h}_\Sigma \circ \Phi^{-1}$ be its pushforward onto $\mathbb{R}^{2n}$ by the symplectic embedding $\Phi : \Sigma \times \mathbb{R} \to \mathbb{R}^{2n}$ provided by the flow φ of Y. Then, the Hamiltonian vector field X_h of h_Σ is of the form

$$X_h = C_x \cdot X_x + C_y \cdot X_y + C_z \cdot X_z,$$

where $C_x, C_y, C_z \in C^\infty(\mathbb{R}^{2n})$ are functions satisfying $C_x, C_z > 0,\ C_y < 0$.

Remark. The assumptions on Σ are satisfied, if $\Sigma = \psi^{-1}(c)$ for a function ψ on x, y, z with $\partial_x\psi\big|_\Sigma, \partial_z\psi\big|_\Sigma > 0$ and $\partial_y\psi\big|_\Sigma < 0$ and $0 \notin \Sigma$.

Proof: As $X_{\tilde{h}} = \alpha \cdot R$ on $\Sigma \times \mathbb{R}$, it follows that on $\mathbb{R}^{2n}$ holds $X_h|_{\varphi^t(\Sigma)} = \alpha \cdot e^t \cdot R_t$, where R_t is the Reeb vector field on $\varphi^t(\Sigma)$. Recall the definitions of x, y, z and X_x, X_y, X_z from 2.2. By assumption, the Reeb vector field R on Σ satisfies

$$R = c_x X_x + c_y X_y + c_z X_z = \sum_{j=1}^{k} \left(c_x\, 2q_j \frac{\partial}{\partial p_j} - c_y\, p_j \frac{\partial}{\partial q_j} \right) + c_z \sum_{j=k+1}^{n} \left(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j} \right).$$

Moreover, Y is given by

$$Y = \sum_{j=1}^{k} \left(2q_j \frac{\partial}{\partial q_j} - p_j \frac{\partial}{\partial p_j} \right) + \sum_{j=k+1}^{n} \frac{1}{2} \left(q_j \frac{\partial}{\partial q_j} + p_j \frac{\partial}{\partial p_j} \right).$$

The flow φ^t of Y is hence given by

$$\varphi^t(q,p) = \Big(\underbrace{\dots, e^{2t} \cdot q_j, e^{-t} \cdot p_j, \dots}_{j=1,\dots,k}, \underbrace{\dots, e^{t/2} \cdot q_j, e^{t/2} \cdot p_j, \dots}_{j=k+1,\dots,n} \Big).$$

As $\mathcal{L}_Y \lambda = \lambda$ and $\mathcal{L}_Y \omega = \omega$, we find for R and any $\xi \in T_{\varphi^t(p)}\varphi^t(\Sigma)$ that

$$\begin{aligned} \lambda_{\varphi^t(p)}\big(D\varphi^t_p R\big) &= \big(\varphi^{t*}\lambda\big)_p(R) &&= e^t \cdot \lambda_p(R) &&= e^t, \\ \omega_{\varphi^t(p)}\big(D\varphi^t_p R\,,\, \xi\big) &= \big(\varphi^{t*}\omega\big)_p\big(R, (D\varphi^t_p)^{-1}(\xi)\big) &&= e^t \cdot \omega_p\big(R, (D\varphi^t_p)^{-1}(\xi)\big) &&= 0, \end{aligned}$$

as R is the Reeb vector field and $(D\varphi^t_p)^{-1}(\xi) \in T\Sigma$. This shows that $e^{-t} \cdot D\varphi^t R$ is the Reeb vector field R_t of $\varphi^t(\Sigma)$. Hence we find for X_h that

$$X_h|_{\varphi^t(\Sigma)} = \alpha \cdot e^t \cdot R_t = \alpha \cdot D\varphi^t(R) = \alpha e^{-t} c_x X_x + \alpha e^{2t} c_y X_y + \alpha e^{t/2} c_z X_z,$$

which is exactly of the announced form. □

Lemma 10. *Let (Σ, α) be a contact manifold with contact form α and symplectization $\big(\Sigma \times \mathbb{R}, d(e^r\alpha)\big)$. Let $\varepsilon, \delta, c > 0$ be constants and let J be an almost complex structure[5] compatible with $\omega = d(e^r\alpha)$ such that the norm $||R||$ of the Reeb vector field R satisfies*

$$\sup_{y \in \Sigma} ||R(y)|| = \sup_{y \in \Sigma} \sqrt{\omega\big(R(y), JR(y)\big)} =: d < \infty.$$

[5]For example, any cylindrical J would do, i.e. if $d(e^r) \circ J = -\lambda$.

Then there exists a smooth monotone function $g : \mathbb{R} \to [0,1]$ such that

$$g(e^r) = 0 \quad \textit{for} \quad r \le -\varepsilon \qquad \textit{and} \qquad g(e^r) = 1 \quad \textit{for} \quad r \ge 0 \tag{$*$}$$

and for all $\phi, \psi \in C^1(\Sigma \times \mathbb{R})$ with $\phi|_{\Sigma\times\{0\}} = \psi|_{\Sigma\times\{0\}}$ and $|\partial_r\phi(y,r) - \partial_r\psi(y,r)| < c$ for all $(y,r) \in \Sigma \times [-\varepsilon, 0]$, holds for their Hamiltonian vector fields X_ϕ, X_ψ that

$$\sup_{(y,r)\in\Sigma\times\mathbb{R}} \Big|\Big| X_{\phi+(\psi-\phi)g}(y,r) - \Big(X_\phi(y,r) + \big(X_\psi(y,r) - X_\phi(y,r)\big)\cdot g(e^r)\Big)\Big|\Big| \le \delta. \tag{$**$}$$

In other words, we can interpolate between ϕ and ψ along $\Sigma \times [-\varepsilon, 0]$, such that the Hamiltonian vector field $X_{\phi+(\psi-\phi)g}$ of the interpolation is arbitrary close to the interpolation of the Hamiltonian vector fields X_ϕ and X_ψ.

Proof: As the Hamiltonian vector field of e^r is the Reeb vector field R, we calculate

$$X_{\phi+(\psi-\phi)g}(y,r) = X_\phi(y,r) + \big(X_\psi - X_\phi\big)(y,r)\cdot g(e^r) + (\psi-\phi)(y,r)\cdot g'(e^r)\cdot R(y).$$

Therefore, $(**)$ translates to

$$||(\psi-\phi)(y,r)\cdot g'(e^r)\cdot R(y)|| \le \delta \qquad \forall (y,r) \in \Sigma\times[-\varepsilon,0].$$

Using $\phi|_{\Sigma\times\{0\}} = \psi|_{\Sigma\times\{0\}}$, we can estimate the left hand side as follows:

$$\begin{aligned} ||(\psi-\phi)(y,r)\cdot g'(e^r)\cdot R(y)|| &= \Big|\Big| - \int_r^0 \partial_s(\psi-\phi)(y,s)\,ds \cdot g'(e^r)\cdot R(y)\Big|\Big| \\ &\le c\cdot(-r)\cdot g'(e^r)\cdot d. \end{aligned}$$

If we write $z = e^r$, we find that $(**)$ is satisfied, if $0 \le g'(z) \le \frac{-\delta}{cd\log z}\ \forall\, z \in [e^{-\varepsilon}, 1]$. As $\int_{e^{-\varepsilon}}^1 \frac{-\delta}{cd\log z}\,dz = \infty$, we can choose a smooth function $\tilde g$ satisfying

$$0 \le \tilde g(z) \le \frac{-\delta}{cd\cdot\log z}, \qquad \tilde g \equiv 0 \quad \text{for} \quad z \le e^{-\varepsilon} \quad \text{or} \quad z \ge 1 \qquad \text{and} \qquad \int_{e^{-\varepsilon}}^1 \tilde g(z)\,dz = 1.$$

Setting $g(e^r) = g(z) := \int_{e^{-\varepsilon}}^z \tilde g(s)\,ds$ then gives the desired function. □

Now, we construct ψ in two steps, first defining ψ on $\Phi^-\big(\Sigma^-\times(-\infty,0]\big)\cup H_k^{2n}$ and then extending it to $\Phi^+\big(\Sigma^+\times[0,\infty)\big)$.

- Step 1: Recall that the isotropic sphere $S \subset \Sigma^- = \{\phi = -1\}$ is given by

$$S := \{x = z = 0,\ y = 1\}.$$

Consider the function h_Σ^- as defined on page 19. As the Reeb vector field R_{Σ^-} of $(\Sigma^-, \lambda|_{T\Sigma^-})$ coincides with the Hamiltonian vector field X_ϕ on S, we find $X_{h_\Sigma^-} = R_{\Sigma^-} = X_\phi$ and hence $dh_\Sigma^- = d\phi$ on S. As also $h_\Sigma^-(\Sigma^-) = \phi(\Sigma^-) = -1$, we find that h_Σ^- and ϕ coincide up to first order on S. Therefore, given any neighborhood U of

S, there exists a function $\hat{\phi}$ of x, y, z and a neighborhood $\hat{U} \subset U$, such that $\hat{\phi} \equiv h_\Sigma^-$ on $\mathbb{R}^{2n} \setminus U$, $\hat{\phi} \equiv \phi$ on $\hat{U}$ and $\hat{\phi}$ is arbitrarily C^1-close to h_Σ^-. As $X_\phi = X_x - X_y + X_z$ and $X_{h_\Sigma^-} = C_x^- \cdot X_x + C_y^- \cdot X_y + C_z \cdot X_z$ with $C_x^-, C_z^- > 0, C_y^- < 0$ by Lemma 9, we can additionally arrange that

$$X_{\hat{\phi}} = \hat{C}_x \cdot X_x + \hat{C}_y \cdot X_y + \hat{C}_z \cdot X_z \quad \text{with} \quad \hat{C}_x, \hat{C}_z > 0,\ \hat{C}_y < 0.$$

Let H_k^{2n} be defined by a function $\tilde{\psi}$ as in (2) and so thin, such that the lower boundary $H_k^{2n} \cap \Sigma^-$ lies in $\hat{U}$. Then set

$$\hat{\psi} : \{\phi \le -1\} \cup H_k^{2n} \to \mathbb{R} \qquad\qquad \hat{\psi} = \begin{cases} \tilde{\psi} & \text{on } \big(\hat{U} \cap \{\phi \le -1\}\big) \cup H_k^{2n} \\ \hat{\phi} & \text{on } \big(U \cap \{\phi \le -1\}\big) \setminus \hat{U} \\ h_\Sigma^- & \text{on } \{\phi \le -1\} \setminus U \end{cases}.$$

Since $\tilde{\psi} = \hat{\phi}$ outside a small neighborhood of H_k^{2n} and $\hat{\phi} = h_\Sigma^-$ outside U, we find that $\hat{\psi}$ is smooth on its domain. Moreover, as $\tilde{\psi}, \hat{\phi}$ and h_Σ^- satisfy ($\psi 1$), so does $\hat{\psi}$. See Figure 8 for the areas where $\hat{\psi}$ is defined.

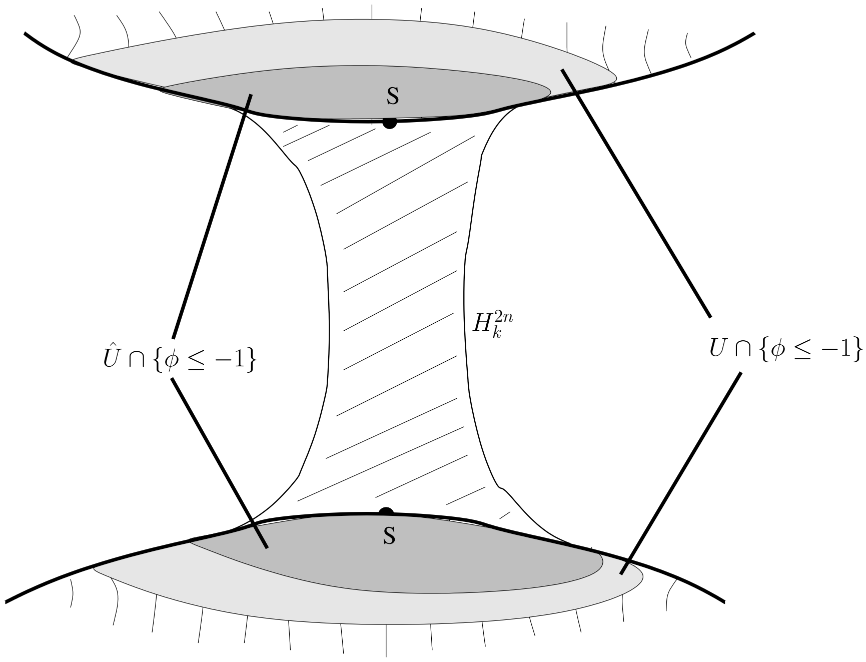

Fig. 8: Areas, where $\hat{\psi}$ is defined

- Step 2 Consider the function h_Σ^+ associated to $\Sigma^+ = \{\hat{\psi} = -1\}$ as defined on page 19. We will now define ψ as an interpolation between $\hat{\psi}$ and h_Σ^+, i.e.

$$\psi := \hat{\psi} + (h_\Sigma^+ - \hat{\psi}) \cdot g(h_\Sigma^+), \tag{3}$$

where g is a function given by Lemma 10 such that for $\varepsilon > 0$ small holds

$$g(e^r-2)=0 \quad \text{for} \quad r\le -\varepsilon \qquad \text{and} \qquad g(e^r-2)=1 \quad \text{for} \quad r\ge 0, \qquad (*)$$

and such that ψ satisfies $(\psi 1)$, i.e. $X_\psi = C_x \cdot X_x + C_y \cdot X_y + C_z \cdot X_z$, where $C_x, C_z, C_y \in C^\infty(\mathbb{R}^{2n})$ and $C_x, C_z > 0$, $C_y < 0$.
For the following definitions of the areas $A, B, B^\pm$ and B' see Figure 9. Recall that $\Sigma^+ = \Sigma^-$ outside U, so that $h^+_\Sigma = h^-_\Sigma = \hat{\psi}$ on $A := \Phi\big((\Sigma^- \setminus U) \times \mathbb{R}\big)$. Consequently, we have by (3) that $\psi = h^+_\Sigma$ on A for any g, so that $(\psi 1)$ holds there by Lemma 9.
Now consider $B := \Phi\big(\overline{\Sigma^+ \cap (H^{2n}_k \cup U)} \times \mathbb{R}\big)$ with the following subsets

$$B^\pm := \Phi\big(\overline{\Sigma^+ \cap (H^{2n}_k \cup U)} \times \mathbb{R}^\pm\big),$$
$$B' := \Phi\big(\overline{\Sigma^+ \cap (H^{2n}_k \cup U)} \times [-\varepsilon, 0]\big) \subset B^-.$$

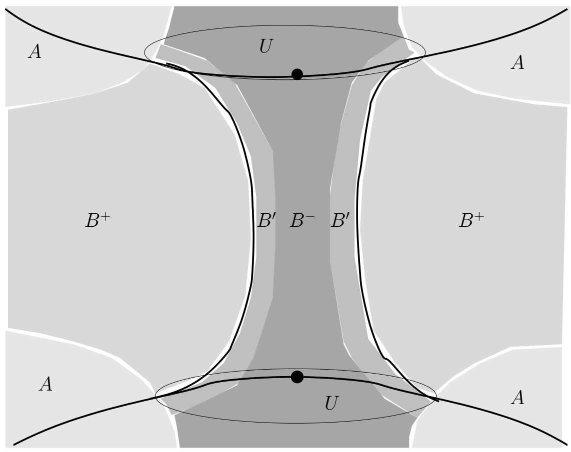

Fig. 9: Interpolation areas

Note that B' is compact so that the following quantities are finite:

$$c := \sup_{p\in B'} \big|Y\cdot \hat{\psi}(p) - Y \cdot h^+_\Sigma(p)\big| < \infty, \qquad d := \sup_{p\in \Sigma^+\cap B'} \big|\big|R(p)\big|\big| < \infty.$$

Here, R is the Reeb vector field on Σ^+ and Y the Liouville vector field which is also the vector field corresponding to the partial derivative ∂_r in the cylindrical coordinates (y, r) on the symplectization $\Sigma^+ \times \mathbb{R}$. As $h^+_\Sigma = e^r - 2$ in these coordinates we find by Lemma 10 for $\delta > 0$ arbitrary a function g satisfying $(*)$ such that with ψ given by (3), we have

$$\sup_{p\in B'} \Big|\Big| X_\psi(p) - \Big(X_{\hat{\psi}}(p) + \big(X_{h^+_\Sigma}(p) - X_{\hat{\psi}}(p)\big)\cdot g(h^+_\Sigma(p))\Big)\Big|\Big| \le \delta, . \qquad (**)$$

Note that $X_{g(h^+_\Sigma)} = g'(h^+_\Sigma)\cdot X_{h^+_\Sigma}$ and that by Step 1 and Lemma 9 holds

$$X_{\hat{\psi}} = \hat{C}_x \cdot X_x + \hat{C}_y \cdot X_y + \hat{C}_z \cdot X_z$$
$$X_{h^+_\Sigma} = C^+_x \cdot X_x + C^+_y \cdot X_y + C^+_z \cdot X_z,$$

where all coefficient functions satisfy $(\psi 1)$.

As $X_\psi = X_{\hat\psi}$ on $B^- \setminus B'$ and $X_\psi = X_{h^+_\Sigma}$ on B^+, we find with $(**)$ for δ small enough that ψ satisfies $(\psi 1)$ also on B and hence everywhere.

2.4 Closed orbits and Conley-Zehnder indices

Set $\psi' := \alpha \cdot \psi + \beta$. In the following we will determine all 1-periodic orbits of $X_{\psi'}$ near the handle H^{2n}_k and calculate their Conley-Zehnder indices. Since ψ satisfies $(\psi 1)$ and consequently also ψ', Discussion 7 guarantees that the only periodic orbits of $X_{\psi'}$ near the handle lie in $\{x = y = 0\}$. By $(\psi 1)$, the Hamiltonian vector field X_ψ is given by

$$X_\psi = \alpha \cdot \big(C_x X_x + C_y X_y + C_z X_z\big),$$

where $C_x, C_y, C_z \in C^\infty(\mathbb{R}^{2n})$ are functions with $C_x, C_z > 0$, $C_y < 0$ and X_x, X_y and X_z are the Hamiltonian vector fields of the functions x, y, z and given by

$$X_x = \sum_{j=1}^{k} 2q_j \frac{\partial}{\partial p_j}, \qquad X_y = \sum_{j=1}^{k} -p_j \frac{\partial}{\partial q_j}, \qquad X_z = \sum_{j=k+1}^{n} A_j \left(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j} \right).$$

On $\big(\Sigma^+ \times [0,\infty)\big) \cap \{x = y = 0\}$, we have by our construction $\psi = h^+_\Sigma$ and hence that X_ψ equals R, the Reeb vector field of Σ^+. To calculate R on $\Sigma^+ \cap \{x = y = 0\}$ note that $\lambda(X_z) = z$ and $\omega(X_z, \cdot) = -dz = -d\tilde\psi$ on $\Sigma^+ \cap \{x = y = 0\}$. The value of z on $\Sigma^+ \cap \{x = y = 0\} = \tilde\psi^{-1}(-1) \cap \{x = y = 0\}$ is given by (2) as

$$-1 = 0 - 0 + z - (1 + \varepsilon/2) \qquad \Leftrightarrow \qquad z = \varepsilon/2.$$

Hence we find that R on $\Sigma^+ \cap \{x = y = 0\}$ is given by

$$R = \frac{2}{\varepsilon} X_z = \frac{2}{\varepsilon} \cdot \sum_{j=k+1}^{n} A_j \left(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j} \right).$$

Still for $x = y = 0$ and z slightly smaller then $\varepsilon/2$ on the other hand, we have by the construction of ψ that $\psi = \hat\psi = \tilde\psi$. Hence (2) gives $\psi = \hat\psi = \tilde\psi = z - (1 + \varepsilon/2)$ and thus $X_\psi = X_z$ on $\{x = y = 0, z < \varepsilon/2\}$. As ψ is a convex interpolation between $\hat\psi$ and h^+_Σ it follows that on $\{x = y = 0\}$ we have $X_\psi = C_z X_z$, where C_z is a z-dependent interpolation between the constants 1 and $2/\varepsilon$. It follows that $X_{\psi'}$ on $\{x = y = 0\}$ is given by

$$X_{\psi'} = \alpha C_z \cdot X_z = \alpha C_z \sum_{j=k+1}^{n} A_j \left(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j} \right).$$

Now we can calculate the flow φ^t of $X_{\psi'}$ on $\{x = y = 0\}$. First we calculate for $z(\varphi^t(p,q))$

$$\frac{d}{dt} z\big(\varphi^t\big) = dz(X_{\psi'}) = \alpha \cdot C_z\, dz(X_z) = 0.$$

It follows that z is constant along the flow lines of φ. Now, consider for $j = k+1, \dots, n$ the complex coordinates $z_j = q_j + ip_j$. Then, we have on $\{x = y = 0\}$:

$$X_{\psi'} = \alpha C_z \cdot \Bigg(\underbrace{0, \dots, 0}_{j=1,\dots,k}, \underbrace{\dots, iA_j \cdot z_j, \dots}_{j=k+1,\dots,n} \Bigg).$$

As $z(\varphi^t)$ is independent from t and hence $\frac{d}{dt} C_z(z(\varphi^t)) = 0$, we obtain that the flow φ^t of $X_{\psi'}$ on $\{x = y = 0\}$ is given by

$$\varphi^t(0, z_{k+1}, \dots, z_n) = \Big(0, \dots, 0, \exp\big(i\alpha C_z A_{k+1} t\big) \cdot z_{k+1}, \dots, \exp\big(i\alpha C_z A_n t\big) \cdot z_n\Big). \qquad (4)$$

By choosing the constants A_j linear independent over $\mathbb{Q}$, we can arrange that the 1-periodic orbits of $X_{\psi'}$ on $\{x = y = 0\}$ are isolated. They are all of the form

$$\gamma(t) = \big(0, \dots, 0, e^{i\alpha C_z A_{j_0}} \cdot z_{j_0}, 0, \dots, 0\big)\,, \qquad (5)$$

where $\alpha C_z(\frac{A_{j_0}|z_{j_0}|^2}{2}) \in 2\pi\mathbb{Z}$. The only exception is the one constant orbit at the origin. For α appropriately chosen, we can assume that there are only finitely many such orbits.

Now, we want to calculate the Conley-Zehnder indices μ_{CZ} of these orbits. Recall that μ_{CZ} is a Maslov type index for paths Φ in the group of symplectic matrices $Sp(2n)$. By [13] the index is calculated as follows. Every smooth path $\Phi : [a, b] \to Sp(2n)$ can be uniquely expressed as a solution of an ODE of the form

$$\frac{d}{dt}\Phi(t) = J_0 S(t) \Phi(t), \qquad \Phi(a) \in Sp(2n),$$

where $t \mapsto S(t) = S(t)^T$ is a smooth path of symmetric matrices
A time t is called a crossing if $\det(\Phi(t) - \mathbb{1}) = 0$. The index μ_{CZ} is now the sum over all crossings t of the signatures of $S(t)$ restricted to $\ker(\Phi(t) - \mathbb{1})$. Note that if the end points a and b are crossings, then only half the signature is added (see [13] for details). It follows easily from this definition that $\mu_{CZ}(\Phi) = 0$ if $sign(S) = 0$ everywhere. Moreover, the Conley-Zehnder index of the path $\Phi : [0, T] \to Sp(2n)$, $\Phi(t) = e^{it}$ is given by

$$\mu_{CZ}(\Phi) = \left\lfloor \frac{T}{2\pi} \right\rfloor + \left\lceil \frac{T}{2\pi} \right\rceil.$$

Another important property of μ_{CZ} that we shall need is the additivity under products: If $Sp(2n) \oplus Sp(2n')$ is understood as the obvious subgroup of $Sp\big(2(n + n')\big)$, then

$$\mu_{CZ}(\Phi \oplus \Phi') = \mu_{CZ}(\Phi) + \mu_{CZ}(\Phi').$$

To calculate μ_{CZ} in our situation, let γ be a 1-periodic orbit of $X_{\psi'}$ as in (5). In order to calculate $\mu_{CZ}(\gamma)$, we identify $T_{\gamma(t)}\mathbb{R}^{2n}$ with $\mathbb{R}^{2n}$ in the natural way. This yields a path

Φ_ψ in $Sp(2n)$ given by $\Phi_\psi(t) = D\varphi^t(z^0)$. We differentiate Φ_ψ on $\{x = y = 0\}$ as

$$\begin{aligned}\frac{d}{dt}\Phi_\psi(t) &= \frac{d}{dt}D\varphi^t(z^0) = D\left(\frac{d}{dt}\varphi^t(z^0)\right) = DX_{\psi'}\big(\varphi^t(z^0)\big)\\ &= D\big(C_xX_x + C_yX_y + C_zX_z\big)\big(\varphi^t(z^0)\big)\\ = \alpha \cdot diag&\left(\underbrace{\begin{pmatrix} 0 & -C_y \\ 2C_x & 0\end{pmatrix}}_{j=1,\dots,k}, \underbrace{A_j\begin{pmatrix} 0 & -C_z \\ C_z & 0\end{pmatrix}}_{j>k, j\neq j_0}, A_{j_0}\begin{pmatrix} -C_z'q_{j_0}p_{j_0} & -C_z'q_{j_0}^2 - C_z \\ C_z'q_{j_0}p_{j_0} + C_z & C_z'p_{j_0}^2\end{pmatrix}\right) \circ \Phi_\psi(t).\end{aligned}$$

Here, we write $C_z' := \partial_z C_z$. Note that no derivatives of C_x, C_y, C_z are involved except for $j = j_0$, as $q_j = p_j = 0$ for $j \neq j_0$ along γ. It follows that Φ_ψ is of block form $\Phi_\psi = diag\big(\Phi_\psi^1, \dots, \Phi_\psi^n\big)$, where the paths of 2×2 matrices Φ_ψ^j are solutions of an ordinary differential equation with initial value $\Phi_\psi^j(0) = \mathbb{1}$ and

$$\begin{aligned}\tfrac{d}{dt}\Phi_\psi^j(t) &= \alpha\begin{pmatrix} 0 & -C_y \\ 2C_x & 0\end{pmatrix}\Phi_\psi^j(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\alpha\begin{pmatrix} 2C_x & 0 \\ 0 & C_y\end{pmatrix}\Phi_\psi^j(t) && j = 1, \dots, k\\ \tfrac{d}{dt}\Phi_\psi^j(t) &= \alpha A_j\begin{pmatrix} 0 & -C_z \\ C_z & 0\end{pmatrix}\Phi_\psi^j(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\alpha A_j\begin{pmatrix} C_z & 0 \\ 0 & C_z\end{pmatrix}\Phi_\psi^j(t) && j > k, j \neq j_0\\ \tfrac{d}{dt}\Phi_\psi^{j_0}(t) &= \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\alpha A_{j_0}\begin{pmatrix} C_z'q_{j_0}^2 + C_z & C_z'q_{j_0}p_{j_0} \\ C_z'q_{j_0}p_{j_0} & C_z'p_{j_0}^2 + C_z\end{pmatrix}\Phi_\psi^{j_0}(t) && j = j_0.\end{aligned}$$

As the matrix $\left(\begin{smallmatrix} 2\alpha C_x & 0 \\ 0 & \alpha C_y\end{smallmatrix}\right)$ has for all t one positive and one negative eigenvalue, it follows that its signature is always zero and hence by the Robbin-Salamon definition of μ_{CZ} that $\mu_{CZ}(\Phi_\psi^j) = 0,\ j = 1, \dots, k$. For $j > k,\ j \neq j_0$ on the other hand, we find by the explicit formula from above that

$$\mu_{CZ}\big(\Phi_\psi^j\big) = \left\lfloor\frac{\alpha A_j C_z}{2\pi}\right\rfloor + \left\lceil\frac{\alpha A_j C_z}{2\pi}\right\rceil, \qquad j > k,\ j \neq j_0.$$

For j_0 finally, it is not difficult to calculate, that the eigenvalues of the matrix are C_z and $C_z + C_z'(q_{j_0}^2 + p_{j_0}^2)$. So depending on C_z' it has either signature 2 or 0. Hence we find that $\mu_{CZ}(\Phi_\psi^{j_0}) \geq 0$ and therefore

$$\mu_{CZ}(\gamma) = \mu_{CZ}(\Phi_\psi) \geq \sum_{j>k, j\neq j_0}\left(\left\lfloor\frac{\alpha A_j C_z}{2\pi}\right\rfloor + \left\lceil\frac{\alpha A_j C_z}{2\pi}\right\rceil\right) \geq \frac{\alpha}{2\pi}\cdot\sum_{j>k, j\neq j_0} A_j, \tag{6}$$

as $1 \leq C_z \leq 2/\varepsilon$. If α tends to $+\infty$, we have therefore for all 1-periodic Hamiltonian orbits γ near the handle that $\mu_{CZ}(\gamma)$ becomes arbitrarily large.

2.5 Proof of the Main Theorem

Recall that we have in 1.6 constructed the transfer maps

$$\pi_*(W, V) : SH_*(V) \to SH_*(W) \quad \text{and} \quad \pi^*(W, V) : SH^*(W) \to SH^*(V),$$

for an exactly embedded subdomain $W \subset V$. This was done in Corollary 5 by showing the isomorphisms $SH_*(W) \cong SH_*^{>0}(W\subset V)$ and $SH^*(W) \cong SH^*_{>0}(W\subset V)$ and then applying the truncation maps $SH(V) \to SH_*^{>0}(W\subset V)$ and $SH^*_{>0}(W\subset V) \to SH^*(V)$. The main theorem, that we are going to prove now, states that the maps $\pi_*(W,V)$ and $\pi^*(W,V)$ are isomorphisms if V is obtained from W by attaching a subcritical handle.

Proof of Theorem 1.
The idea of the proof is to construct yet another cofinal sequence of Hamiltonians $(H_n) \subset Ad^w(V) \cap Ad(W\subset V)$ for which we can directly show that

$$SH_*(W) \overset{(*)}{\cong} SH_*^{>0}(W\subset V) \overset{(**)}{=} \lim_{n\to\infty} FH_*^{>0}(H_n) \overset{(1)}{\simeq} \lim_{n\to\infty} FH_*(H_n) \overset{(**)}{=} SH_*(V)$$
$$SH^*(W) \overset{(*)}{\cong} SH^*_{>0}(W\subset V) \overset{(**)}{=} \lim_{n\to\infty} FH^*_{>0}(H_n) \overset{(2)}{\simeq} \lim_{n\to\infty} FH^*(H_n) \overset{(**)}{=} SH^*(V).$$

Note that the identities $(**)$ are given by the construction of SH_* and SH^*, while the isomorphisms $(*)$ have been shown in Corollary 5.

To start the contruction fix sequences $k(n) \notin Spec(\partial W, \lambda)$, $k(n) \to \infty$ and $\varepsilon(n) \to 0$. Then choose an increasing sequence of non-degenerate Hamiltonians H_n on W that is on $\partial W \times (-\varepsilon(n), 0]$ of the form

$$H_n|_{\partial W \times (-\varepsilon(n),0]} = k(n) \cdot e^r - \big(1 + \varepsilon(n)\big)$$

and extend H_n over the handle by a function ψ with $\alpha = k(n)$ and $\beta = -1 - \varepsilon(n)$ as described in Section 2. For each n choose the handle so thin[6] such that each trajectory of X_{H_n} which leaves and reenters the handle has length greater than 1. Thus we obtain a cofinal weakly admissible sequence (H_n), whose 1-periodic orbits having positive action are all contained in W. Recall that we have the long exact sequences

$$\cdots \to FH_{j+1}^{>0}(H_n) \to FH_j^{\leq 0}(H_n) \to FH_j(H_n) \to FH_j^{>0}(H_n) \to \dots$$
$$\cdots \to FH^{j-1}_{\leq 0}(H_n) \to FH^j_{>0}(H_n) \to FH^j(H_n) \to FH^j_{\leq 0}(H_n) \to \dots$$

and note that $FH_j^{>0}(H_n)$ is generated by all 1-periodic orbits of H_n inside W, while $FH_j^{\leq 0}(H_n)$ is generated by all other orbits. The orbits of negative action all lie on the handle and are explicitly given in (4). Observe that H_n is on the handle time-independent. The orbits there are therefore of Morse-Bott type. We can now use either the definition of SH with Morse-Bott techniques, as described in [2], or perturb H_n near these orbits to make it non-degenerate, as described in [4]. In both cases we obtain for each orbit γ two generators in the chain complex whose indices are $\mu_{CZ}(\gamma)$ and $\mu_{CZ}(\gamma) + 1$. We have shown in Section 2.4 that the possible values of $\mu_{CZ}(\gamma)$ increase to ∞ as the slope $\alpha = k(n)$ tends to ∞. Therefore, $FH_j^{\leq 0}(H_n)$ becomes eventually zero for n large enough, as well as $FH_{j+1}^{\leq 0}(H_n)$. This implies for n large enough that

$$FH_j(H_n) \to FH_j^{>0}(H_n)$$

[6] Note that different choices of ψ give different handles H_k^{2n}, however the completions $\widehat{W\# H_k^{2n}}$ of the resulting symplectic manifolds are symplectomorphic.

is an isomorphism. As the direct limit is an exact functor, these maps converge to an isomorphism in the limit, proving (2). In the cohomology case, the line of arguments is the same. Even though taking inverse limits is not exact, it still takes the isomorphism

$$FH^j_{>0}(H_n) \to FH^j(H_n)$$

to an isomorphism in the limit, as it is left exact (see [8], Thm. 5.4 or [1], §6, no.3, prop. 4). This proves (5). $\square$

2.6 The invariance of Rabinowitz-Floer homology

Given a Liouville domain V, Rabinowitz-Floer homology $RFH_*(V, \partial V)$ was defined in [5] as a Floer-type homology associated to the Rabinowitz action functional

$$\mathcal{A}^H_{Rab}(x, \eta) := \mathcal{A}^{\eta H}(x),$$

a Lagrange multiplier version of the Hamiltonian action functional $\mathcal{A}^H$. Here, $x : S^1 \to V$ is a loop, $\eta \in \mathbb{R}$ and $H : \widehat{V} \to \mathbb{R}$ is a Hamiltonian such that $\partial V = H^{1-}(0)$ is a regular hypersurface and $X_H|_{\partial V} = R$. In [6], it was shown that $RFH(V, \partial V)$ is isomorphic to the symplectic homology $\check{SH}(V)$ of V-shaped Hamiltonians. In [7], it is denoted as $SH(\partial V)$ and is defined as follows:
Consider (weakly) admissible Hamiltonians $H \in Ad^w(V)$ as in 1.2 and require that $H|_{\partial V} < 0$ (see figure 10). For homotopies H_s require that they are globally monotone decreasing, so that for $\lim_{s\to\pm\infty} H_s = H_\pm$ the resulting continuation map $\sigma_*(H_-, H_+)$ respects action truncation as in 1.4. Then define

$$\check{SH}_*(\partial V) := \varinjlim_{b\to\infty} \varprojlim_{a\to-\infty} \varinjlim_{\substack{H\in Ad(V),\\ H|_{\partial V}<0}} FH^{(a,b)}(H).$$

One big difference in the definitions of $SH(\partial V)$ and $SH(V)$ is that all 1-periodic orbits

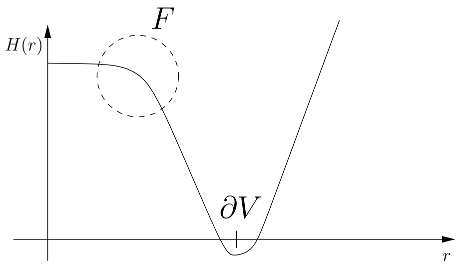

Fig. 10: A V-shaped Hamiltonian

in the region F are discarded due to the restriction to a fixed action window.
In [7], Cieliebak and Oancea introduce even more versions of SH, namely symplectic homology of symplectic cobordisms. Given a Liouville domain V and a subdomain $W \subset V$, we say that $C := \overline{V \setminus W}$ is an exact Liouville cobordism between ∂W and ∂V with filling W. In order to define $SH_*(C)$ we consider the subset $Ad^w(C) \subset Ad^w(V)$

given by the condition $H \in Ad^w(C) \Leftrightarrow H|_C < 0$. Again, we consider only globally monotone decreasing homotopies such that action windows are respected. Then we define

$$SH_*(C) := \varinjlim_{b\to\infty} \varprojlim_{a\to-\infty} \varinjlim_{H\in Ad(C)} FH^{(a,b)}(H).$$

There are yet two more flavors of SH. Up to now, we have had H tend to $+\infty$ on certain subsets of V. However, we can let H also tend to $-\infty$, but then we have to use inverse limits, as we have continuation maps $\sigma_*(H_-, H_+) : FH(H_+) \to FH(H_-)$ only if $H_- > H_+$ due to the maximum principle. Let $C^- := \partial W$ and $C^+ := \partial V$ denote the "lower" respectively "upper" boundary of C. Following [7], we define for $H \in Ad^w(C)$:

$$SH_*(C, C^-) := \varinjlim_{b\to\infty} \varprojlim_{a\to-\infty} \varinjlim_{\substack{H\to\infty,\\ \text{on } \widehat{V}\setminus V}} \varprojlim_{\substack{H\to-\infty,\\ \text{on } W}} FH^{(a,b)}(H),$$

$$SH_*(C, C^+) := \varinjlim_{b\to\infty} \varprojlim_{a\to-\infty} \varinjlim_{\substack{H\to\infty,\\ \text{on } W}} \varprojlim_{\substack{H\to-\infty,\\ \text{on } \widehat{V}\setminus V}} FH^{(a,b)}(H).$$

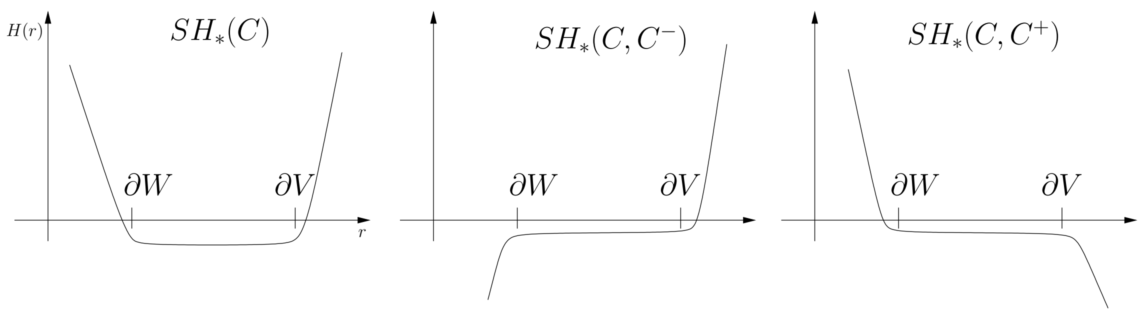

Fig. 11: Shapes of H for different versions of SH

In [7], $SH(C, C')$ is also defined for a pair of filled Liouville cobordisms $C' \subset C$, a version which we shall not need. We only remark that for the Liouville domains $W \subset V$ and $C = \overline{V \setminus W}$ it holds that $SH(V, W) = SH(C, C^-)$ directly by definition.

Theorem 11 (Invariance of RFH under subcritical surgery)**.**
Let W and V be as in Theorem 1. Then it also holds that

$$RFH_*(V, \partial V) \cong RFH_*(W, \partial W).$$

Proof: Set $C = H_k^{2n} = \overline{V \setminus W}$ and $C^- = \partial W$, $C^+ = \partial V$ as above. Our proof follows closely the one given in [7], prop. 9.14. However, our arguments differ slightly as the essential vanishing of $SH_*(C, C^-)$ is shown differently. The key tool for the demonstration is the long exact sequence in symplectic homology associated to a pair of Liouville cobordisms (see [7], prop 7.3). First applied to $W \subset V$, this sequence reads as

$$SH_*(V) \xrightarrow{\pi} SH_*(W) \to SH_{*-1}(V, W) \to SH_{*-1}(V) \xrightarrow{\pi} SH_{*-1}(W),$$

where π is the transfer map. As we have shown in the proof of Theorem 1, π is an isomorphism if V is obtained from W by attaching a subcritical handle. It follows hence that $SH_*(V, W) = 0$ for all $* \in \mathbb{Z}$. As mentioned above $SH_*(V, W) = SH_*(C, C^-)$. A duality argument over the field $\mathbb{Z}_2$ (see [7], prop. 3.4, 3.5) then shows that $SH_{-*}(C, C^+) = 0$ as well. Secondly, the long exact sequences for the pairs $\partial W = C^- \subset C$ and $\partial V = C^+ \subset C$ give

$$SH_*(C, C^-) \to SH_*(C) \to SH_*(\partial W) \to SH_{*-1}(C, C^-),$$
$$SH_*(C, C^+) \to SH_*(C) \to SH_*(\partial V) \to SH_{*-1}(C, C^+).$$

As in both sequences the most left and right groups vanish, we get isomorphisms

$$RFH(W, \partial W) \cong SH_*(\partial W) \cong SH_*(C) \cong SH_*(\partial V) \cong RFH_*(V, \partial V). \qquad \square$$

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