Nelson

SFT VIII Hamiltonian & Lagrangian Floer homology I & II

Lecture I: Def of HF — ref: Salamon, Audin-Damian,
Novikov Rings — PSS (Piunikhin-Salamon-Schwarz)
pair of pants product — Hofer-Salamon Novikov & Floer homology

Lecture II: Def of LF — ref: Seidel, Auroux
A_∞ relations
Product str on LF

exercises I: Morse theory:
ex. 1.16, 1.17, 1.18 Salamon

warm up! For $W = \mathbb{R}^{2n}$ w/ $\omega = \sum dp_i \wedge dq_i$ (ex 6.3.1 AD pg 159)
compute $\mathcal{A}_H(x)$.

ex. 1.19, 1.20 Salamon

Prove that a loop $x \in \mathrm{Crit}(\mathcal{A}_H) \Leftrightarrow t \mapsto x(t)$
is a periodic sol. of $\dot{x} = H_t(x(t))$ (prop 6.3.3 AD pg 160)

Prop 1.21 & ex. 1.22 from Salamon (pg 12)
ex 1.23 from Salamon

Crossing form compute $CZ(\{e^{it}\}|_{t\in[0,2\pi n]})$ pg 100 thesis

read §3 of Salamon

bubble: 6.6.9 in AD (pg 182)

Hofer-Salamon ex. 7.2

Check that the finiteness condition
$$\#\{A \in \Gamma \mid \lambda_A \neq 0,\ \omega(A) \leq c\} < \infty$$
is preserved under multiplication
$$\lambda * \mu = \sum_{A,B} \lambda_A \mu_B e^{2\pi i(A+B)}$$

exercises II: Nick §1 Combinatorial Floer homology
Nick §2 The Action functional
Nick §5 Displacement energy
Nick §6 Morse traj. & hol. strips.

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Lecture I: HF & pair of pants

SKIP $\boxed{\text{0. Recollection of Morse theory}}$ ~15 min

Let M^n be a smooth closed mfld, metric g.
Let $f: M \to \mathbb{R}$ be a smooth Morse function
Consider the negative grad flow $-\nabla_g f(x) = \dot{x}$
$\rightsquigarrow$ write ϕ as the negative gradient flow
$$\phi : M \times \overset{\text{time, } s}{\mathbb{R}} \to M$$
for $q \in \text{Crit}(\phi)$

def: Define the stable & unstable mflds of $q \in \text{Crit}(f)$ to be
$$W^u(q;f) = \{ x \in M \mid \lim_{s \to -\infty} \phi_s(x) = q \}$$
$$W^s(q;f) = \{ x \in M \mid \lim_{s \to +\infty} \phi_s(x) = q \}$$

def: f is said to be Morse whenever all the critical pts of f are nondegenerate, e.g. the Hessian $d^2 f(x) : T_x M \times T_x M \to \mathbb{R}$ is non deg.

ex: Prove that $W^u(q)$ & $W^s(q)$ are smooth submflds of M for every critical pt q of a Morse function f.

def: The Morse index of a critical pt q is defined by
$$\text{ind}_q f = \#\text{ negative evals } (\text{Hess}(f)|_q)$$
$$= \nu^-(d^2 f(q))$$

Prop: $\dim W^u(q;f) = \text{ind}_q f$

make donut example an exercise

def: The gradient flow is called a Morse-Smale system if for any $q \in \text{Crit}(f)$ the stable & unstable mflds intersect transversely

transversally if you're german or a student of Hofer / Salamon

Prop: If (f,g) is a Morse-Smale pair then for $p,q \in \mathrm{Crit}(f)$,
$\hat{\mathcal{M}}(p,q;f) = W^u(p;f) \cap W^s(q;f)$
the set of grad lines which connect p to q is a smooth submfld of M whose dim is given by

$$\dim \mathcal{M}(p,q;f) = \mathrm{ind}_p f - \mathrm{ind}_q f$$

Remark: $\hat{\mathcal{M}}(p,q;f) = \{ u : \mathbb{R} \to M \mid \dot{u} = -\nabla_g f(u),\ p = \lim_{s\to -\infty} u(s),\ q = \lim_{s \to +\infty} u(s) \}$
$\mathbb{R}$ acts on $\mathcal{M}(p,q;f)$ by translation

Prop: $\mathcal{M}(p,q;f) = \hat{\mathcal{M}}(p;q;f)/\mathbb{R}$ is a mfld of $\dim \mathcal{M}(p,q;f) = \mathrm{ind}_p f - \mathrm{ind}_q f - 1$, whenever it is nonempty.

Cor: Morse Smale $\Rightarrow$ $\mathrm{ind}_q f < \mathrm{ind}_p f$ whenever there is a flow line connecting p to q. Thus the index decreases strictly along flow lines.

Prop: There is a natural way to orient flow lines connecting crit pts of index difference 1, even if M is not orientable. See ex. 1.11 Salamon.

Thm: (Morse-Smale, Witten)
Let (f,g) be a Morse-Smale pair for closed M.
Define

$$CM_k(f) = \bigoplus_{\substack{df(p)=0 \\ \mathrm{ind}_p f = k}} \mathbb{Z}\langle p \rangle,$$

$$\partial_k^M p = \sum_{\substack{q \in \mathrm{Crit}(f) \\ \mathrm{ind}_q f = k-1}} \sum_{u \in \mathcal{M}(p,q)} \varepsilon(u) q$$

Then $\partial_k^M \circ \partial_{k+1}^M = 0$ & $\exists$ natural iso

$$HM_k(M,f;\mathbb{Z}) = \ker \partial_k^M / \ker \partial_{k+1}^M \to H_k(M;\mathbb{Z})$$

where $H_k(M;\mathbb{Z})$ denotes the singular homology of M.

Lecture I: HF & pair of pants

1. Hamiltonian Floer theory: Moduli Spaces

→ gromov threw a hissy fit & stole Zehnder's chalk when he tried to explain Floers ideas at a conference.

Arnold Conj: A symplectomorphism of a closed symplectic mfld that is generated by a time dependent Ham VF should have at least as many fixed pts as a function on a mfld must have crit. pts.

→ Floer: used the following action functional on $\mathcal{L}M$ that Moser had declared to be useless for existence proofs in 1976 b/c it is too degenerate.

Set up: Let (M^{2n}, ω) be a closed symp mfld & J a ω-compatible a.c.s. ⇝ e.g. $\langle X, Y \rangle = \omega(X, JY)$ defines a Riem. metric on M.

The space of ω-compatible J, $\mathcal{J}(M,\omega)$ is nonempty & contractible. Thus $c_1 = c_1(TM, J) \in H^2(M;\mathbb{Z})$ is indep of choice of J.

(M,ω) is said to be monotone whenever $\exists \tau \in \mathbb{R}^{>0}$ s.t. $\int_{S^2} u^* c_1 = \tau \int_{S^2} u^* \omega$ for every smooth $u: S^2 \to M$

→ e.g. $\omega(A) = \lambda c_1(A)$ $\forall A \in \pi_2(M)$.

HS: $N \geq n$
Ono: $N \geq n-2$] includes all compact (M,ω), dim 4 or 6 — semipositive

Remark: Floer assumed monotone. Hofer-Salamon introduced weakly monotone: $c_1(A)=0$ $\forall A \in \pi_2(M)$, or minimal chern $N \geq 0$ def by $c_1(\pi_2(M^{2n})) = N\mathbb{Z}$ is $\geq n-2$, or monotone. Requires Novikov rings. Else $\exists$ pseudo hol sphere w/ neg chern #... need other methods.

Key

The contractible 1-periodic sol. of $\dot{x}(t) = X_{H_t}(x(t))$ can be interpreted as crit pts of a circle valued symplectic action functional on the space of contractible loops $\mathcal{L}M$. Write $\mathcal{P}(H)$ = set of contr. orbits x

Symplectic Action

Think of a loop as $x : \mathbb{R} \to M$ s.t. $x(t+1) = x(t)$, for $t \in \mathbb{R}$. A tan vector to $\mathcal{L}M$ at x is a VF ξ along x.

$\omega(X_{H_t}, \cdot) = dH_t(\cdot)$

explicitly: $\xi : \mathbb{R} \to TM$ smooth s.t. $\xi(t) \in T_{x(t)}M$
$\xi(t+1) = \xi(t)$ for $t \in \mathbb{R}$.

Denote the space of such VF's by
$$T_x \mathcal{L}M = C^\infty(\mathbb{R}/\mathbb{Z}, x^* TM)$$

For each 1-periodic Hamiltonian $H_t = H_{t+1}$ the loop space $\mathcal{L}M$ carries the natural 1-form $\Psi_H : T\mathcal{L}M \to \mathbb{R}$ def. by
$$\Psi_H(x,\xi) = \int_0^1 \omega(\dot{x}(t) - X_t(x(t)), \xi(t))\, dt.$$

exercise: The zeroes of Ψ_H are 1-per. sol of $\dot{x}(t) = X_{H_t}(x(t))$.

exercise: Ψ_H is closed

why we will need Novikov coeff!

exercise: Ψ_H is not exact. But it is the differential of a circle valued action $\mathcal{A}_H : \mathcal{L}_0 M \to \mathbb{R}/\mathbb{Z}$
$$\mathcal{A}_H(x) = -\int_D v^* \omega - \int_0^1 H_t(x(t))\, dt$$
for $x \in \mathcal{L}M$, $v : D = \{z \in \mathbb{C} \mid |z| \leq 1\} \to M$ smooth s.t. $v(e^{2\pi i t}) = x(t)$ for $t \in \mathbb{R}$. Restrict to $\mathcal{L}_0 M$ = space of contr. loops.

if x is contractible such a v exists

Fix a time dep. Ham. s.t. all the 1-per. orbits are nondegenerate. We want to study "grad" flow lines of $\mathcal{A}_H$. Metric comes from a 1-per. choice of a.c.s. compatible w/ ω. Define
$$\langle \xi, \eta \rangle_t = \omega(\xi, J_t \eta)$$
&
$$\langle \xi, \eta \rangle_{T_x \mathcal{L}M} = \int_0^1 \langle \xi(t), \eta(t) \rangle_t \, dt$$

postpone

Two such maps v_0 & v_1 are called equivalent if $v_0 \# (-v_1)$ is a torsion class in $H_2(M, \mathbb{Z})$. Use the notation $[x, v_0] = [x, v_1]$ for equivalent pairs & denote by $\tilde{\mathcal{L}}M$ the space of equivalence classes. Elements: $[x, v] \in \tilde{\mathcal{L}}M$. Group of deck transf. of $\tilde{\mathcal{L}}M \downarrow \mathcal{L}M$ is the image $\Gamma \subset H_2(M)$ of $\pi_2(M) \to H_2(M)$

Lecture I: HF & pair of pants.

Since $d\mathcal{A}_H = \Psi_H$ then wrt above metric

$$\operatorname{grad} \mathcal{A}_H(x)(t) = J_t(x(t))\dot{x}(t) - \nabla H_t(x(t))$$

$\rightsquigarrow$ $d\mathcal{A}_H(\cdot) = \langle \operatorname{grad} \mathcal{A}_H, \cdot \rangle$

A gradient flow line of $\mathcal{A}_H$ is a smooth 1-par family of loops $u: \overset{s}{\mathbb{R}} \times \overset{t}{S^1} \to M$ s.t.

$$\frac{\partial u}{\partial s} + \operatorname{grad} \mathcal{A}_H(u(s,\cdot)) = 0$$

e.g.

$$\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} - \nabla H_t(u) = 0 \qquad ✩$$

Remark: If $J, H,$ & u are indep of t then get pos. grad flow of H

Def: The energy of a sol to ✩ is

$$E(u) = \frac{1}{2}\int_0^1 \int_{-\infty}^{\infty} \left(\left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial t} - X_t(u)\right|^2 \right) ds\, dt$$

Prop: Let $u: \mathbb{R} \times S^1 \to M$ be a sol of ✩. Then tfae

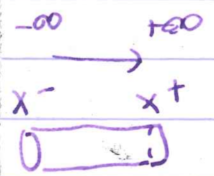
i) $E(u) < \infty$

ii) $\exists$ per. sol $x^{\pm} \in \mathcal{P}(H)$ (1-per. sol. of H_t) s.t. $\lim_{s \to \pm\infty} u(s,t) = x^{\pm}(t)$
& $\lim_{s \to \pm\infty} \frac{\partial u}{\partial s}(s,t) = 0$ where both lim are uniform in t variable

iii) $\exists$ const. $\delta > 0$, $c > 0$ s.t.

$$\left|\frac{\partial u}{\partial s}(s,t)\right| \leq c e^{-\delta |s|} \qquad \forall s,t \in \mathbb{R}$$

"exponential decay"



Define: $\widehat{\mathcal{M}}^J(x^-, x^+) = \{ u: \mathbb{R} \times S^1 \mid ✩ \text{ \& } \lim_{s \to \pm\infty} u(s,t) = x^{\pm}(t) \}$

Ex: for $u \in \widehat{\mathcal{M}}^J(x^+, x^-)$, $E(u) = \mathcal{A}_H(x^-, u^-) - \mathcal{A}_H(x^+, u^+)$
where $u^{\pm}: D \to M$ are smooth s.t. $u^{\pm}(e^{2\pi i t}) = x^{\pm}(t)$
& $u^+ = u^- \# u$.

Remark: There is a beautiful Fredholm theory associated to the linearized perturbed CR eqn

$$\bar{\partial}_{H,J}(u) = \frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} - \nabla H_t(u).$$

map u capped by disk v w/ opp or. reps a torsion homology class
Say what this is in $H_2(M,\mathbb{Z})$.

Thm: For $u \in \tilde{\mathcal{M}}^J(x^-, x^+)$ & $v_\pm : D \to M$ satisfying $v_\pm(e^{2\pi i t}) = x^\pm(t)$ & $v^+ = v_- \# u$ Then
$$D_u : W^{1,p}(u^*TM) \to L^p(u^*TM)$$
is a Fredholm op & its Fredholm index is
$$\operatorname{index} D_u = CZ_{v_-}(x^-) - CZ_{v^+}(x^+)$$
Define $\#$ (from earlier)

The CZ index describes the spectral flow of D_u. It is a winding number associated to any path of symp matrices which unique up to htpy provided $1 \notin \operatorname{Spec}(\Phi(1))$, $\Phi(1)$ linearized return map of x.

* note about language *

Note: Depends on linearization of a periodic orbit &
$$CZ_{A\#v}(x) = CZ_v(x) + 2c_1(A) \quad \text{for } A \in \pi_2(M).$$
$CZ : \mathcal{P}(H) \to \mathbb{Z}/2N\mathbb{Z}$ for $N\mathbb{Z} = c_1(\pi_2(M))$ minimal chern #

Transversality Thm (monotone vs semi pos)
There exists a Baire subset $\mathcal{HJ}_{reg} = \mathcal{HJ}_{reg}(M,\omega)$ s.t. for a generic $(H,J) \in \mathcal{HJ}_{reg}(J)$ The space $\tilde{\mathcal{M}}^J(\tilde{x}^-, \tilde{x}^+)$ is a fin dim smooth mfld $\forall \tilde{x}^\pm \in \tilde{\mathcal{P}}(H)$ of dimension
$$\dim \tilde{\mathcal{M}}^J(\tilde{x}^-, \tilde{x}^+) = CZ_{v_-}(\tilde{x}^-) - CZ_{v^+}(\tilde{x}^+)$$

* why it's ok to restrict to $\partial_0 M$ in closed case *

Remark If $H : M \to \mathbb{R}$ is a time indep. Morse function w/ suff small 2^{nd} deriv, x is a crit pt of H & $u(z) \equiv x$ is the const. disk then $CZ_u(x) = 2n - \operatorname{ind}_x H$

stated as needed for semi pos case (stronger in monotone)

Compactness Thm (monotone vs semi pos)
For a generic pair $(H,J) \in \mathcal{HJ}_{reg}$ we have
$$\text{key } \sum_{\substack{\boxed{\omega(A) \le C} \\ c_1(A) = 0}} \#\{\tilde{\mathcal{M}}(\tilde{x}, A\#\tilde{y})/\mathbb{R}\} < \infty,$$
For all $\tilde{x}, \tilde{y} \in \tilde{\mathcal{P}}(H)$ w/ $CZ(\tilde{x}) - CZ(\tilde{y}) = 1$

Floer's Def: * check PD *
If M closed, monotone
$$C_k = \bigoplus_{\substack{CZ_v(x) = k \bmod 2N \\ x \in \mathcal{P}(H)}} \mathbb{Z}\langle x \rangle$$
$\partial_k : C_k \to C_{k-1}$ is # of 0 dim comp of $\tilde{\mathcal{M}}(x,y)/\mathbb{R}$ whenever $CZ_{v^-}(x^-) - CZ_{v^+}(x^+) = 1 \bmod 2N$, counted w/ signs.
$$\bigoplus_{j = k \bmod 2N} H_j(M) = HF_k(M, H, J) = \ker \partial_k / \operatorname{im} \partial_{k+1} \text{ (graded mod } 2N) = \bigoplus_{j = k+n \bmod 2N} H^j(M)$$

Lecture I : HF & pair of pants product

2. Novikov Rings & HF

Or, how Helmut & Dietmar rediscovered a wonderful ring.

possibly infinite

Motivation: (1995) without strict monotonicity one can have a sequence of connecting orbits w/ index difference 1 whose energy converges to ∞. Hofer & Salamon took care of these seq by constructing a suitable coeff. ring Λ_ω which algebraically incorporates the period map

anticipated by Floer

$$\pi_2(M) \to \mathbb{R}$$
$$A \mapsto \int \omega\, A$$

Such a ring was used by Novikov in his generalization of Morse theory for closed 1-forms. (1981)

In other words, $\mathcal{M}^J(x,y)$ w/ $CZ(x) - CZ(y) = 1$ might not be a finite set but there are finitely many connecting orbits in each homology class. Counting the connecting orbits in their homology classes leads to defining HF groups of $\mathcal{L}M$ form a module over Novikov's ring of generalized Laurent series.

Difficulties: ① Presence of J-hol spheres of $c_1 = 0$. Such bubbling no longer leads to connecting orbits of strictly lower index.

To overcome in semi-positive case vs monotone case.

② $\mathcal{A}_H$ is only well-defined on $\tilde{\mathcal{L}_0 M}$ w/ a possibly dense set of critical values.

Setup: Let $\mathcal{L}_0 M$ be space of contr. loops $x: S^1 = \mathbb{R}/\mathbb{Z} \to M$. For every contr. x $\exists$ smooth $v: D \to M$, $v(e^{2\pi i t}) = x(t)$. v_0 & v_1 are equivalent if $v_0 \# (v_1)$ is a torsion class in H_2. Let $[x, u_0] = [x, u_1]$ denote equiv pairs &

$\tilde{\mathcal{L}}_0 = \tilde{\mathcal{L}}_0 M$ be the space of equivalence classes.
$\downarrow$
$\mathcal{L}M$ $\tilde{\mathcal{L}}_0$ is the unique covering space of $\mathcal{L}$ whose group of deck transf. is the image $\Gamma < H_2(M)$ of the Hurewicz homomorphism $\pi_2(M) \to H_2(M;\mathbb{Z})/\text{torsion}$.

Note $\Gamma = \pi_2(M)/\ker \phi_{c_1} \cap \ker \phi_\omega$ where

check

$$\phi_\omega : \pi_2(M) \to \mathbb{R} \qquad \phi_{c_1} : \pi_2(M) \to \mathbb{R}$$
$$S^2 \mapsto \int_{S^2} \omega \qquad S^2 \mapsto \int_{S^2} c_1 .$$

$$\Gamma \times \tilde{\mathcal{L}}_0 \to \tilde{\mathcal{L}}_0$$
$$[x, u] \mapsto [x, A\# u]$$

$A\# u$ denotes equiv class of $v \# u$ for $v \in A$.
Consider $\omega : \Gamma \to \mathbb{R}$ homomorphism. $A \mapsto \int_{S^2} u^*\omega$

def Associated to this homo is Λ_ω, the <u>Novikov ring</u> whose elements are formal sums

$$\lambda = \sum_{A \in \Gamma} \lambda_A e^{2\pi i A} \quad \text{w/ } \lambda_A \in \mathbb{Q} \text{ satisfying finiteness:}$$

$$\#\{A \in \Gamma \mid \lambda_A \neq 0, \omega(A) \leq c\} < \infty \qquad \forall c > 0.$$

Multiplication is given by
$$\lambda * \mu = \sum_{A,B} \lambda_A \mu_B e^{2\pi i (A+B)}, \quad \text{so } (\lambda * \mu)_A = \sum_B \lambda_{A-B} \mu_B$$

This ring comes w/ a natural grading def by
$$\deg(e^{2\pi i A}) = 2c_1(A).$$

We denote Λ_k to be the elements of degree k.
Λ_0 is a subring & $\Lambda_k \neq \emptyset$ if $k \in 2N\mathbb{Z}$
$$* : \Lambda_j \times \Lambda_k \to \Lambda_{j+k}.$$

Quantum cohomology:
$$QH^k(M) = \bigoplus_j H^j(M) \otimes \Lambda_{k-j}; \quad H(M) = H^*(M,\mathbb{Z})/\text{torsion}$$

Let $\tilde{P}(H) = \{\tilde{x} = [x, u] \mid x \in P(H),\ u : D \to M \text{ smooth},\ u(e^{2\pi i t}) = x\}$

whenever $CZ_{u^-}(\tilde{x}^-) - CZ_{u^+}(\tilde{x}^+) = 1$ we denote

$$n(\tilde{x}^-, \tilde{x}^+) = \#\{\mathcal{M}(\tilde{x}^-, \tilde{x}^+)/\mathbb{R}\}$$

where the connecting orbits are counting w/ signs determined by a system of coherent or. C_k is the space of formal sums

$$\xi = \sum_{\substack{\tilde{x} \in \tilde{P}(H) \\ CZ_u(\tilde{x}) = k \bmod 2N}} \xi_{\tilde{x}} \langle \tilde{x} \rangle$$

where $\xi_{\tilde{x}} \in \mathbb{Q}$ satisfying finiteness:

$$\{\tilde{x} \in \tilde{P}(H) \mid \xi_{\tilde{x}} \neq 0,\ \mathcal{A}_H(\tilde{x}) \geq c\} < \infty.$$

infinite dimensional over $\mathbb{Z}_2$ or $\mathbb{Z}$ but finite dimensional over Λ_ω

note: $\mathcal{A}_H([x, A\#u]) - \mathcal{A}_H([x,u]) = \omega(A)$

C_* is a module over Λ_ω via

$$\lambda * \xi = \sum_{\tilde{x}} \sum_{A} \lambda_A \xi_{(-A)\#\tilde{x}} \langle \tilde{x} \rangle, \text{ thus}$$

$$C_k = \bigoplus_{\substack{\tilde{x} \in \tilde{P}(H) \\ CZ_u(\tilde{x}) = k \bmod 2N}} \Lambda_\omega x$$

$$\partial_k x = \sum_{\substack{\tilde{y} \in \tilde{P}(H) \\ CZ_u(y) = k-1}} n(\tilde{x}, \tilde{y})\, \tilde{y}, \text{ linear over } \Lambda_\omega.$$

For generic H, J

Thm ∂ is well defined & $\partial^2 = 0$

b/c finitely many connecting orbits in a homology class

b/c 1-par. families of connecting orbits of index difference 2 avoid J-spheres w/ $c_1 = 0$ b/c they form a 3D set in M while the spheres form a set of codim 4.
J hol. spheres of $c_1 \geq 1$ can only bubble off if they intersect a per. sol & for generic J
dim {per sol} = 1, codim $\{S^2, c_1 = 1\} = 2$.
Same for $c_1 \geq 2$

$$HF_*(M, H, J) = \ker \partial_k / \operatorname{im} \partial_{k+1}.$$

remark 6.4(v) of H-S. if time permits pg 25

Thm: Given regular pairs $(H^\alpha, J^\alpha), (H^\beta, J^\beta)$ there exists a natural isomorphism

$$\Phi^{\beta\alpha} : HF_*(M, \omega, H^\alpha, J^\alpha) \to HF_*(M, \omega, H^\beta, J^\beta)$$

which preserves the grading given by CZ index
If (H^γ, J^γ) is also regular then

$$\phi^{\gamma\beta} \circ \phi^{\beta\alpha} = \phi^{\gamma\alpha}, \quad \phi^{\alpha\alpha} = id$$

These iso's are linear over Λ_ω.

3 Poincaré Duality

At the level of (co)chains,

PD: $CF^k(H) \cong \mathrm{Hom}(CF_k(H), \underset{\text{deg } 0}{\Lambda_0}) \cong CF_{2n-k}(\bar{H})$

where $\bar{H}_t = -H_{-t}$.

check: $\mathcal{A}_{\bar{H}}([\bar{x}, \bar{v}]) = -\mathcal{A}_H([x,v])$ $\qquad \bar{v}(z) = v(\bar{z})$

$CZ_{\bar{v}}\,\bar{x} = 2n - CZ_v\, x$

$CF^k(H)$ is defined as space of formal sums

$$\eta = \sum_{CZ_v(\tilde{x}) = k \bmod 2N} \eta_{\tilde{x}} \langle \tilde{x} \rangle$$

which satisfy the opposite finiteness cond.

$$\{ \tilde{x} \in \tilde{P}(H) \mid \eta_{\tilde{x}} \neq 0,\ \mathcal{A}_H(\tilde{x}) \le c \} < \infty.$$

action of Novikov ring on this group is given by

$$\lambda * \eta = \sum_{\tilde{x}} \sum_{A} \lambda_A \eta_{A\#\tilde{x}} \langle \tilde{x} \rangle$$

There is a pairing $CF^k \times CF_k \to \Lambda_0$ def by

$$\langle \eta, \xi \rangle = \sum_{\substack{A \\ c_1(A)=0}} \Big(\sum_{\tilde{x}} \eta_{\tilde{x}} \xi_{A\#\tilde{x}} \Big) e^{2\pi i A}$$

$\to$ this gives iso $\quad CF^k(H) \cong \mathrm{Hom}(CF_k(H), \Lambda_0)$

The natural iso yields a PD iso

$$PD_F : HF^k(H,J) \to HF_{2n-k}(H,J)$$

In view of $CF^* = \mathrm{Hom}(CF_*(H), \Lambda_0)$ get a pairing $HF_{2n-k}(H,J) \otimes HF^{2n-k}(H,J) \to \Lambda_0$.

In combo w/ Poincaré duality iso we get a PD pairing

$$HF^k(H,J) \times HF^{2n-k}(H,J) \to \Lambda_0$$

& similarly for homology.

4. Products

While product $M = M_1 \times \dots \times M_\ell$ of semi-pos. symp. mflds is not in general semi pos its Floer homology groups are well-defined for every generic product a.c.s. $J = J_1 \times \dots \times J_\ell$ & arbitrary $H_t : M \to \mathbb{R}$.

WHY? the J-hol curves are all products of J_i-hol curves in M_i & so cannot have neg c_1 for generic J_i's.

Consider $M_1 = \dots = M_\ell = M$ & $H = \sum_{i=1}^{\ell} H_i(t,x)$

Let $\tilde{x} = [\tilde{x}_1, \dots, \tilde{x}_\ell] \sim [A_1 \# \tilde{x}_1, \dots, A_\ell \# \tilde{x}_\ell]$ whenever $A_i \in \Gamma$ & $\sum A_i$ is torsion.

Γ acts on $\tilde{P}(H_1, \dots, H_\ell)$ by

$$A \# \tilde{x} = [A \# \tilde{x}_1, \dots, \tilde{x}_\ell].$$

$$cz(\tilde{x}) = \sum cz(\tilde{x}_i) \qquad \mathcal{A}_H(\tilde{x}) = \sum \mathcal{A}_{H_i}(\tilde{x}_i)$$

With this notation the tensor product

$$CF_*(H_1, \dots, H_\ell) = CF_*(H_1) \otimes \dots \otimes CF_*(H_\ell)$$

is the set of formal sums

$$\xi = \sum_{\substack{\tilde{x} \in \tilde{P}(H_1, \dots, H_\ell) \\ cz(\tilde{x}) = k \bmod 2N}} \xi_{\tilde{x}} \langle \tilde{x} \rangle$$

w/ $\xi_{\tilde{x}} \in \mathbb{Q}$ satisfying $\#\{\tilde{x} \in \tilde{P}(H_1, \dots, H_\ell) \mid \xi_{\tilde{x}} \neq 0, \mathcal{A}_H(\tilde{x}) \geq c\} < \infty$

For $\tilde{x}, \tilde{y}$ w/ $cz(\tilde{x}) - cz(\tilde{y}) = 1$ the corresponding entry of ∂ can only be nonzero if $\exists$ reps

$$\tilde{x} = [\tilde{x}_1, \dots, \tilde{x}_\ell] \quad \& \quad \tilde{y} = [\tilde{y}_1, \dots, \tilde{y}_\ell]$$

s.t. $cz(\tilde{x}_j) - cz(\tilde{y}_j) = 1$ for some j & $\tilde{x}_i = \tilde{y}_i \ \forall i \neq j$.

Then $n(\tilde{x}, \tilde{y}) = n(\tilde{x}_j, \tilde{y}_j)$.

(rhs) is indep of choice of rep's b/c $n(A \# \tilde{x}_j, A \# \tilde{y}_j) = n(\tilde{x}_j, \tilde{y}_j) \ \forall A \in \Gamma$.

The resulting HF groups are

$$HF_*(H_1, \dots, H_\ell, J) = HF_*(H_1, J) \otimes \dots \otimes HF^*(H_\ell, J)$$

where the rhs is the graded tensor prod. over Λ_0. At this place the choice of rational coeff is essential.

in case of $\mathbb{Z}$-coeff Floer Homology of product is given by Künneth Formula.

5. Relative Donaldson type invariants.

Consider J-hol curves $u: \Sigma \to M$ def. on Σ_g w/ ℓ cylindrical ends $Z_i = \phi_i((0,\infty) \times S^1) \subset \Sigma$

fix j on Σ so $\phi_i^* j$ agrees w/ std str on cyl's.

Fix ℓ time dependent Ham's

$$H_i = H_i(s,t,x) = H_i(s, t+1, x)$$

which vanish near $s=0$ & are indep. of s for $s \geq 1$.

Assume per. sol $\dot{x} = X_i(1, t, x)$ are nondeg

$\tilde{P}(H_i) \subset \tilde{\mathcal{L}}$ lift of such per. sol.

Given $\tilde{x}_i = [x_i, v_i] \in \tilde{P}(H_i)$ consider

$$\mathcal{M}_\Sigma(\tilde{x}_1, \dots, \tilde{x}_\ell) = \mathcal{M}_\Sigma(\tilde{x}_1, \dots, \tilde{x}_\ell, H_1, \dots, H_\ell, J)$$

of all smooth $u: \Sigma \to M$ s.t.

a) u is J-hol on $\Sigma_0 = \Sigma - \bigcup_i Z_i$

b) $u_i = u \circ \phi_i$ satisfy

$$\partial_s u_i + J(u)\partial_t u_i - \nabla H_i(s,t,u_i) = 0$$

$$x_i(t) = \lim_{s \to \infty} u_i(s,t)$$

c) the map u capped off by the disks v_i w/ opp. or. represent a torsion homology class in $H_2(M, \mathbb{Z})$.

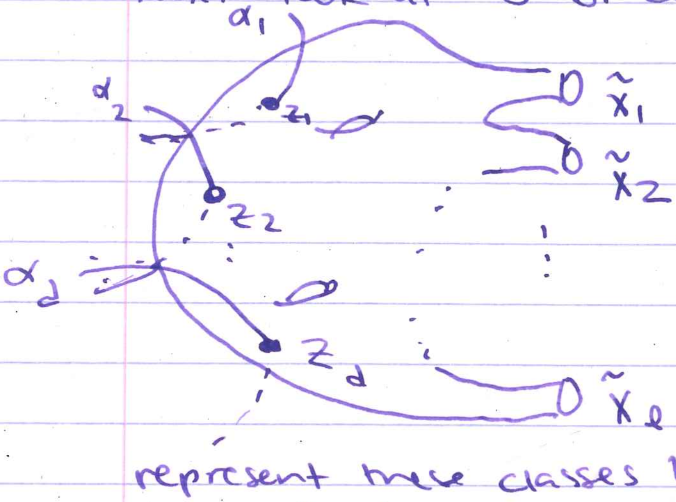
<u>Remarks:</u> (c) & hence $\mathcal{M}_\Sigma$ depend only on equiv class of $[\tilde{x}_1, \dots, \tilde{x}_\ell]$. $\mathcal{M}_\Sigma$ is a fin dim mfld for generic choice of H_i

$$\dim \mathcal{M}_\Sigma = 2n(1-g) - \sum_{i=1}^{\ell} \mu(\tilde{x}_i)$$

see also Schwarz thesis.

S. cont

Next look at J-hol curves w/ mk'd pts



Fix d distinct pts $z_1, \dots, z_d \in \Sigma$ & $\alpha_1, \dots, \alpha_d \in H_*(M)$ s.t.

$$\sum_{i=1}^{\ell} \mu(\tilde{x}_i) = 2n(1-g) - \sum_{\nu=1}^{d} (2n - \deg(\alpha_\nu)) \quad (\star)$$

represent these classes by generic cycles (still denoted by α_ν) & define

$$\mathcal{M}_\Sigma(\alpha_1, \dots, \alpha_d, \tilde{x}_1, \dots, \tilde{x}_\ell)$$

to be the set of all curves $u \in \mathcal{M}_\Sigma(\tilde{x}_1, \dots, \tilde{x}_\ell)$ w/ $u(z_\nu) \in \alpha_\nu$.

This is a finite set for generic choices & we denote

$$n_\Sigma(\alpha_1, \dots, \alpha_d, \tilde{x}_1, \dots, \tilde{x}_\ell) = \# \mathcal{M}_\Sigma(\alpha, \tilde{x})$$

where pts are counted w/ signs.

→ in general n_Σ depend on choice of H_i & J. but they define a HF class which is indep of choices. HF extracts the inv. info from these moduli spaces.

to be precise, define the cycle $\psi_\Sigma(\alpha_1, \dots, \alpha_d) = \sum_{\tilde{x}_i} n_\Sigma(\alpha_1, \dots, \alpha_d, \tilde{x}_1, \dots, \tilde{x}_\ell) \langle \tilde{x}_1, \dots, \tilde{x}_\ell \rangle$ in $CF_*(H_1, \dots, H_\ell) = CF_*(H_1) \otimes \dots \otimes CF_*(H_\ell)$.

Here sum runs over all equiv. classes of ℓ-tuples $[\tilde{x}_1, \dots, \tilde{x}_\ell]$ satisfying $(\star)$.

The mk'd surface $(\Sigma, \{z_i\})$ determines a multi-linear map

$$\psi_\Sigma : H_*(M) \otimes \dots \otimes H_*(M) \to HF_*(H_1) \otimes \dots \otimes HF_*(H_\ell).$$

this can be interpreted as a symplectic version of a relative Donaldson inv. In original context they are def for X^4 w/ $\partial X \neq \emptyset$ & take values in $HF_*(\partial X)$. Symplectically the rel. Donaldson inv of Σ take values in $HF_*(\partial \Sigma)$.

* original def of donaldson inv

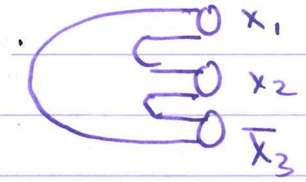
thm ① Ψ_Σ is a floer homology cycle. The corresponding HF class is $\Psi_\Sigma(\alpha_1, \ldots, \alpha_d) \in HF_*(H_1) \otimes \cdots \otimes HF_*(H_d)$

it has degree $2n(1-g) - \sum_{v=1}^{d} (2n - \deg(\alpha_v))$.

② If one of α_v is a boundary then Ψ_Σ is a bdry.
Ψ_Σ is indep of choice of $\{z_i\}$.

③ Ψ_Σ is natural under variation of H_i & J.

Def: (Pair of pants product)
Let Σ =



$\bar{H}_t = -H_{-t}$

$H_1 = H_2 = \bar{H}_3 = H$. Then
$\Psi_\Sigma([M]) \in HF_*(H) \otimes HF_*(H) \otimes HF_*(\bar{H})$
is a class of degree $2n$. PD says we can interpret this class as a map
$HF^j(H) \otimes HF^k(H) \to HF^{j+k}(H)$
$(\eta, \xi) \mapsto \eta \cup_F \xi$

what else to say?

obtained by contracting $\eta, \xi \in HF^k(H)$ w/ first 2 factors in $\Psi_\Sigma([M])$. This is the pair of pants product.

→ email Mohammed & Tobias to say what I covered

Nelson

Lecture II: LF & A_∞-relations

what to assume geometrically?

LF Setup: Let X be a closed symp mfld.
Let L & F be closed embedded Lagrangians
Assume $L \pitchfork F$ — typically achievable by a C^∞-small perturbation of either L or F.
$CF^*(L,F)$ is supposed to be a chain cx w/ generators labelled by $x \in L \pitchfork F$, rk = intersection #. (we eventually need more geometric restrictions)

Further assume L & F are oriented.
compare or. of T_xX. & $T_xL \oplus T_xF$.
Then the degree of x as a generator of $CF^*(L,F)$ will be $\{^0_1\}$.

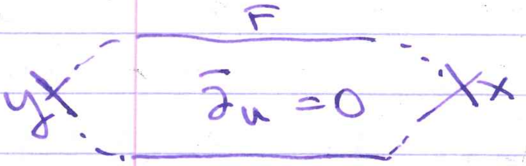
clarify = look at Nick notes

$$CF^0(L,F) \underset{d}{\overset{d}{\rightleftarrows}} CF^1(L,F)$$

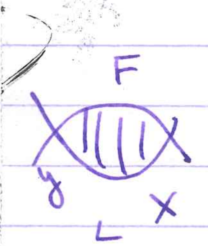
upgradable...

we define a differential on this $\mathbb{Z}/2\mathbb{Z}$ graded complex: Let J be ω-compatible. d will "count" elements of moduli spaces of J-hol maps, given $x, y \in L \pitchfork F$, $\mathcal{M}^J(y, x, \beta)$ in a free htpy class β.

always a mouthful, look at the picture!



$B = \mathbb{R} \times [0,1]$ strip
$[u] = \beta \leftarrow$ some htpy class



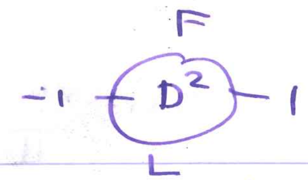
$$\mathcal{M}^J(y,x,\beta) = \left\{ u: B \to X \;\middle|\; \begin{array}{l} \bar\partial_J u = 0 \\ u(\mathbb{R}\times\{0\}) \subset L \\ u(\mathbb{R}\times\{1\}) \subset F \end{array} \quad \begin{array}{l} \lim_{s\to+\infty} u(s,t) = x \\ \lim_{s\to-\infty} u(s,t) = y \end{array} \quad \forall t \in [0,1] \right\}$$

$\mathbb{R}$ acts by translation $u_S(s,t) = u(s+S, t)$.

$\mathbb{R}$ source of trouble

→ in literature impose finite energy: Analysis shows this is equiv to conv. at ends.

② Eine kleines bisschen Analysis...



Also: B is bihol to $D^2 \setminus \{\pm 1\}$

$\rightsquigarrow$ version of remorable sing thm exists & is applicable

$\mathcal{F} = C^\infty(D^2, (X, L, F)) \ni u$ $\qquad u^*(TX)$

$\mathcal{E} = \coprod_u C^\infty(D^2, u^*(TX))$ is naturally a Frechet bdl over $\mathcal{F}$.

$$\bar\partial \curvearrowright \begin{matrix} \mathcal{E} & \subset & \mathcal{E}^p \\ \downarrow & & \downarrow \\ \mathcal{F} & \subset & \mathcal{F}^{1,p} \end{matrix} \curvearrowleft \bar\partial u = J\partial_s u - \partial_t u$$

{space of sol} $\rightsquigarrow$ set of sol. to a nonlinear elliptic PDE.

irritating to work w/ Fréchet spaces.

Appeal to Sobolev completion $W^{1,p}(D^2, (X, L, F)) = \mathcal{F}^{1,p}$

$\bar\partial$ takes values in $L^p(u^*(TX))$.

elliptic regularity $\Rightarrow$ set of $W^{1,p}$ sol. is the set of smooth sol.

In Sobolev land the linearization of $\bar\partial$ is fredholm

eg. im is closed

ker & coker are finite rank.

adverb: modifies verb, adj, noun phrase

adj: describing word.

So we can assign an integer (so called virtual dim) to every $[u] \in \mathcal{M}(y, x)$.

$D_u : W^{1,p}((D^2, L, F), u^*(TX)) \to L^p(D^2, u^*(TX))$.

$\mathrm{vdim}([u]) = \dim(\ker(D_u)) - \dim(\mathrm{coker}(D_u)) - 1$.

grammar pt.

Assume momentarily that $\mathrm{coker}(D_u) = 0$

this is so nice it has a name! regularity

Then we can appeal to the implicit function thm & conclude that the space of sol. near u is cut out transversely, hence a mfld of dim $= \mathrm{rk}(D_u))$ & really $\ker(D_u)$ "=" tan space.

<u>Caveat</u>: But we need to check $/\mathbb{R}$ is not hideous (e.g. Hausdorff )
$\Rightarrow \mathcal{M}(y,x)$ is a mfld of dim $\mathrm{rk}(\ker(D_u))-1$.

J is regular if all D_u are surjective.
This is typically true for generic J w/ some other assumptions.

want to define $d\underset{\text{gen of } CF^*(L,F)}{x} = \sum_y \underbrace{\# \overset{\text{dimension}}{\mathcal{M}^\circ}(y,x;\beta)}_{\text{count mod 2}} \cdot y$

To ensure d is well-defined we assume

① $\omega|_{\pi_2(X)}=0$ symplectic asphericity
$u: S^2 \to X \quad \int_{S^2}\omega = 0$.

<u>note</u>: tameness $\omega(v,Jv)>0$ tells us
if $u: \mathbb{CP}^1 \to X$, $\bar\partial_J u = 0$ then $\int u^*\omega > 0$
unless u is constant

① $\Rightarrow \nexists$ no non const. J-hol spheres

② $\exists$ no J-hol disks w/ ∂ on L & F
e.g. $\nexists\, u: D^2 \to X$, $\partial D^2 \to L$ (or F).

not many closed X satisfy ①
even fewer if we enforce ②.

Gromov + Floer <u>Prop</u>: Assume ① & ②. If $[u_i] \in \mathcal{M}^\circ(y,x;\beta)$ is a seq of distinct elements then $\lim_{i\to\infty} \underbrace{\int u_i^*(\omega)}_{\text{topological energy}} \to \infty$

topological energy, agrees w/ geometric formulation too

in this setting
$0 \le \int u^*\omega = \int \underbrace{\|du\|^2}_{\text{geometric energy}}$
geometric energy aka. L^2-energy

proof: By contradiction. Say we have $0 < E$ & a seq. u_i s.t. energy of u_i is $\leq E$. we want to show $\exists$ limit in $\mathcal{M}^0$

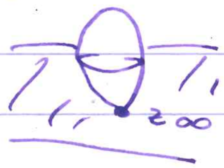
Gromov: compactify $\mathcal{M}(y, x, \beta) \subset \overline{\mathcal{M}}(y, x, \beta)$.
A few things can happen:

case I C^0 - norm of $\|du_i\|^2$ is unbounded
$\Rightarrow u_i$ blows up at some point.

a) $\times z_\infty$ $\rightarrow$ zoom in & rescale at z_∞ by changing coor.

if you want $\int \|du\|^2 < \infty$ it had better be the case that as we $\rightarrow$ end of strips at $\pm\infty$ we $\rightarrow 0$ exponentially fast.

thus we conclude in limit $\exists$ nonconst hol. sphere.



But we assumed there are none.

$\rightarrow$ if it appeared we'd have to count it.

b) blow up occurs on ∂ $\rightarrow$ J hol disks, also excluded geometrically.

case II C^0 norm is uniformly bdd
note: $\|du\|^2$ goes to 0 at ∞ (exp fast)
this means we could see broken strips but Fredholm index + regularity controls dim of these broken strips so we can rule out breaking when we start w/ a 0-dim strip ... we do see breaking for a 1-dim strip

* if R is a field $\alpha: H_*(X)\otimes H_*(Y) \to H_*(X\otimes Y)$ is a natural iso
else $H_*(X)\otimes H_*(Y)$ & $H_*(X\otimes Y)$ are not the same b/c the natural ses seq unnaturally splits. working w/ J-curves necessitates we work w/ $C_*(X)\otimes C_*(Y)$. so we get $H_*(C_*(X)\otimes C_*(Y))$
But we really want to work w/ $H_*(X)\otimes H_*(Y)$. so need the iso, e.g. need k to be a field!

② LF differential

J generic & no sphere or disk bubbling & $x, y \in L \pitchfork F \subset X$

then $\mathcal{M}^0(y, x_i)$ consists of finitely many elts below a given energy level.

$\mathcal{M}^{k,E}(y, x,\beta) = \{ \text{sol } u \text{ s.t. } \int u^*\omega \le E \ \& \ \underbrace{\text{ind}(D_u) = k}_{\text{say more}} \} \subset \mathcal{M}(y, x_i)$.

idea: $\langle dx, y\rangle = \sum_{[u]\in \mathcal{M}^0(y,x,\beta)} \pm e^{-\int u^*\omega}$

→ very delicate to understand convergence.

→ unless we know there are finitely many terms (e.g. monotonicity) convergence is an ad-hoc computation.

Note
$\mathcal{A}_H$ was circle valued! Hence $\mathcal{E}_A \in 2\pi i A$

So we replace $e^{-\int u^*\omega}$ by a formal variable $T \sim \frac{1}{e}$. Define $\langle dx, y\rangle = \sum \pm T^{\int u^*\omega}$ RHS lies in the Novikov ring.

the point is that to apply Gromov compactness need bound on energy.

energy in HF is given by $\mathcal{A}_H$ which is S^1 valued.

energy in LF is symplectic area

Novikov coeff let us get a bound on energy by refining our construction to see homotopy/homology class

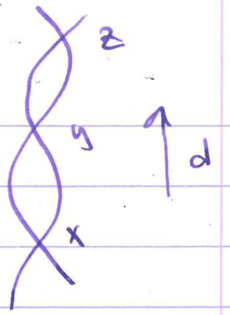
yay, no signs!

Let k be a ring e.g. $\mathbb{Z}$ or $\mathbb{Z}_2$ or $\mathbb{C}$ (conn to MS)
$\mathbb{Z}$ if gradable
or $\mathbb{Q}$ (if sphere bubbles) (if want product str on H_* b/c of AT see *.)

$$\Lambda_0^k = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in k,\ \lambda_i \in \mathbb{R},\ 0 \le \lambda_0 < \lambda_1 < \dots,\ \lim_{i\to\infty} \lambda_i = +\infty \right\}$$

Remark: A series w/ "real" powers lies in Λ_0^k if there are only finitely many terms of bded exponent. Gromov compactness + regularity ⇒ only finitely many contr. to $\langle dx, y\rangle$ of bded exponent.

$$\Rightarrow \quad CF^*(L F, \Lambda_0) = \bigoplus_{x\in L\cap F} \Lambda_0 \cdot \langle x\rangle.$$



tameness of $J \Rightarrow \int u^* \omega > 0$ if u is not const.
a_0 ~ contribution of constant disks.
by $\langle\!\uparrow \models \langle dx, y\rangle \in \Lambda_+$.

Why does $d^2 = 0$?

$$\langle d^2 x, z\rangle = \sum_{\substack{y \\ [v]\in \mathcal{M}^0(z,y)}} T^{E(v)} \sum_{\substack{y \\ [u]\in \mathcal{M}^0(y,x)}} T^{E(u)}$$

$$= \sum_{\substack{y \\ (u,v)\in \mathcal{M}^0(z,u)\times \mathcal{M}^0(y,x)}} T^{E(u)+E(v)}$$

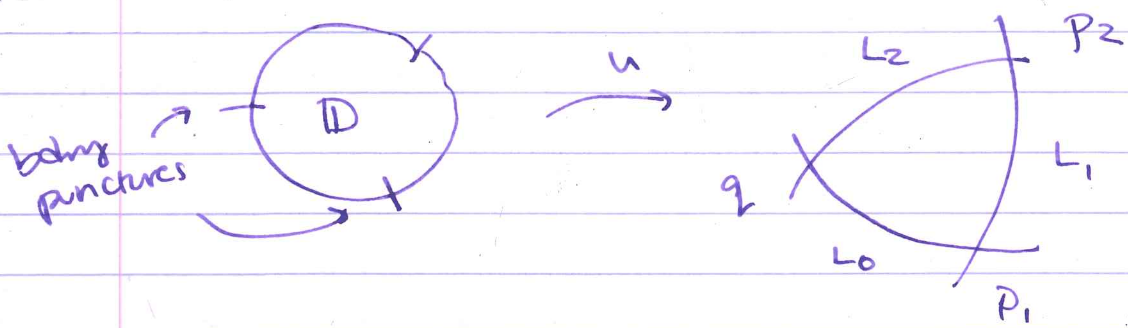
lemma: The bdry of $\mathcal{M}^{1,E}(z,x)$ is $\coprod_y \mathcal{M}^{0,E_1}(z,y)\times \mathcal{M}^{0,E_2}(y,x)$ where $E_1 + E_2 = E$. $\mathcal{M}^{1,E}$ is a 1-dim mfld.

It remains to show $\overline{\mathcal{M}}^1(z,x)$ is a mfld w/ ∂
the most painful ingredient in any Floer theory: gluing. Go read A-D.

$\rightarrow$ nice example in Nick talk I.

③ Product Structure. At this point we will remember that we should have kept track of the htpy class of strips, call it β.

Consider J-hol triangles:



Again have a moduli space $\mathcal{M}(p_1, p_2, q; \beta, J)$ of such J hol. Δ's w/ appropriate bdry / asymptotics in htpy class $[u] = \beta$.

more on index

we allow $J = J_z$ to depend on $z \in D$. Then for generic J_z this moduli space is a smooth mfld of dim given by Fred index.

we define a map

$$m^2 : CF(L_1, L_2) \otimes_\Lambda CF(L_0, L_1) \to CF(L_0, L_2)$$

$$m^2(p_2, p_1) := \sum_{q, \beta} \# \mathcal{M}_J(p_1, p_2, q, \beta) \cdot T^{\omega(\beta)} q.$$

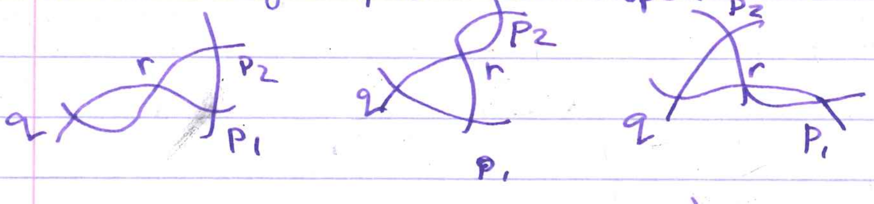
<u>lemma</u>: If the L_i are graded then

$$i(\beta) = \deg(q) - \deg(p_1) - \deg(p_2).$$

In particular, b/c we only count the 0-dim part of the moduli space, $i(\beta) = 0$ so m^2 <u>respects</u> the grading.

To work out what algebraic relations m^2 satisfies, we look at Gromov compactification of the 1-D component of the moduli space. Again one can ensure it is a compact 1-mfld w/ ∂.

Its bdry pts correspond to



→ we again assume $\omega|_{\pi_2(M, L_i)} = 0$ to rule out disc & sphere bubbling.

The fact that their signed count is 0 means

$$\partial m^2(p_2, p_1) + m^2(\partial p_2, p_1) + m^2(p_2, \partial p_1) = 0$$

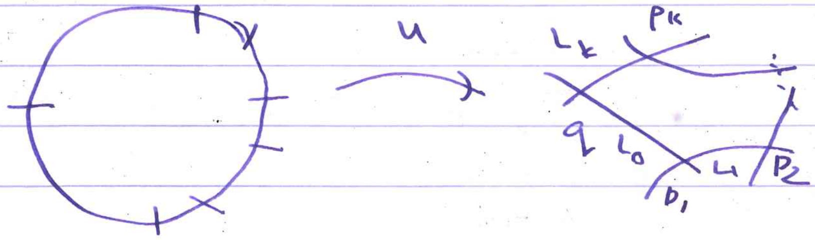
so m^2 defines a map

$$HF(L_1, L_2) \otimes_{\Lambda_0} HF(L_0, L_1) \to HF(L_0, L_2).$$

④ A_∞-relations

Wrapped Fukaya category
Donaldson-Fukaya category
Fukaya category.

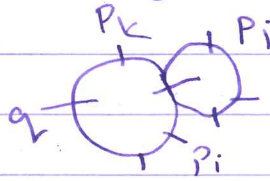
More generally we define the moduli space $\mathcal{M}(p_1, \dots, p_k, q, \beta, J_z)$ of J hol maps



up to biholomorphism counting the 0-dim pieces of $\mathcal{M}$ defines a map

$$m^k : CF(L_{k-1}, L_k) \otimes \dots \otimes CF(L_0, L_1) \to CF(L_0, L_k)$$

of degree $2-k$; counting the bdry pts of the Gromov compactification of the 1-d component of $\mathcal{M}$ shows

$\sum_q$  $= 0$.

$\Leftrightarrow \sum m^*(p_k, \dots, m^*(p_j, \dots), p_i, \dots, p_1) = 0$

These are the A_∞ relations! (5b) later

Note: If we define $\lambda(L) := \sum_{\mu(\beta)=2} n_\beta T^{\omega(\beta)} \in \Lambda$ (m^0) (π_2 L)

Then we have $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$
$\partial^2(p) = (\lambda(L_0) - \lambda(L_1)) \cdot p$

More

we want to define the Fukaya category $Fuk(M, \omega)$ to have

- objects $L \subset M$ Lag, graded, $\omega|_{\pi_2(M,L)} = 0$ & spin if we want to use a Novikov field of char $\neq 2$.
- morphism spaces $hom^\circ(L_0, L_1) := CF^*(L_0, L_1)$
- A_∞ str maps m^k.

char$(\mathbb{K}) \neq 2$
spin L_i

If no bubbling then we can define
$m^k : CF(L_{k-1}, L_k) \otimes \cdots \otimes CF(L_0, L_1) \to CF(L_0, L_k)$
It is the Λ_0 linear map def. by

$$m^k(p_k, \dots, p_1) = \sum_{\substack{q \in L_0 \pitchfork L_k \\ [u] \mid \text{ind}([u]) = 2-k}} \# \mathcal{M}^J(p_1, \dots, p_k, q; [u]) T^{\omega([u])} q$$

the moduli space of conformal structures $\mathcal{M}_{0,k+1}$ on D admits a natural compactification to a $(k-2)$ dim polytope $\overline{\mathcal{M}}_{0,k+1}$, the Stasheff associahedron whose top dim facets correspond to nodal deg of D to a pair of disks w/ each comp carrying at least 2 marked pts. higher codim correspond to nodal deg w/ more comp.

Prop: The operations m^k satisfy the A_∞ relations

$$0 = \sum_{\ell=1}^{k} \sum_{j=0}^{k-\ell} (-1)^* m^{k+1-\ell}(p_k, \dots, p_{j+\ell+1}, m^\ell(p_{j+\ell}, \dots, p_{j+1}), p_j, \dots, p_1)$$

where $* = j + \deg(p_1) + \dots + \deg(p_j)$

$k=1$ is $\partial^2 = 0$
$k=2$ is Leibniz rule
$k=3$ this expresses that the Floer product m^2 is associative up to an explicit htpy given by m^3.

And we would like to define the Donaldson-Fuk category $H^*(Fuk(M,\omega))$ to be its cohom cat, e.g. the one w/ morphism spaces $HF^*(L_0, L_1)$.
It's an honest Λ-linear $\mathbb{Z}$-graded cat – one has to check it has id. morphisms $e \in H^0(L) \cong HF^0(L,L)$.

Achtung: there is a problem: we only def. $CF^*(L_0,L_1)$ when $L_0 \pitchfork L_1$.
A subcat won't work (for stuff that does $\pitchfork$) b/c $CF^*(L,L)$ won't be defined.

to resolve: use Ham isotopy invariance: we choose $\psi \in Ham(M)$ s.t. $\psi(L_1) \pitchfork L_0$ & define

$$HF^*(L_0, L_1) := HF^*(L_0, \psi(L_1)).$$

by invariance the result is indep. of ψ.
L & $\psi(L)$ will be iso (resp. q-iso) objects of the Donaldson-Fukaya (resp Fukaya) cat.
D-F not bad. Defining Fuk is harder b/c $CF^*(L_0, \psi(L_1))$ depends on ψ but it can be done, however the final result is only indep. of choice up to A_∞-quasi-equiv.

Note

Define

this is so we can relax to monotone setting.

Define $\lambda(L) := \sum_{\mu(\beta)=2} n_\beta T^{\omega(\beta)} \in L$ → constant + disk w/ bdry on L
Then. we have $\partial: CF(L_0,L_1) \to CF(L_0,L_1)$; $\partial^2(p) = (\lambda(L_0) - \lambda(L_1)) \cdot p$
to express this we can introduce a curvature term in Fuk category:

$$m^0 := \lambda(L) \cdot e_L \in CF(L,L)$$

$\forall L$, so second A_∞ rel. reads

$$m^1(m^1(p)) + m^2(m^0, p) + m^2(p, m^0) = 0$$

$$\Leftrightarrow \quad \partial^2(p) + \lambda(L_1)p - \lambda(L_0)p = 0.$$

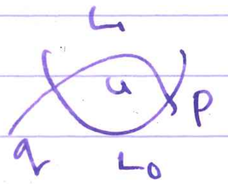
Thus if $\lambda(L_0) = \lambda(L_1)$ we can define $HF(L_0,L_1)$
~~e.g.~~ in particular if $L_0 = L_1$.

So we get $Fuk(M,\omega)_\lambda$: ob: $L \subset M$ w/ $\lambda(L) = \lambda$ } get a separate Fuk cat for each $\lambda \in \Lambda$. (15)

~~Gradings & Coefficients & Novikov~~...

It is not $\mathbb{Z}$-graded: $\mu_0 \neq 0$.
But if L_i are or we can define a $\mathbb{Z}_2$ grading: $|p|$ = sign with which L_0 intersects L_1 at p.

L' is Ham isotopic to L if $\exists\, \psi \in Ham(M,\omega)$ s.t. $L' = \psi(L)$.



Gradings: Let β be a htpy class of strips

trivialize $u^*TM \cong \mathbb{R} \times [0,1] \times \mathbb{C}^n$ as a CX VB
choose a path $\rho_p : T_pL_0 \to T_pL_1$
$\rho_q : T_qL_0 \to T_qL_1$
concatenate to get a loop
$\tilde{\rho} : S^1 \to \mathcal{G}(n)$ Lag. Grassmannian

$\mathcal{G}(n) \cong U(n)/O(n)$, $\pi_1(\mathcal{G}(n)) \cong \mathbb{Z}$.
This iso is called maslov index, call it μ.
It can be extended to a maslov index for paths, well-def integral grading depends on choice of nowhere vanishing section of $(\Lambda^n_{\mathbb{C}}(T^*M))^{\otimes 2}$ a quad ~~tot.~~ CX vol. form, which determines a map
$\eta : \mathcal{G}M \to S^1$
where $\mathcal{G}M$ is the bundle of Lag. subspaces of

but this is complicated & not always absolutely defined

Go read Nick's lecture notes I.