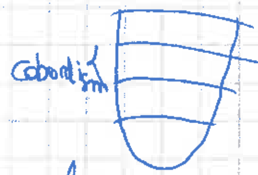
Alexandru Oancea

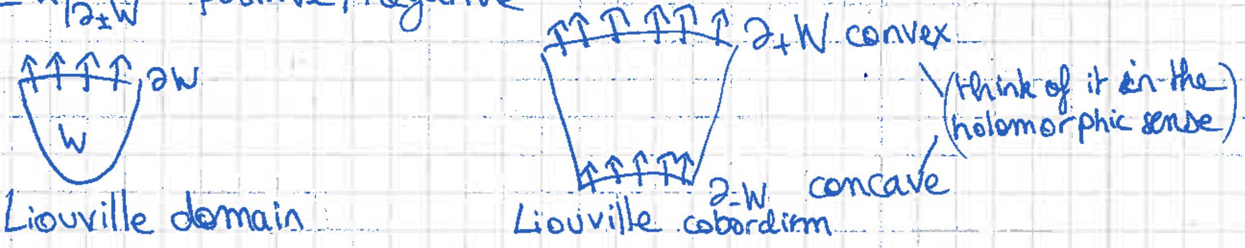
Symplectic homology and the Eilenberg Steenrod axioms.

Motivation: Understand SH_* (cf A. Fauck's precourse) behave under the simplest decompositions of a Liouville domain as a composition of cobordisms?

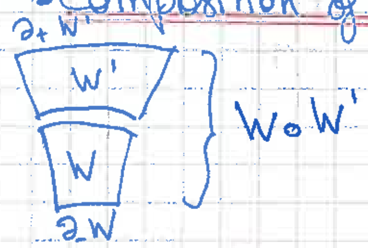


Recall (S. Courte's precourse) that a Liouville domain is (W, λ) cpact manifold with boundary $\lambda \in \Omega^1(W)$, $\omega = d\lambda$ symplectic, $\alpha = \lambda|_{\partial W}$ positive contact form ($\Leftrightarrow$ Liouville v. f. points outwards)

• Liouville cobordism: (W, λ) compact mfd w/ bdy $\lambda \in \Omega^1(W)$, $\omega = d\lambda$, $\alpha = \lambda|_{\partial W}$ contact and $\partial W = \partial_+ W \sqcup \partial_- W$ s.t $\alpha_\pm = \alpha|_{\partial_\pm W}$ positive/negative



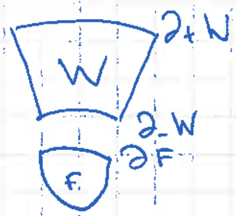
• Composition of cobordisms: $W \circ W' = W \sqcup W'$
$(\partial_+ W, \alpha_+) \simeq (\partial_- W', \alpha_-)$



Note: trivial cobordism: $(M, \xi = \ker(\alpha))$, $W = I \times M$, $\lambda = r\alpha$ where $r \in I \subseteq (0; \infty)$. To do the composition, use the fact that near $\partial_+ W$, W looks like a trivial cobordism and glue.

Change of contact form $\rightsquigarrow$ holomorphic Liouville 1-form on $I \times M$.

• filling of (the neg. boundary) of a Liouville cobordism (W) is a Liouville domain (F, λ_F) s.t. $(\partial F, \alpha_F) \simeq (\partial_- W, \alpha_-)$



filling of W Liouville cobordism

- pair of Liouville cobordism (with filling) (W, V)

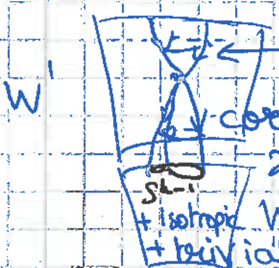
$W = W^{bottom} \circ V \circ W^{top}$ ← can be empty

} W (compatible with fillings)

Examples: • trivial cobordism $W = I \times M$ where M ctct mfd.

• handle attachment (Weinstein): subcritical / critical

$W^{2n} \sqcup H_k$, k index $\in \{0, \dots, n\}$ ← critical case



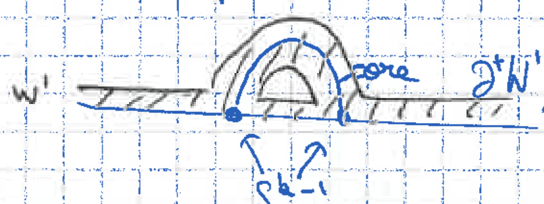
= cocore = coisotropic D^{2n-k}, dim = coindex = $2n-k$.

core = isotropic D^k, dim = index k.

+ isotropic

+ trivialization of $CSN(S^{k-1} \subset \partial W)$ ← data for handle attachment
isotropic

This gives rise to cobordism W' s.t. $\partial_- W' = \partial_+ W$ and $\partial_+ W'$ called surgery of the contact manifold $\partial_+ W$ along the isotropic sphere S^{k-1}.



$\partial^+ W = \partial_- W'$

- W Liouville domain: we get a pair $(W, \partial W) \underset{\text{convention}}{=} (W, I \times \partial W)$

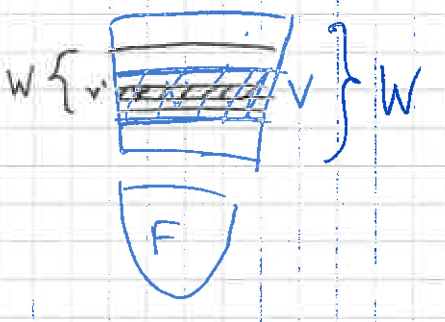
- W cobordism $(W, \partial_- W)$, $(W, \partial_+ W)$ ⚠ $(W, \partial W)$ is not formally as before because ∂W is not "connected horizontally"

In general, we shall identify in notation a contact manifold M and the trivial cobordism $I \times M$.

<u>Note</u>: the extension of the def of SH. to such Liouville cobordism ("sandwiched pairs") is formal.

With K. Cieliebak, we associate to a pair (W, V) of L. cob. with filling a module (over an arbitrary ring) noted $SH_*(W,V)$ (or $SH(W,V;\underset{\text{filling}}{F})$) with the following <u>properties</u>:

(0) <u>(VITERBO FUNCTORIALITY)</u> an embedding $(W,V) \overset{f}{\hookrightarrow} (W',V')$ of cobordisms which is exact ($f^*\lambda' = \lambda$), codim 0, and compatible with fillings, induces $f_! : SH(W',V') \longrightarrow SH(W,V)$



<u>Remark</u>: M. Abouzaid, P. Seidel call these kind of invariants cohomology. Algebraically, it looks like a cohomology because there is a natural product of degree 0 and functorial maps go this way.

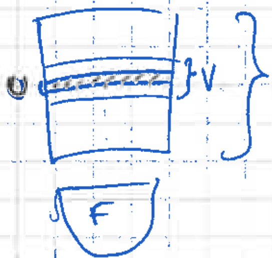
Geometrically, I think of SH_* as homology of a free loop space over some space (recall $SH(T^*M) \simeq H_*(\mathcal{L}M)$). Also $SH = \varinjlim$ (finite dim vector space) which does not look like cohomology

(i) <u>(HOMOTOPY INVARIANCE)</u>, f, g homotopic through such "admissible" embeddings then $f_! = g_!$.

(ii) <u>(EXACT TRIANGLE OF A PAIR)</u>: denote $V \overset{i}{\hookrightarrow} W \overset{j}{\hookrightarrow} (W,V)$ where (W,V) pair. Then there is a functorial exact triangle

$$\begin{tikzcd} SH(W,V) \arrow[rr, "j_!"] & & SH(W) \arrow[dl, "i_!"] \\ & SH(V) \arrow[ul, "{[-1]}"] & \end{tikzcd}$$

(iii) (EXCISION) (W, V, U) triple of L. cob. with filling.

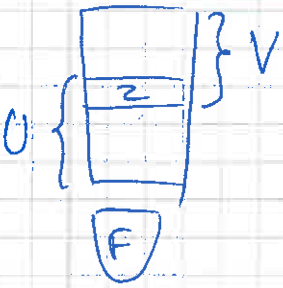


Then the transfer map induced by the inclusion $(W\setminus \mathrm{int}(U), V\setminus \mathrm{int}(U)) \overset{i}{\hookrightarrow} (W,V)$ is isomorphism $i_!: SH(W,V) \longrightarrow SH(W\setminus \mathrm{int}(U), V\setminus \mathrm{int}(U))$

(We interpret $(W\setminus \mathrm{int}(U), V\setminus \mathrm{int}(U))$ as the union of the two disconnected component and $SH(W\setminus \mathrm{int}(U), V\setminus \mathrm{int}(U))$ is the $\oplus$ of the SH of two disconnected component)

Consequence:

(iv) (MAYER-VIETORIS) there is an exact sequence



$$\begin{tikzcd} SH(U\cup V) \arrow[rr, "(j_{U!}, j_{V!})"] & & SH(U)\oplus SH(V) \arrow[dl, "(i_{U!}, -i_{V!})"] \\ & SH(Z) \arrow[ul, "{[-1]}"] & \end{tikzcd}$$

Motivations: * interpret from unified point of view various long exact sequence & Floer homology groups.

Examples: * (T. Ekholm's talk)

$$\begin{tikzcd} LH^{Ho}(\Lambda) \arrow[rr] & & SH(W^{2n}\sqcup H_n) \arrow[dl, "\text{transfer map}"] \\ & SH(W) \arrow[ul, "{[-1]}"] & \end{tikzcd}$$

→ interpret $LH^{Ho}(\Lambda)$ as "computation" of $SH(W\sqcup H, W)$

* (SH^{+,S^1}: of J. Gutt, McLean):

$$\begin{tikzcd} SH^{+,S^1}(D^{2(n-k)}) \arrow[rr] & & SH^{+,S^1}(W^{2n}\sqcup H_k) \arrow[dl] \\ & SH^{+,S^1}(W^{2n}) \arrow[ul, "{[-1]}"] & \end{tikzcd}$$

(consequence of this formalism)

(for $k=1$: previously obtained by Bourgeois & Van Koert in the context of CH^{lin}.)

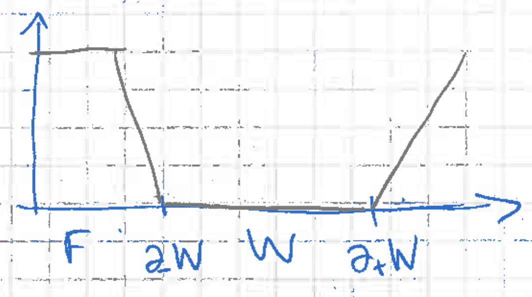
* Study cobordisms

Theorem (w/ P. Albers & K. Cieliebak): $\exists$ Liouville cobordisms W s.t. $\partial_+ W$ is subcritically Stein fillable & $\partial_- W$ is hypertight (i.e. $\exists$ contact form with no contractible closed periodic Reeb orbits)

Note: this generalizes Weinstein conjecture for subcritical Stein fillable contact manifolds: such a manifold cannot be hypertight

Definition: W cobordism with filling F. Consider hamiltonian $H: \widehat{F \circ W} \longrightarrow \mathbb{R}$, $H \equiv 0$ on W with the shape:

$\widehat{F\circ W} = F \circ W \sqcup \partial_+ W \times [1; +\infty[$



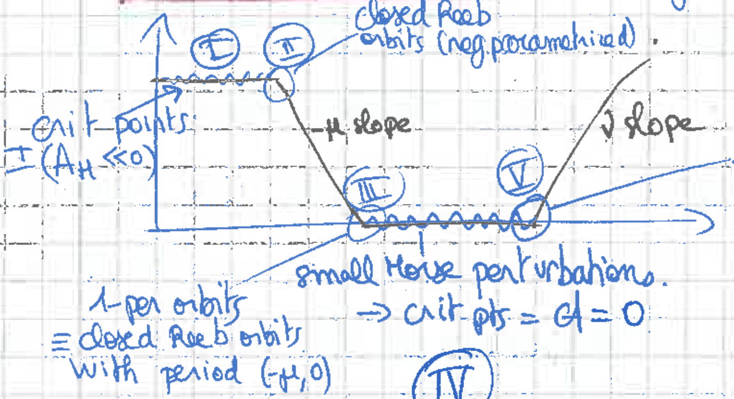
$H(r,x) = \nu r + \nu_0$ on $[1; +\infty[\times \partial_+ W$ (constants)

Then define $SH_*(W) = \underset{b\to\infty}{\underrightarrow{\lim}}\ \underset{a\to-\infty}{\underleftarrow{\lim}}\ \underset{\substack{H\equiv 0 \text{ on } W \\ H\to\infty \text{ on } \widehat{F\circ W}\setminus W}}{\underrightarrow{\lim}} \overbrace{FH^{(a,b)}(H)}^{\text{complex generated by orbits with action } \in (a,b)}$

Convention: $\omega(X_H, \cdot) = -dH \Rightarrow X_{h(r)} = h'(r) R$

$\mathcal{A}_H : C^\infty(S^1, \widehat{F\circ W}) \longrightarrow \mathbb{R}$, $\mathcal{A}_H(\gamma) = \int \gamma^*\lambda - H_t(\gamma(t))\,dt$

What are the generators of $FC^{(a,b)}(H)$?



There are 5 types of generators in $FC_*^{(a,b)}(H)$ but as $H \to \infty$, type I and II disappear because their actions is $\ll 0$ and go out of the interval (a,b)

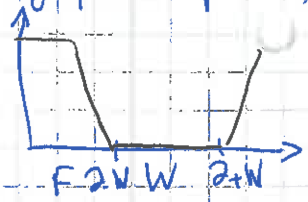
(Remark: $FC^{(-\infty, b)}(H)$ subcomplexes since ∂ decreases action)
→ explain the sense of direct/indirect limits)

And when we take $\underset{b\to\infty}{\underrightarrow{\lim}}\ \underset{a\to-\infty}{\underleftarrow{\lim}}$ we see "all" of III, IV, V.)

Lecture 2.

• $SH_\cdot(W)$ is ring with unit (w.r.t. pair of pants product)

(Recall $SH_\cdot(W) = \varinjlim_{b\to\infty} \varprojlim_{a\to-\infty} \varinjlim_{H} FH^{(a,b)}(H)$ with H:



Pair of pants: at chain level:

$$FC^{(-\infty,b)}(H) \otimes FC^{(-\infty,b')}(H) \longrightarrow FC^{(-\infty,b+b')}(2H)$$

$$FC^{(-\infty,b)} \otimes FC^{(-\infty,2b)}(H) \longrightarrow FC^{(-\infty,2b)}(2H)$$

$$FC^{(-\infty,a)} \otimes FC^{(-\infty,b)} \longrightarrow FC^{(-\infty,a+b)}$$

$a<b$, $FC^{(a,b)} = FC^{(-\infty,b)}/FC^{(-\infty,a)}$ and $FC^{(a,b)} \otimes FC^{(a,b)} \longrightarrow FC^{(a+b,2b)}$

Now take limits: $\varinjlim_{H}$: $SH^{(a,b)} \otimes SH^{(a,b)} \longrightarrow SH^{(a+b,2b)}$

$\varprojlim_{a\to-\infty}$: $SH^{(-\infty,b)} \otimes SH^{(-\infty,b)} \longrightarrow SH^{(-\infty,2b)}$

$\varinjlim_{b\to+\infty}$: $SH \otimes SH \longrightarrow SH$

$\to$ 2nd motivation for adopting this order in the direct/inverse limit:
- pair of pants product would not have worked in the other sense
- direct limit is an exact limit

(disclaimer):
- inverse limit is not an exact function. However it is exact when applied to systems of finite dim vector space.

↳ for this reason, all exact triangles have been established with coefficient in a field. But definition of $SH_\cdot$ works on any ring.

About pair-of-pants product in noncompact setting:

$S = \mathbb{C}P^1 \setminus \{z_1^+, z_2^+, z^-\}$: choose "cylindrical ends" near the punctures

i.e. identifications of nghbds of:
holomorphic

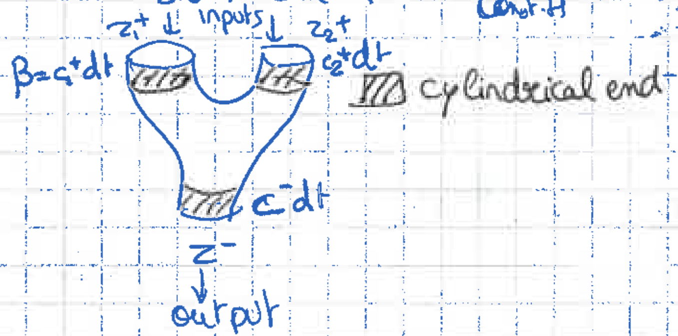
- z_1^+, z_2^+ with $[0,\infty[_s \times S^1_t$
- z^- with $]-\infty, 0]_s \times S^1_t$

Pick Hamiltonian H as before, and before and $\beta \in \Omega^1(S,\mathbb{R})$ such that $\beta = c\,dt$ near each puncture (c = constant) and consider the equation $(du - X_H \otimes \beta)^{0,1} = 0$.

$$(du - X_H \otimes \beta)^{0,1} = \frac{1}{2}\big(du - X_H \otimes \beta + J(du - X_H\otimes\beta)\, i\big)$$

Near the punctures in coord $s+it$ equivalent to

$\partial_s u + J(\partial_t u - X_{\text{const}\cdot H}) = 0$ ($X_{c_1^+ H}$ near z_1^+, $X_{c_2^+ H}$ near z_2^+ and $X_{c^- H}$ near z^-).



<u>Fact:</u> solutions $u : S \to \hat{W}$ of this equation (with asymptotes $\gamma_1^+, \gamma_2^+, \gamma^-$ at the corresponding punctures) obey the max principle provided <u>$d\beta \leq 0$</u> ($\forall z \in S$, $d\beta_z$ is non-positive multiple of area form on S).

Necessary (and sufficient) condition for the existence of such a β is <u>$c_1^+ + c_2^+ - c^- \leq 0$</u> (Stokes formula).

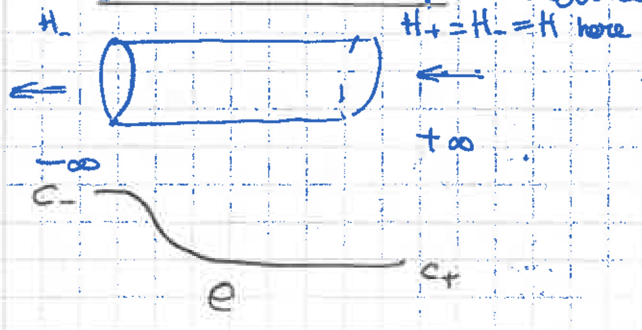
<u>Rmk:</u> More precisely $\psi(\gamma_1^+ \otimes \gamma_2^+) := \sum \#\mathcal{M}(\gamma_1^+, \gamma_2^+; \gamma^-)\, \gamma^-$
(outputs / inputs)
$\mathcal{M}$: moduli space of solutions to our equation $(du - X_H \otimes \beta)^{0,1} = 0$ with asymptotic condition $\gamma_1^+, \gamma_2^+, \gamma^-$

<u>Rmk (continuation maps / increasing homotopies):</u> $S = \mathbb{CP}^1 \setminus \{z^+, z^-\} = \mathbb{R} \times S^1$

Given H, we wish to consider 1-forms $\beta(s,t) = \rho(s)\,dt$.

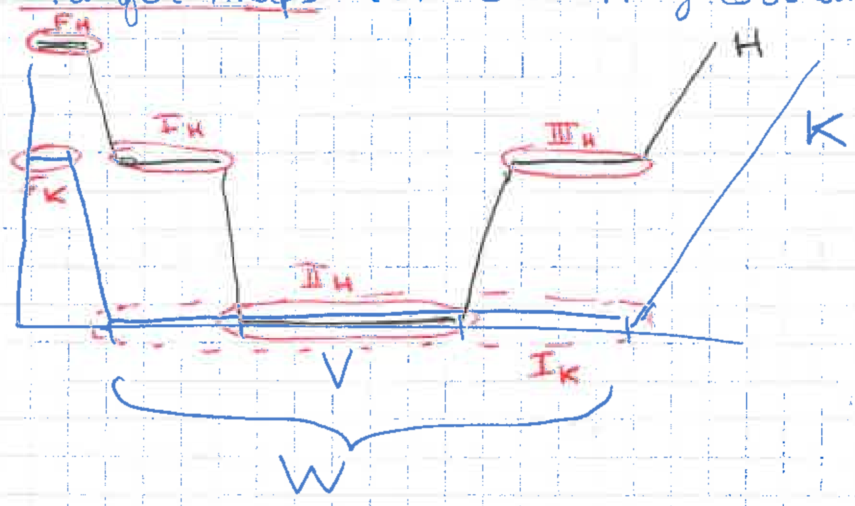
The condition $d\beta \leq 0$ amounts to $\rho'(s)\, ds \wedge dt \leq 0$ i.e. $\rho'(s) \leq 0$

The count of solutions to $(du - X_H \otimes \beta)^{0,1} = 0$ correspond to <u>continuation maps</u> induced by <u>increasing homotopies</u>



$\exists \rho$ s.t. $\rho'(s) < 0 \Leftrightarrow$ <u>$c^+ \leq c^-$</u>

Transfer maps (in the context of cobordisms): $V \subset W$, (W,V) pair



We can arrange slopes such that. $\underbrace{F_K < I_K}_{\text{actions are} <}$ (for K)

Moreover we can achieve

$F_H \prec I_H \prec III_H \prec II_H$ where $\prec$ means that there are no Floer trajectories from the smaller group of orbits to the larger ones.

This last statement cannot be achieved solely by action requires further arguments (will be explained later).

and $F_K < II_H$

$$\text{Now:}\quad \begin{tikzcd} FC(K) \arrow[r, "\text{continua}^\circ"', "\text{map}"] \arrow[d, "\text{quotient}"'] & FC(H) \arrow[d, "\text{(quotient map)}"] \\ FC(I_K) \arrow[r, dashed, "\text{induced contin. map}"'] & FC(II_H) \end{tikzcd}$$

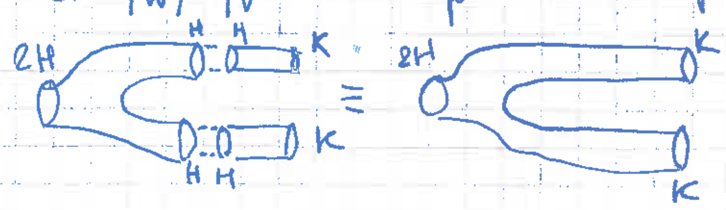
after limit(s) produces $SH(W)$ after limit(s), produces $SH(V)$

Viterbo transfer map $i_! : SH(W) \longrightarrow SH(V)$ obtained in this way is fundamentally a continuation map.

As such it respects the product structure & preserves the unit.

Proof by picture: Let ψ_W, ψ_V be the product maps for W, V.

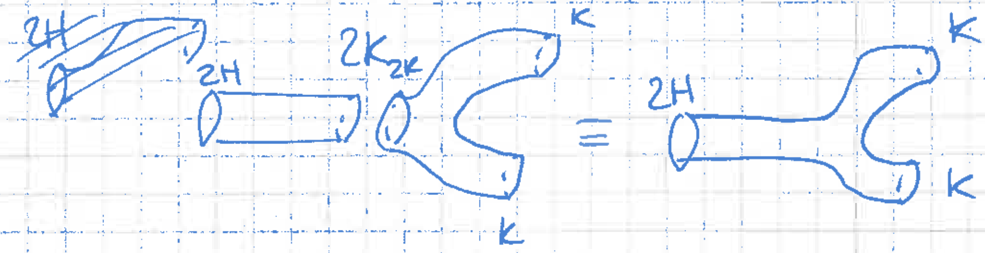
$\psi_V(i_!(\alpha) \otimes i_!(\beta))$:



Here we have some family of Hamiltonians $\tilde{H}_z$, $z \in S$, $\beta \in \Omega^1(S)$ and $\boxed{d(\hat{H}_z(x)\beta) \leq 0 \text{ for each } x \in \hat{W}}$

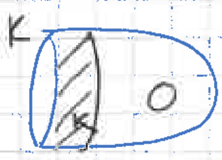
(generalized form of max principle see Ekholm-O.)

$i_!(\psi_W(\alpha \otimes \beta))$:

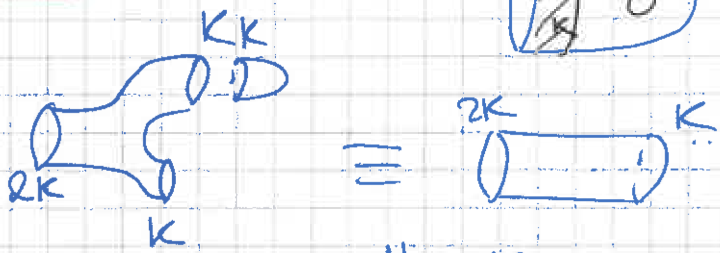


The max principle condition is a convex condition $\rightarrow$ we can interpolate between the two conditions.

Unit: $S = \mathbb{C}P^1 \setminus \{z^-\}$.



1_W



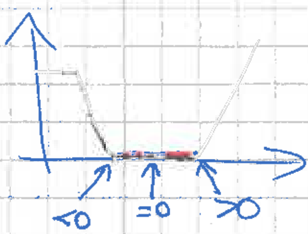
(proof of $1_W \cdot x = x$)

$1_V = i_!(1_W)$:

Action filtered versions: motivation: compare $SH(W)$ and $H(W)$

Indeed, $H^{n-*}(W) \cong SH_*^{=0}(W)$. This can only be achieved by two successive truncations which give rise to

$SH^{<0}$, $SH^{\leq 0}$, $\underbrace{SH^{>0}}$, $SH^{\geq 0}$.
non-equivariant version of linearized contact homology.



$\rightarrow$ tautological exact triangles:

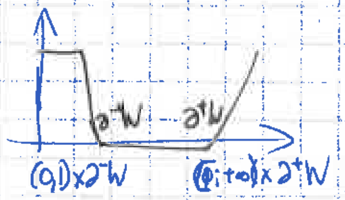
$$\begin{tikzcd} SH^{\leq 0} \arrow[r] & SH \arrow[dl] \\ SH^{>0} \arrow[u, "{[-1]}"] & \end{tikzcd} , \qquad \begin{tikzcd} SH^{<0} \arrow[r] & SH^{\leq 0} \arrow[dl] \\ SH^{=0} \arrow[u, "{[-1]}"] & \end{tikzcd}$$

Since continuation maps (increasing) preserve action filtration, have: $i_! : SH^{\heartsuit}(W) \longrightarrow SH^{\heartsuit}(V)$, $\heartsuit = <0, \leq 0, =0, >0, \geq 0$

& all axioms written yesterday hold for $\forall \heartsuit$

<u>Recall</u>: $\nexists$ Liouville cobordism W s.t. $\partial_+ W$ subcritically Stein fillable
$\partial_- W$ hypertight.

<u>Proof</u>: Note that $SH_*(W)$ is defined in exactly the same way for a cobordism W s.t. $\partial_- W$ is hypertight.



Indeed there can be no trajectories (Floer) escaping into the negative end of the symplectization. Otherwise SFT compactness would produce contractible Reeb orbits.

Assume for simplicity, $c_1(W)=0$: integral grading (though proof goes also if $c_1 \neq 0$).

$$\begin{tikzcd} & W & \\ \partial^- W \arrow[ur] & & \partial^+ W \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & SH(W) \arrow[dl] \arrow[dr] & \\ SH(\partial^- W) & & SH(\partial^- W) \end{tikzcd}$$

Write long exact sequence:

$$\begin{tikzcd}
SH^{>0}_{n+1}(\partial_- W) \arrow[r] & SH^{=0}_n(\partial_- W) \cong H^0(\partial_- W) \arrow[r] & SH^{\geq 0}(\partial_- W) \arrow[r] & SH^{>0}_n(\partial_- W) \\
SH^{>0}_{n+1}(W) \arrow[u] \arrow[d, "\wr"] \arrow[r, "\alpha"] & SH^{=0}_n(W) \cong H^0(W) \arrow[u] \arrow[d, "\text{injective}"] \arrow[r] & SH^{\geq 0}_n(W) \arrow[u] \arrow[d, "\cancel{\text{surjective}}"] \arrow[r] & SH^{>0}_n(W) \arrow[u] \arrow[d, "\wr"] \\
SH^{>0}_{n+1}(\partial_+ W) \arrow[r, "\alpha"] & SH^{=0}_n(\partial_+ W) \cong H^0(\partial_+ W) \arrow[r] & SH^{\geq 0}_n(\partial_+ W) \arrow[r] & SH^{>0}_n(W)
\end{tikzcd}$$

$1_{\partial_- W} \mapsto 0$, $1_W \mapsto 0$ because α injective, $1_{\partial_+ W} \mapsto 0$ ($\partial_+ W$ Stein fillable)

<u>Fact</u>: for subcritically fillable contact manifolds, SH vanishes and does not depend on the filling

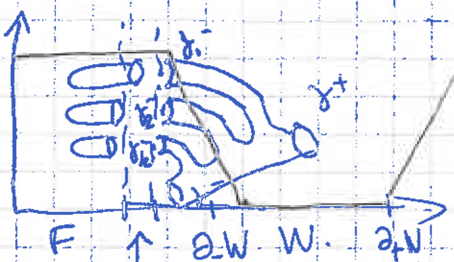
$\Rightarrow 1_{\partial_- W} = 0 \Rightarrow SH(\partial_- W) = 0$. But for hypertight $\underset{0}{SH} \cong \underset{0}{H}$ with $SH = 0$, $H \neq 0$.

$\rightarrow$ Contradiction

Lecture 3.

1) Dependance / independance w.r.t filling.

Prop: $SH_\cdot(W)$ is indep of the filling provided $c_1(W)=0$ and $\partial_- W$ admits contact form s.t. all closed contractible Reeb orbits are nondeg & have $CZ(\gamma)+n-3>0$ ($\dim W = 2n$).

Proof: 

$$\dim \mathcal{M}(\gamma^+,\gamma^-;\gamma_1^-,\dots,\gamma_k^-) = \underbrace{CZ(\gamma^+)-CZ(\gamma^-)-1}_{=0} - \underbrace{\left(\sum_{i=1}^k CZ(\gamma_i^-)+n-3\right)}_{<0 \text{ if } k\geq 1}$$

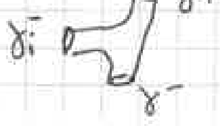
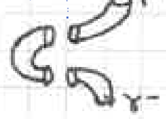
stretch neck here: replace cylindrical almost complex structure on $(1-2\delta,1-\delta)\times\partial_- W$ by cylindrical a.c.s. on $(\varepsilon,1-\delta)\times\partial_- W$ via some diffeo $(1-2\delta,1-\delta)\simeq(\varepsilon,1-\delta)$ and let $\varepsilon\to 0$.

$\to \mathcal{M}(\gamma^+,\gamma^-;\gamma_1^-\dots\gamma_k^-)=\emptyset$ if $k\geq 1$. $\square$

Variant: If $c_1(W)=0$ and $\partial_- W$ has contact form s.t. all closed contractible R. orbits are nondegenerate & have $CZ(\gamma)+n-3>1$ then $SH_\cdot(W)$ can be defined without filling. (ensures well definedness of contin. maps.

Rmk: In the spirit of T. Ekholm's lectures: one could define $SH_\cdot(W)$ in the absence of a filling with coeff in DGA of $\partial_- W$.

Any choice of augmentation defines a "linearized" version.

⚠ Sequence of Floer traj. can degenerate to:  but perhaps to  ← this cannot happen (see later) for convex Ham + max principle

Thm (P. Uebele). Let (M,ξ) contact mfd, $\pi_1(M)=0$ & $c_1(\xi)=0$ admits nondeg contact form s.t. $\forall$ closed Reeb orbit $CZ(\gamma)>3$.
Then $SH(M)$ is well defined as ring.

2) Relationship to other homologies.

- T. Ekholm: $X=\partial Y$, Liouville domain: $C(Y) = \check{P} \oplus \hat{P} \oplus \mathrm{Morse}$
  - $H_\cdot(C(Y)) \simeq SH(Y)$
  - $H_\cdot(\check{P} \oplus \hat{P}) \simeq SH^{>0}(Y)$.

If W is a cobordism then $SH(W)$ admits similar description.

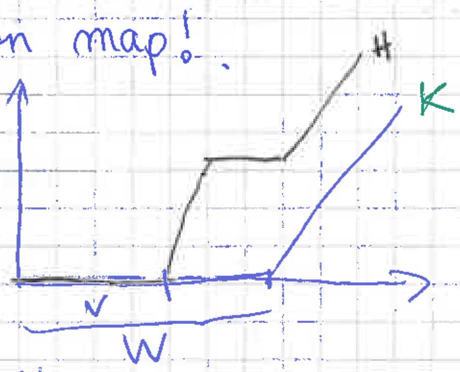
$$\underbrace{\hat{P}_+ \oplus \check{P}_+}_{\text{homological}} \oplus \mathrm{Morse} \oplus \underbrace{\hat{P}_- \oplus \check{P}_-}_{\text{cohomological}}$$

- $CH^{lin}(X) = SH^{S^1,>0}(Y)$ S^1-equivariant symplectic hom.

Through these identification, the transfer map corresponds to the cobordism map.

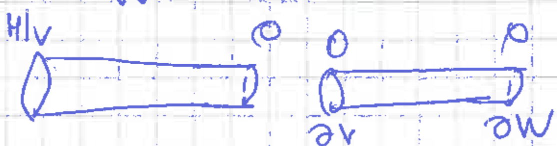
Indeed both transfer and isomorphism CH & SH are continuation map!

Proof:

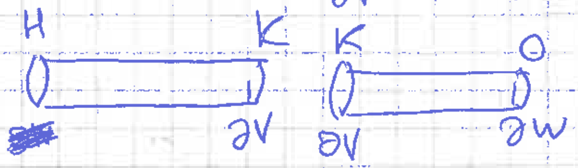


$$\begin{tikzcd} H_\cdot(C(W)) \arrow[r, "cob"] \arrow[d, "iso"] & H_\cdot(C(V)) \arrow[d, "iso"] \\ SH(W) \arrow[r, "transfer"] & SH(V) \end{tikzcd}$$

iso ∘ cob:



transfer ∘ iso:



• Rabinowitz–Floer homology (Cieliebak–Frauenfelder):

$RFH(W, \partial W) \simeq SH(\partial W)$ (W Liouville domain).

RFH sees "leafwise intersection points": $\phi \in Ham_c(\hat{W})$. Call
(Albers–Frauenfelder)

$x \in \partial W$ a leafwise intersection point if $\phi(x) \in \partial W$ and lies on the same Reeb traj as x. (generalize to certain coisotropic Bolle–Kang)

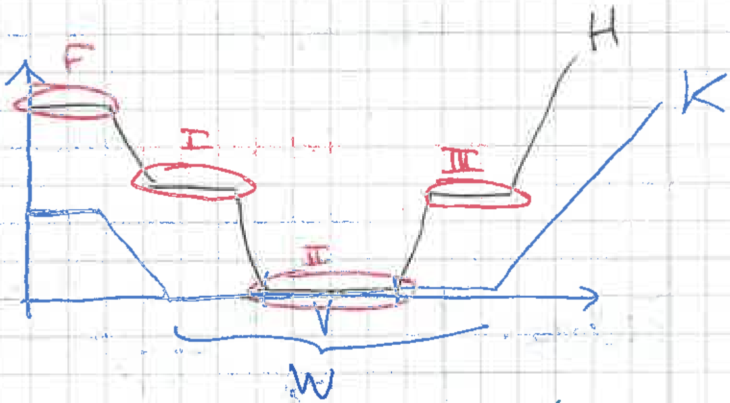
• cf T. Ekholm: Wrapped Floer homology of some exact Lagrangian L

$WH(L)$ falls under same formalism

$\rightsquigarrow$ exact Lag cobordism (Chantraine–Ghiggini–Golovko–Rizell)

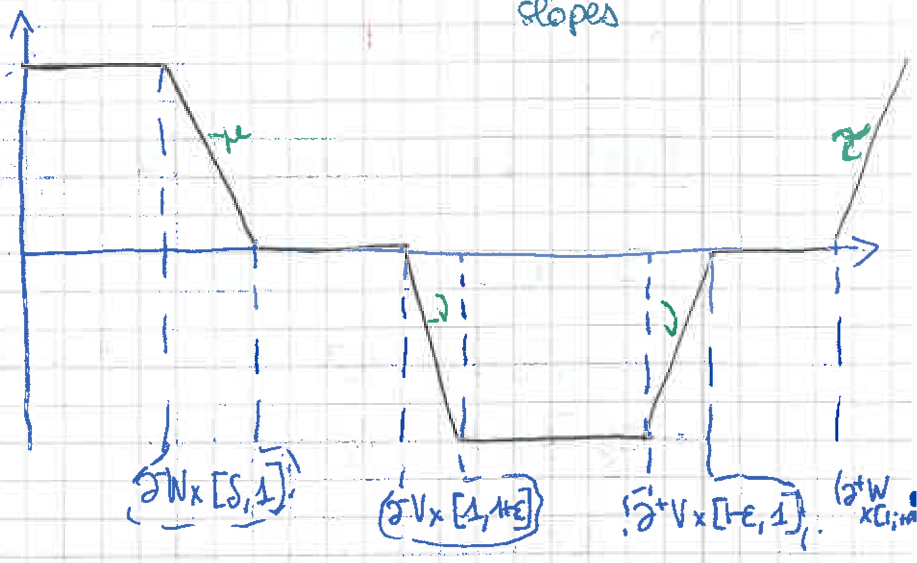
3) Confinement tools.

Recall earlier picture:



and statement: $F \prec I \prec III \prec II$ and also $\underset{\text{slopes}}{III \prec I}$

Redraw picture for H with parameters and shifted for convenience



Prop: Fix $a<b$. If the parameters $\mu, \nu, \tau, \delta, \varepsilon$ satisfy

$(\mu,\nu,\tau \gg 0$, $\varepsilon,\delta>0$ small$)$

- $(1-\delta)\mu > \min(-a, \nu-\eta_\nu)$
- $\varepsilon\nu > \min(b, \tau-\eta_\tau)$

[Here ν,τ are chosen s.t. they are not equal to the period of a closed $O<\eta_\nu$, η_τ is the distance from ν,τ to the set of periods of closed Reeb orbits.

and if we use an a.c.s. structure with long enough neck near $(1-2\varepsilon)\times\partial_+V$ then

$F \prec I \prec III \prec II$, $III \prec I$ for closed R orbits in action window (a,b).

Lemma 1: $(M, \xi=\ker(\alpha))$, $((0,\infty)_r\times M, r\alpha)$ symplectization, $H=h(r)$

(+∞ input, −∞ output)

Consider $u=(a,f): \mathbb{R}\times S^1 \longrightarrow (0,\infty)\times M$ sol of

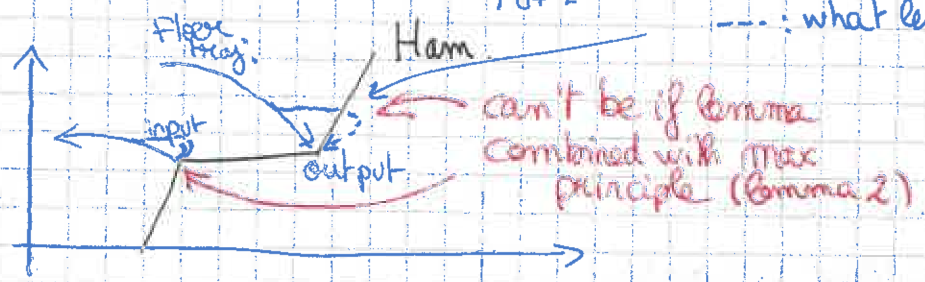
$\partial_s u + J(\partial_t u - X_H) = 0$, $\lim_{s\to\pm\infty} u(s,\cdot) = (r_\pm, \gamma_\pm)$

(i) If $h''(r_-)>0$, then either $\exists (s_0,t_0)\in\mathbb{R}\times S^1$ s.t. $a(s_0,t_0)>r_-$ or u is constant equal to (r_-,γ_-)

(ii) If $h''(r_+)<0$. Then either $\exists(s_0,t_0)$ s.t. $a(s_0,t_0)>r_+$ or u constant equal to (r_+,γ_+)

---- : what lemma 1 says.

Pictures:



Proof: Decompose Floer equation on ∂_r, R, ξ.

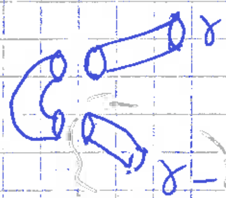
- $\partial_s a - a\,\alpha(\partial_t f) + a h'(a) = 0$ ← divide by $a \rightsquigarrow \partial_s(\log a)$
- $a\alpha(\partial_s f) + \partial_t a = 0$. ← divide by $a \rightsquigarrow \partial_t(\log a)$
- $\pi_\xi \partial_s f + J\pi_\xi \partial_t f = 0$.

and show that $\log a(s) = \int_{S^1} \log a(s,t)\,dt$ is either

- non increasing in case (ii)
- non decreasing in case (i)

For whole proof see Bourgeois-O.

A. Oancea

Remark: Lemma 1 forbids 

of the form seen earlier

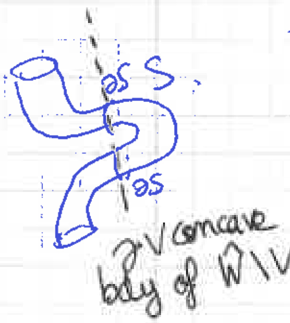
Lemma 2: $\hat{W}$ completion of L. domain, $V \subset \hat{W}$ L. subdomain, H hamiltonian s.t. $H = h(r)$ near $\partial V = \{r = 1\}$.

(reform. of max principle by Abouzaid Seidel)

If both asymptotes of a Floer cyl for H are contained in V, then u is contained in V.

Proof: Assume not and reach contradiction.

Let S piece of cylinder such that $u(\partial S) \subset \partial V$ and $u(S) \subset \hat{W} \setminus V$



$$E(u|_S) = \int_S \|\partial_t u - X_H\|^2 = \int_S u^*\omega - u^* dH \wedge dt$$

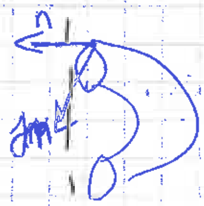
$$= \int_{\partial S} u^*\lambda - u^* H dt$$

$$\leq \int_{\partial S} u^*\lambda - \lambda(X_H)dt = \int_{\partial S} \lambda(du - X_H \otimes dt)$$

$(du - X_H \otimes dt)^{0,1} = 0$

$$= \int_{\partial S} -\lambda J (du - X_H \otimes dt) j = \int_{\partial S} -dr\,(du - X_H \otimes dt) j$$

$$= \int_{\partial S} -dr(du) j \leq 0 \text{ because}$$



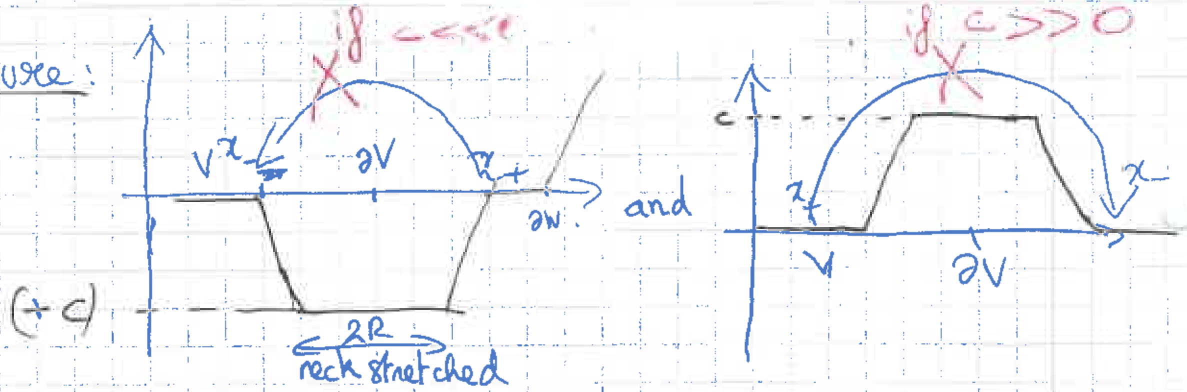
Lemma 3: $\hat{W}$ completion of L. domain, V L. subdomain, $H \equiv c$ near ∂V. Let J_R be a c.s. obtained by inserting cylinder of length $2R$ around V.

Then for R large enough, $\nexists$ J_R-Floer traj. $u : \mathbb{R} \times S^1 \to \hat{W}$ asymptotic at $\pm\infty$ to $x_\pm$ s.t.:

(1) $x_- \subset \mathrm{int}\, V$, $x_+ \subset \hat{W} \setminus V$ and $A_H(x_+) < -c$ or

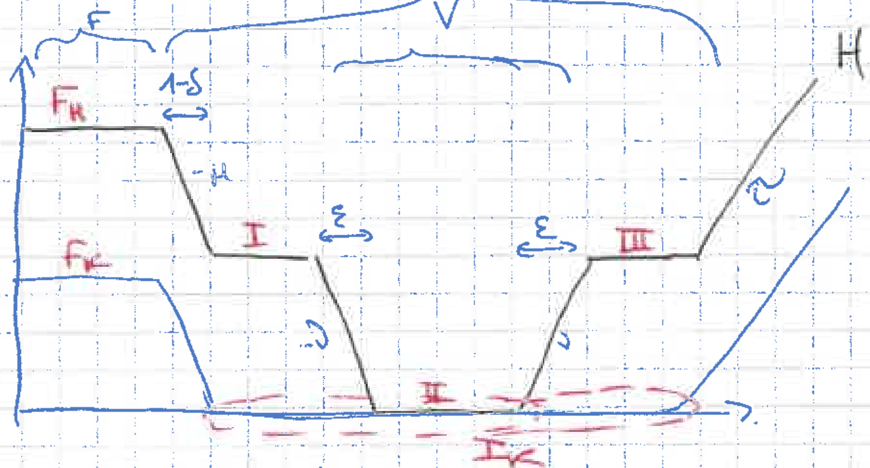
(2) $x_+ \subset V$, $x_- \subset \hat{W} \setminus V$ and $A_H(x_-) > -c$

Picture:



Proof: exhibit (close to the limit) a separating loop on the cylinder s.t. $v(S)$ is close to closed Reeb orbit on ∂V.

Recall:



with conditions: $\cdot\ (1-\delta)\mu > \min\{-a, \nu - \eta_\nu\}$

$\cdot\ \varepsilon\nu > \min\{b, \tau - \eta_\tau\}$

Think from now in terms of "cofinal" families

K_{ij}, H_{ij} $\quad j \in \mathbb{Z}^+$ ("value of pos slope") (For H: ν, τ)

$i \in \mathbb{Z}^-$ "value of neg slope" (For H: μ)

$SH^\heartsuit_*(W) = \varinjlim_j \varprojlim_i FH_{I_K}^\heartsuit(K_{ij})$ (instead of truncating action by a, b, we only consider orbits in class II, I_K)

$SH^\heartsuit_*(V) = \varinjlim_j \varprojlim_i FH_{II}^\heartsuit(H_{ij})$

$f^\heartsuit_{ij} : FC_{I_K}(K_{ij}) \longrightarrow FC_{II}(H_{ij})$ fit in a doubly directed system

f_{ij} continuation map

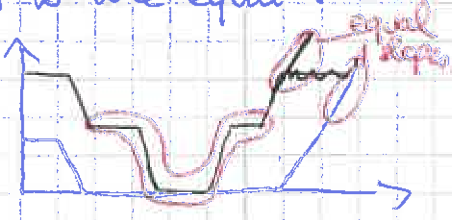
Viterbo transfer map is $f^\heartsuit_! = \varinjlim_j \varprojlim_i f^\heartsuit_{ij}$

Define $\boxed{SH^\heartsuit_*(W,V) = \varinjlim_j \varprojlim_i FH^\heartsuit_{(I,III)}(H_{ij})}$

We can also give a definition in the same spirit as for $SH(W)$: $\underline{SH(W,V)} = \varinjlim_b \varprojlim_a \varinjlim_{\substack{H\to\infty \\ \text{on } \hat{F_0W}\setminus W}} \varprojlim_{\substack{H\to-\infty \\ \text{on } V}} FH^{(a,b)}(H)$
$(H\equiv 0 \text{ on } W\setminus V)$

Mnemotechnic rule: Ham goes to $-\infty$ near region w.r.t. which we compute SH; else too.

Intuitively, we will get long exact sequences since $FC_{I}(K) \simeq FC_{I,II,III}(H)$ if slopes at ∞ are equal!

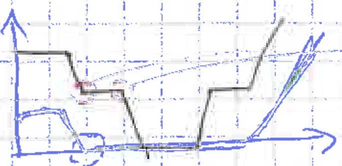


Then, we use short exact sequence:

$$0 \to FC_{I,III}(H) \longrightarrow \underset{\substack{\wr\| \\ FC_I(K)}}{FC_{I,II,III}(H)} \to FC_{II}(H) \to 0$$

& pass to the limit.

⚠ This argument doesn't work with ♡ version as easily for example: $FH^{<0}(K)$



We will define a version $SH^{♡,\text{cone}}(W,V)$ which fits naturally in exact triangle

$$\begin{tikzcd} SH^{♡}(W) \arrow[r] & SH^{♡}(V) \arrow[dl] \\ SH^{♡,\text{cone}}(W,V) \arrow[u, "{[-1]}"] & \end{tikzcd}$$

and we prove $SH^{♡,\text{cone}}(W,V) \simeq SH^{♡}(W,V)[-1]$
↖ algebraic construction.

Lemma: $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ exact sequence of complexes which is split as exact sequence of modules.

(i) a splitting $s: C \to B$ induces canonical identification $B = C(f)$, $f: C[1] \to A$.

(ii) There are isomorphisms in Kom $\phi: C \xrightarrow{\sim} C(i)$, $\phi = \begin{pmatrix}0\\1\\f\end{pmatrix}$, $\tau: A[-1] \xrightarrow{\sim} C(p)$, $\tau = \begin{pmatrix}0\\-1\\0\end{pmatrix}$ realize iso of distinguished triangle

Lemma: (3x3 lemma):

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "s"] & Y \arrow[d, "\psi"] \\ X' \arrow[r, "g"] & Y' \end{tikzcd}$$

commutative diagram in Kom: $\psi f - g\varphi = \partial_Y s + s\partial_X$ where $s: X \to Y'$ degree -1.

This diagram can be completed to a diagram whose rows and columns are distinguished triangles, all squares commute except the bottom right one which anti-commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[r] \arrow[d, "\psi"] & Z \arrow[r] \arrow[d, "\chi"] & X[-1] \arrow[d] \\ X' \arrow[r, "g"] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X'[-1] \arrow[d] \\ X'' \arrow[r, "h"] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] \arrow[dr, phantom, "-"] & X''[-1] \arrow[d] \\ X[-1] \arrow[r] & Y[-1] \arrow[r] & Z[-1] \arrow[r] & X[-2] \end{tikzcd}$$

(If you think as $Z = C(f)$, $Z' = C(g)$ then $\chi = \begin{pmatrix}\psi & s\\ 0 & \varphi\end{pmatrix}$ and the lemma says that $C(\chi) \simeq C(h)$ in Kom.)

Application: l.e.s. of (W,V) is compatible tautological exact sequences (induced by action)

__Later (or not):__ it is not automatic that, if we have a doubly directed system of cpxes in Kom, ~~then~~ $f_{ij} : X_{ij} \to Y_{ij}$, then $C(f_{ij})$ form a doubly directed system.

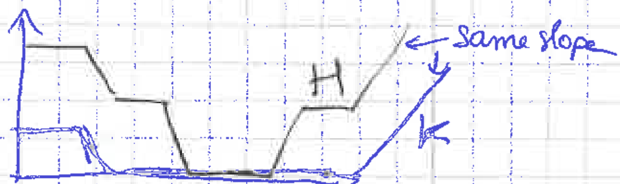
(This will come from Floer theory)

However, this holds for Floer theory (if the maps are induced by continuation) and hence we can define:

$$SH^{\heartsuit, \mathrm{cone}} = \varinjlim_j \varprojlim_i H_*(C(f_{ij}^{\heartsuit}))$$

__Proof that $SH^{\heartsuit, \mathrm{cone}}(W,V) \simeq SH^{\heartsuit}(W,V)[-1]$.__

__Case $\heartsuit = \emptyset$.__ We work with hamiltonians that have the same slope at infinity.



$$\begin{tikzcd} FC_{I}(K) \arrow[r, "f"] \arrow[dr, "h.e."'] & FC_{III}(H) \\ & FC_{I,II,III}(H) \arrow[u, "p"'] \end{tikzcd}$$

produces $SH^{\mathrm{cone}}(W,V)$

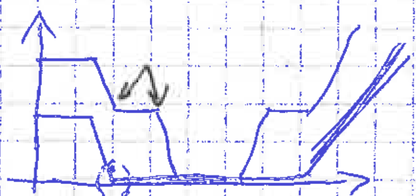
$C(f) \simeq C(p)$

$\wr\!\wr$

$FC_{I,II}(H)[-1]$

produces $SH(W,V)$ □

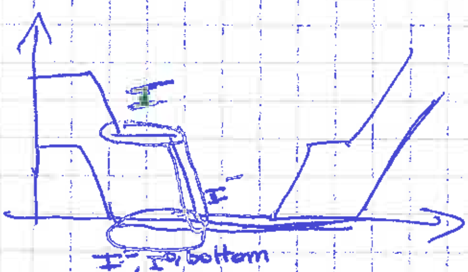
__Case $\heartsuit = <0$__



We want to compare orbits in and in

First step: $FC_{I', I^{0}, \mathrm{bottom}}(K) \xrightarrow{\varphi} FC_{II}(H)$

$$\begin{tikzcd} FC_{I', I^{0}, \mathrm{bottom}}(K) \arrow[r, "\varphi"] \arrow[dr, "h.e."'] & FC_{II}(H) \\ & FC_{I,II}(H) \arrow[u, "p"'] \end{tikzcd}$$



$C(\varphi) \simeq C(p) \simeq FC_{I}(H)[-1]$.

Step 2:

$$\begin{tikzcd}
FC_{I^-}(K) \arrow[r] \arrow[d, "f^{<0}"'] & FC_{I^-, I^0, \text{bottom}}(K) \arrow[r] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\substack{\text{commutes because}\\ \text{right column}\\ \text{consists of cones}}"] & FC_{I^0 \text{bottom}}(K) \arrow[d] \\
FC_{I^-}(H) \arrow[r, equal] \arrow[d] & FC_{I^-}(H) \arrow[r] \arrow[d] & 0 \arrow[d] \\
C(f^{<0}) \arrow[r] & C(\varphi) \simeq FC_{I^-}(H)[-1] \arrow[r, "1 \simeq p[-1]"] & C(0)
\end{tikzcd}$$

$$C(0) \cong FC_{I^0, \text{bottom}}(K)[-1] \cong FC_{I^0}(H)[-1].$$

3x3 lemma: $C(f^{<0}) \simeq C(p[-1]) \simeq C(FC_{I}(H)[-1] \to FC_{I^0}(H)[-1]) \cong FC_{I^-}(H)[-1]$.

3) About subcritical handle attachment.

Thm (Cieliebak): V L. domain, $W = V \cup H_k$, $k<n$ Weinstein subcritical handle

then the transfer map $SH_\cdot(W) \to SH_\cdot(V)$ is an isomorphism.

Reinterpret: $SH(W,V) = 0$.

 $SH(\overline{W \setminus V}, \partial_-(\overline{W \setminus V}))$

Rewrite as follows: If U cobordism (with filling) corresp. to subcritical handle attachment (read $U = \overline{W\setminus V}$) then $SH(U, \partial_- U) = 0$.

Consequences:
- algebraic duality (over k): $SH^\cdot(U,\partial_- U) \simeq SH_\cdot(U,\partial_- U)^\vee$ (works because we have only positive slope)
- Poincaré duality:

$$SH^*(U,\partial_- U) \simeq SH_{-*}(U, \partial_+ U) \leftarrow \text{vanishes!}$$

$\to$ $SH(\partial_- U) \xleftarrow{\sim} SH(U) \xrightarrow{\sim} SH(\partial_+ U)$: invariance of RFH under subcritical handle attachment.

Back: (W,V):

$$\begin{tikzcd}
 & H^*(W,V) \arrow[r] \arrow[d] & H^*(W) \arrow[rr, "i^*"] \arrow[d] & & H^*(V) \arrow[d] & \\
{} \arrow[r] & 0 \arrow[r] \arrow[d] & SH_*(W) \arrow[rr, "\simeq"] \arrow[d] & & SH_*(V) \arrow[r] \arrow[d] & 0 \\
 & SH_*^{>0}(W,V) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & SH_*^{>0}(W) \arrow[rr] & & SH_*^{>0}(V) & \\
 & H^*(D^k, S^{k-1}) & & & &
\end{tikzcd}$$

$\square$