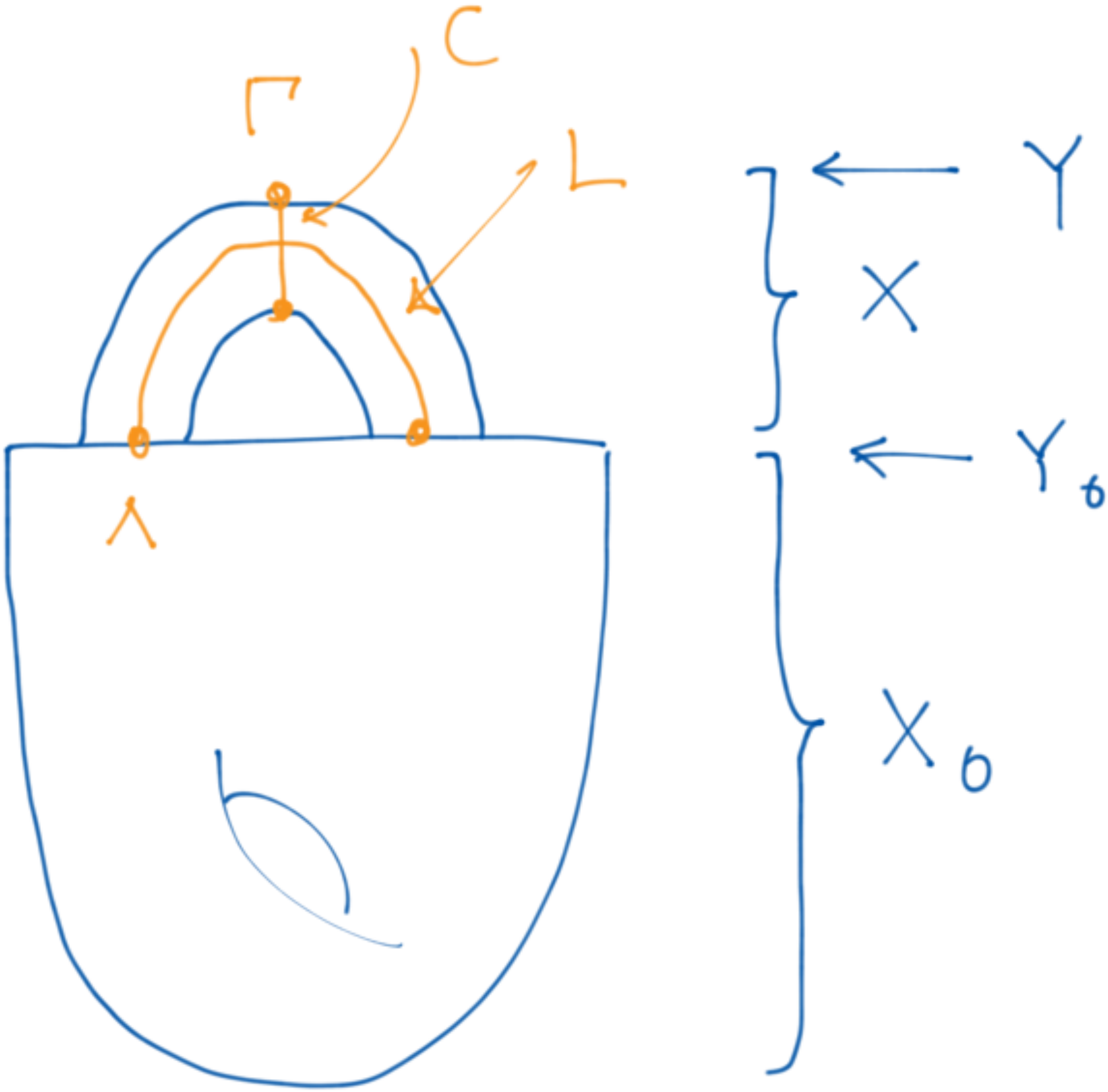
Lecture 1

Set up:



- The main idea in these lectures is to express the symplectic cohomology of X and the wrapped Floer cohomology of C in terms of the Legendrian DGA of Λ.

- We will first focus on the linear chain complexes of symplectic cohomology and wrapped Floer cohomology and look at their product structures later.

Contact homology

Y - contact $(2n-1)$-manifold

α - contact 1-form

$\xi = \ker(\alpha)$ - contact plane field

R - Reeb vector field

$d\alpha(R, \cdot) = 0$; $\alpha(R) = 1$

$\Lambda \subset Y$ - Legendrian submanifold.

For simplicity assume

$c_1(\xi) = 0 \Rightarrow \det(\xi)$ trivial

and that the Maslov class $\mu \in H^1(\Lambda)$ vanishes

The orbit Differential Graded Algebra (DGA)

$Q(Y)$ is the unital graded algebra over the rational numbers generated by the good Reeb orbits in Y, graded by a shifted Conley-Zehnder index, and with differential which counts holomorphic spheres with one positive and several negative punctures.

γ - Reeb orbit

Pick complex trivialization T of ξ along γ.

The linearized Reeb flow along γ defines a path of linear symplectomorphisms

$$\Psi_t : \mathbb{C}^{n-1} \longrightarrow \mathbb{C}^{n-1}$$

The graph of Ψ_t is a path L_t of Lagrangian planes in $\mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1}$. Let L'_t denote L_t closed up by the smallest positive rotation. The Conley-Zehnder index is

$$CZ_T(\gamma) = \mu(L'_t) - \frac{n-1}{2}$$

where μ denotes the Maslov index (intersection number with the Maslov cycle).

A trivialization of $\det(\xi)$ determines a class of trivializations T such that $\det(T) \simeq \det(\xi)$, making CZ well-defined.

An orbit γ is called bad if it is an even multiple of another orbit β such that the parities of $CZ(\gamma)$ and $CZ(\beta)$ are different. (Note that parities are independent of complex trivializations.)

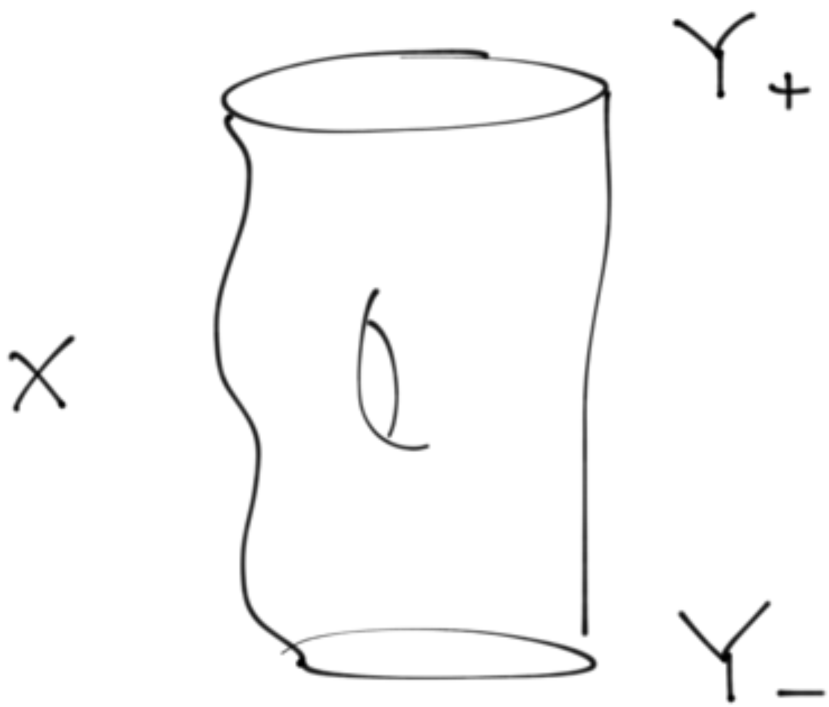
Orbits that are not bad are called good.

Take the contact form generic so that L_1 is transverse to the diagonal for each orbit.

$Q(Y)$ is generated by all good Reeb orbits γ graded by

$$|\gamma| = CZ(\gamma) + (n-3).$$

Consider a Weinstein cobordism



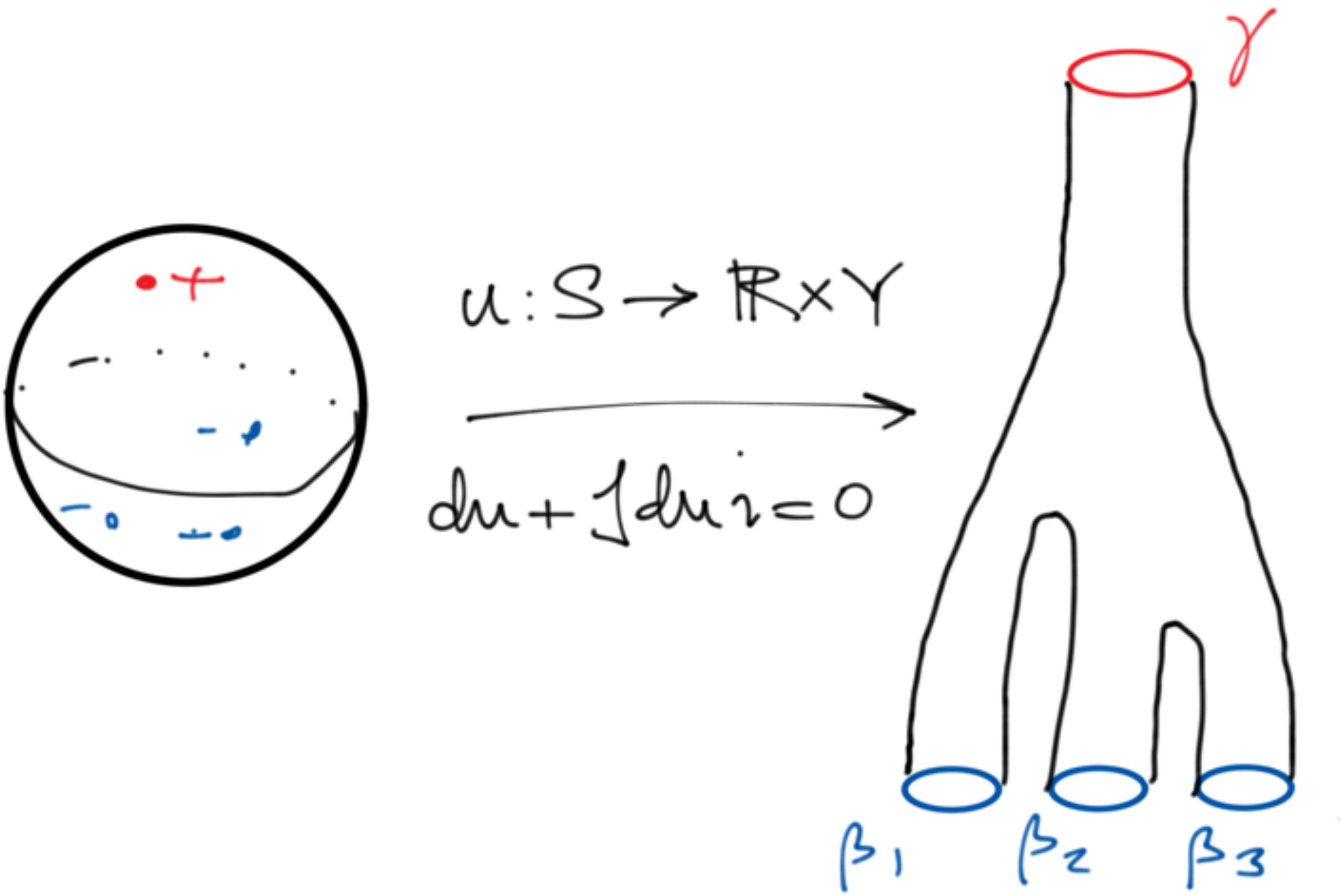
Pick an almost complex structure J on X which is adjusted to the contact structures in the positive and negative end.

$$J(\xi^{\pm}) \subset \xi^{\pm} \; ; \; J(\partial_t^{\pm}) = R^{\pm}$$

For γ a Reeb orbit in Y^+ and $\underline{\beta} = \beta_1^{k_1}\beta_2^{k_2}\dots\beta_s^{k_s}$ a monomial of Reeb orbits in Y^- let

$$\mathcal{M}^X(\gamma, \underline{\beta})$$

denote the moduli space of holomorphic spheres:



The formal dimension of the moduli space is

$$\dim\left(\mathcal{M}^X(\gamma,\beta)\right) = |\gamma| - |\beta| =$$

$$= |\gamma| - \sum_{j=1}^{m} k_j |\beta_j| .$$

The differential $\partial : Q(Y) \to Q(Y)$
satisfies Leibniz rule and is defined on generators by the following curve count:

$$\partial(\gamma) =$$

$$= \sum_{|\gamma|-|\beta|=1} |\mathcal{M}^{\mathbb{R}\times Y}(\gamma,\beta)| \frac{1}{k_1! \dots k_s!} \frac{1}{m(\beta_1)} \cdots \frac{1}{m(\beta_s)} \beta$$

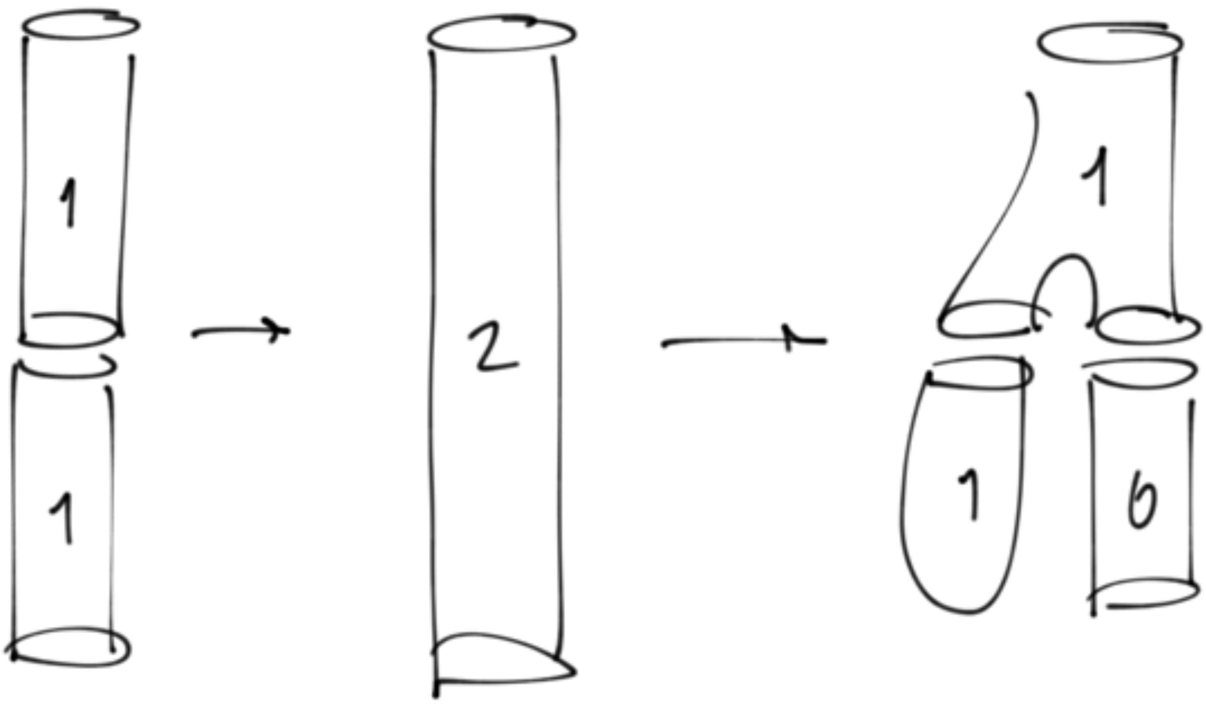
where m(β) is the multiplicity of β.

For a suitable perturbation scheme transversality holds and the boundary of the (reduced) 1-dim moduli space is by SFT-compactness and gluing in 1-1 correspondence with 2-level broken curves of total dimension 2:

This shows that ∂ is a differential:

$$\partial^2 = 0 .$$

Note in particular that there could be splittings of the form



This is the reason for a DGA rather than a linear chain complex.

Similarly the cobordism X defines a DGA chain map

$$\Phi_X : Q(Y^+) \longrightarrow Q(Y^-)$$

$$\Phi_X(\gamma) =$$

$$= \sum_{|\gamma|-|\beta|=0} |\mathcal{M}^X(\gamma;\beta)| \frac{1}{k_1! \ldots k_s!} \frac{1}{\kappa(\beta_1) \ldots \kappa(\beta_s)} \beta$$

Then $\Phi_X \circ \partial^+ = \partial^- \circ \Phi_X$

since

An <u>augmentation</u> is a DGA chain map

$$\varepsilon : Q(Y) \longrightarrow \mathbb{Q} \ ;$$

$$\varepsilon \circ \partial + \cancelto{0}{\partial \circ \varepsilon} = \varepsilon \circ \partial = 0$$

If Y^- is empty then $Q(Y^-) = \mathbb{Q}$ and

$$\varepsilon_X := \Phi_X : Q(Y^+) \to \mathbb{Q}$$

is an augmentation.

$\partial\Big($ 

$\Big) =$ 

$\Rightarrow \varepsilon_X \circ \partial = 0$

The Legendrian DGA

$\mathcal{A}(\Lambda)$ is a unital non-commutative DGA generated by Reeb chords of Λ, graded by Maslov index with differential that counts disks with one positive and several negative boundary punctures.

In general the chord algebra is an algebra over Q(Y), but an augmentation of Q(Y) can be used to reduce the coefficients to $\mathbb{Q}$.

c - Reeb chord with initial point c^- and final c^+

Pick a path λ'_c in Λ connecting c^+ to c^-

Then $c * \lambda'_c$ is a loop λ_c. Pick a trivialization T of ξ along λ_c compatible with $\det(\xi)$.

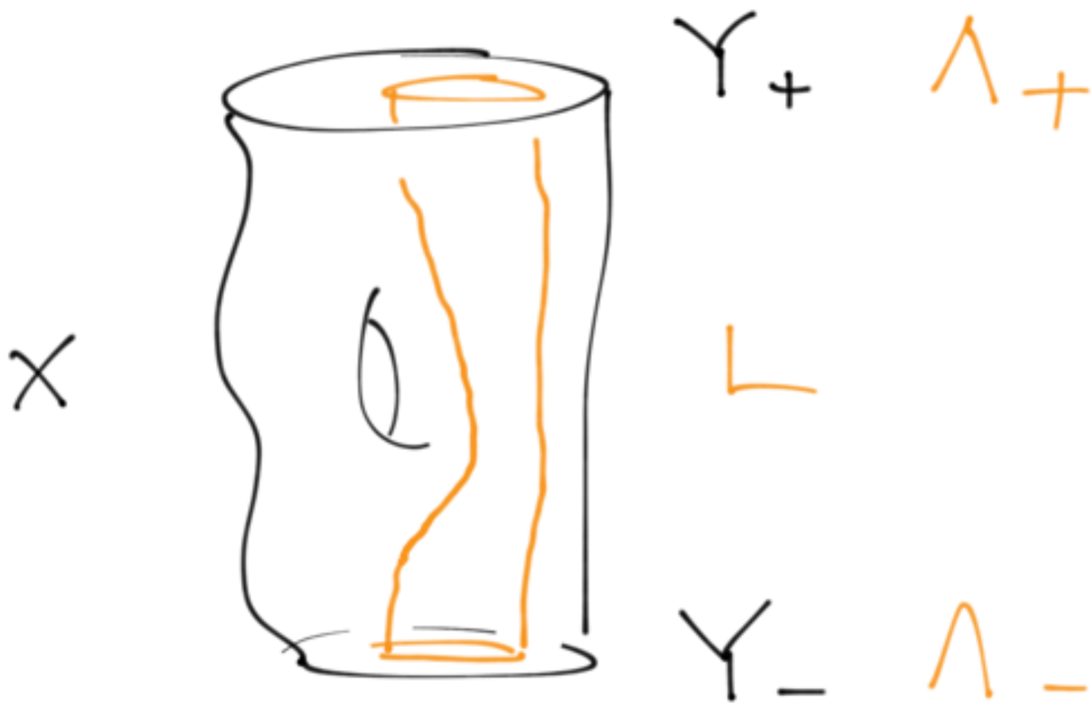
Consider the following loop of Lagrangian planes in ξ

$$L_c = T_{\lambda'_c}\Lambda * \left[d(e^R)\, T_{c^-}\Lambda \right] * \mathrm{Rot}^+$$

The grading of c is then

$$|c| = \mu(L_c) - 1 .$$

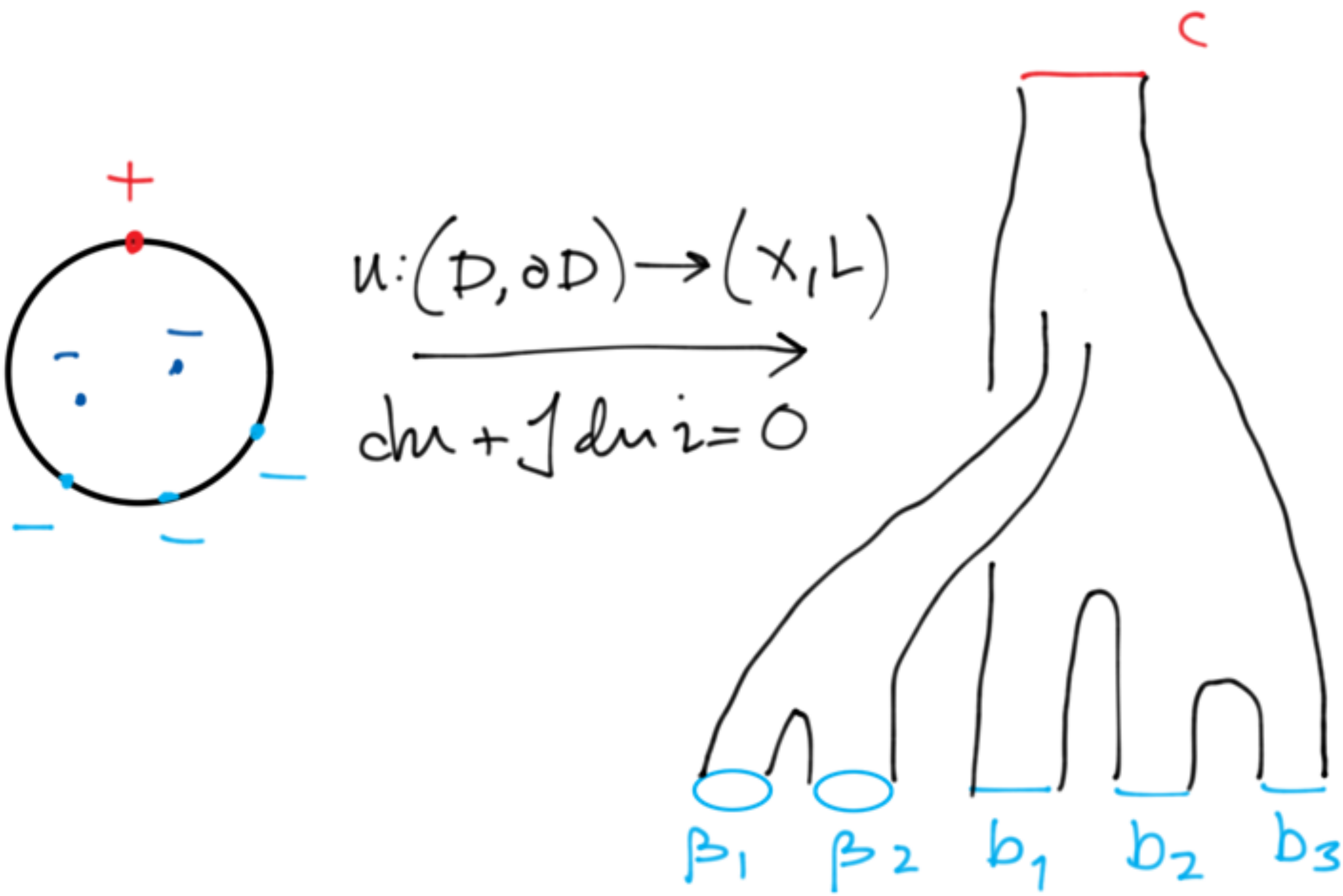
Consider a Weinstein cobordism with an exact Lagrangian cobordism inside



For c a Reeb chord of Λ^+, for $\underline{b} = b_1 b_2 \dots b_r$, a monomial of Reeb chords of Λ^-, and for $\underline{\beta}$ a monomial of Reeb chords, let

$$\mathcal{M}^{(X,L)}(c, \underline{b}; \underline{\beta})$$

denote the moduli space of holomorphic disks:



The formal dimension of the moduli space is

$$\dim\left(\mathcal{M}^{(X,L)}(c, \underline{b}; \underline{\beta})\right) =$$

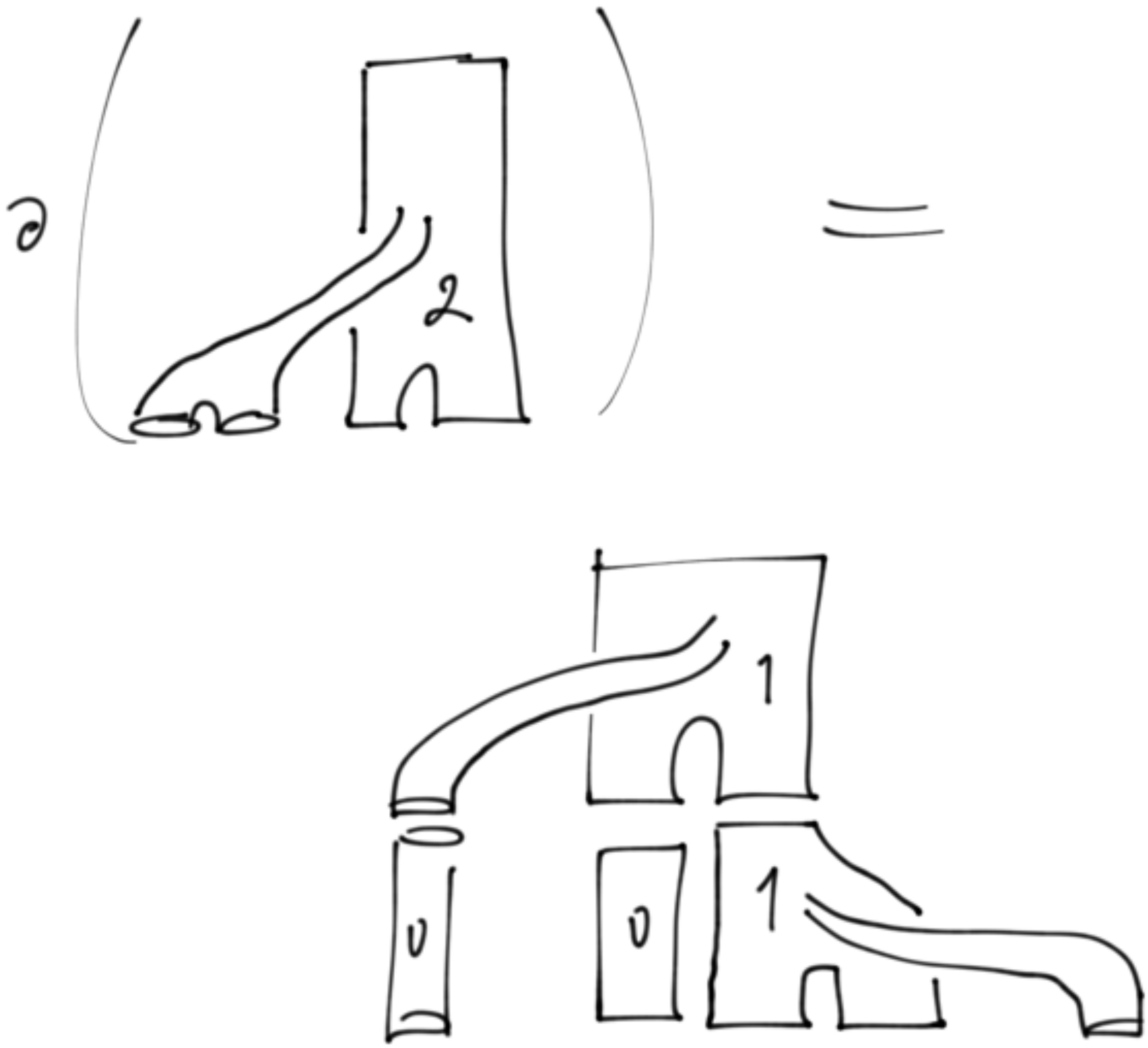
$$= |c| - |\underline{b}| - |\underline{\beta}| .$$

The differential $\partial : \mathcal{A}(\Lambda) \longrightarrow \mathcal{A}(\Lambda)$
satisfies Leibniz rule and is defined on generators by the following curve count:

$$\partial(c) =$$

$$= \sum_{|c|-|\underline{b}|-|\underline{\beta}|=1} \left|\mathcal{M}^{(\mathbb{R}\times Y, \mathbb{R}\times\Lambda)}(c, \underline{b}; \underline{\beta})\right| \frac{1}{k_1! \cdots k_s!} \frac{1}{\kappa(\beta_1)} \cdots \frac{1}{\kappa(\beta_s)} \underline{\beta}\, \underline{b}$$

This gives a differential:



$$\Rightarrow \quad \partial^2 = 0 .$$

Similarly, the exact Lagrangian cobordism (X,L) defines a DGA map

$$\Phi_{(X,L)} : A(\Lambda^+) \longrightarrow A(\Lambda^-)$$

$$\Phi_{(X,L)}(c) =$$

$$= \sum_{|c|-|\underline{b}|-|\beta|=0} \left|\mathcal{M}^{(X,L)}(c, \underline{b}; \beta)\right| \frac{1}{k_1! \dots k_s!} \frac{1}{K(\beta_1)} \dots \frac{1}{K(\beta_s)} t^{\beta} \underline{b}$$

$$\Phi_{(X,L)} \circ \partial^+ = \partial^- \circ \Phi_{(X,L)}$$

As before an augmentation is a chain map to $\mathbb{Q}$

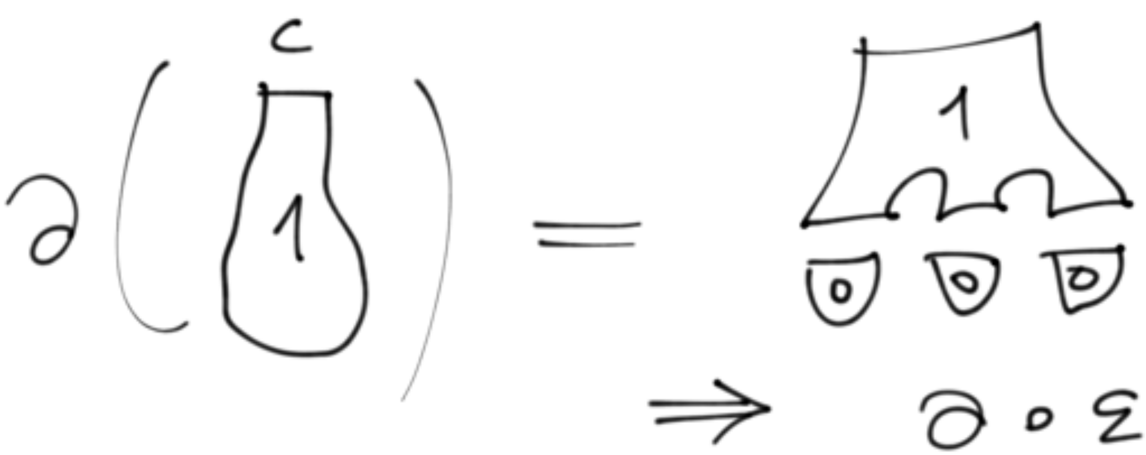
$$\varepsilon : \mathcal{A}(\Lambda) \longrightarrow \mathbb{Q}$$

$$\varepsilon \circ \partial = 0$$

and if (Y^-, Λ^-) is empty then

$$\varepsilon_{(X,L)} := \Phi_{(X,L)}$$

is an augmentation



$$\Rightarrow \quad \partial \circ \varepsilon_{(X,L)} = 0.$$

Examples

Standard contact S^{2n-1}

$$S = \left\{ |z_1|^2 + \frac{1}{a_2}|z_2|^2 + \cdots + \frac{1}{a_n}|z_n|^2 = 1 \right\}$$

$0 < a_j << 1$; a_j's lin indep over $\mathbb{Q}$.

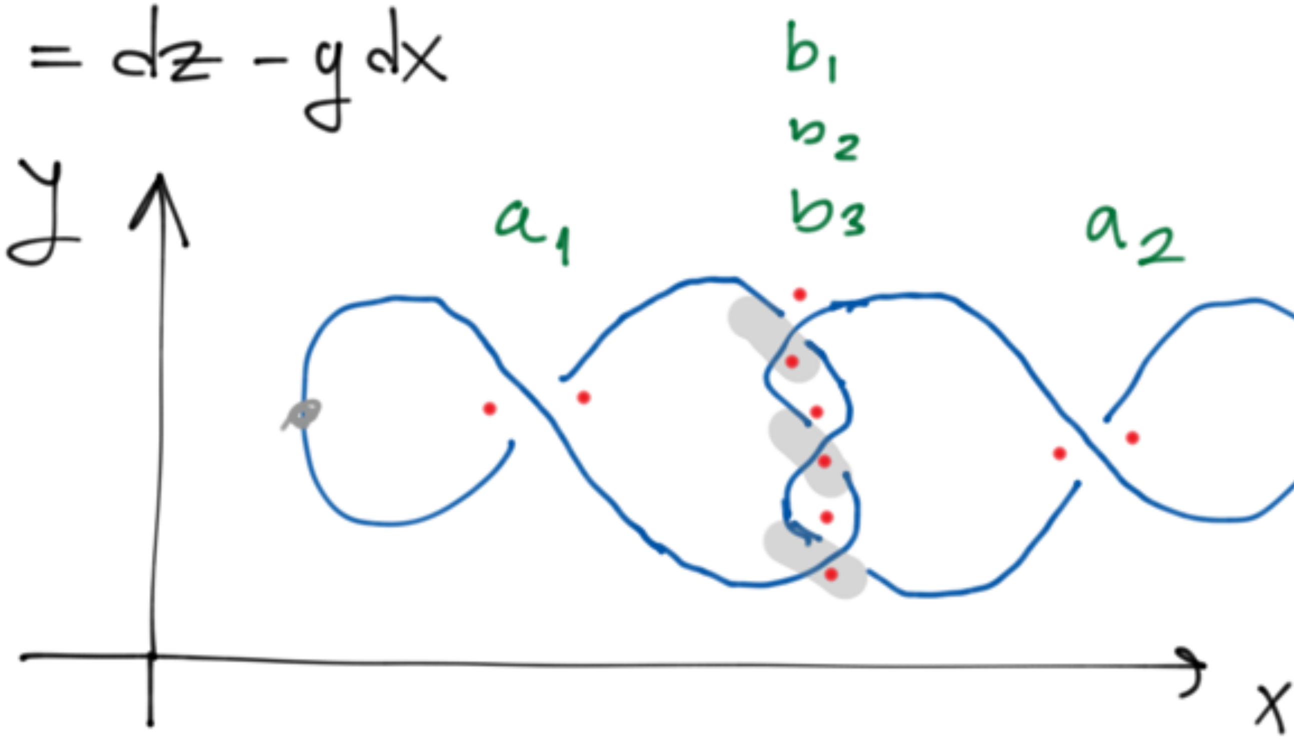
$\Rightarrow$ effectively only one geometric orbit in $z_2 = \cdots = z_n = 0$, γ

Orbit	γ	$\gamma^{(2)}$	$\cdots$	$\gamma^{(m)}$	$\cdots$
$\lvert\cdot\rvert$	$2n-2$	$2n$		$2(n+m-1)$	

$$\Longrightarrow HQ(Y) = \mathbb{Q}[\gamma^{(j)}]_{j=1,2,\ldots}$$

Legendrian trefoil in standard contact 3-space

$$\alpha = dz - y\,dx$$



$$|a_1| = |a_2| = 1$$

$$|b_1| = |b_2| = |b_3| = 0$$

$$\partial a_1 = -1 - b_1 - b_3 - b_3 b_2 b_1$$

$$\partial a_2 = 1 + b_1 + b_3 + b_1 b_2 b_3$$

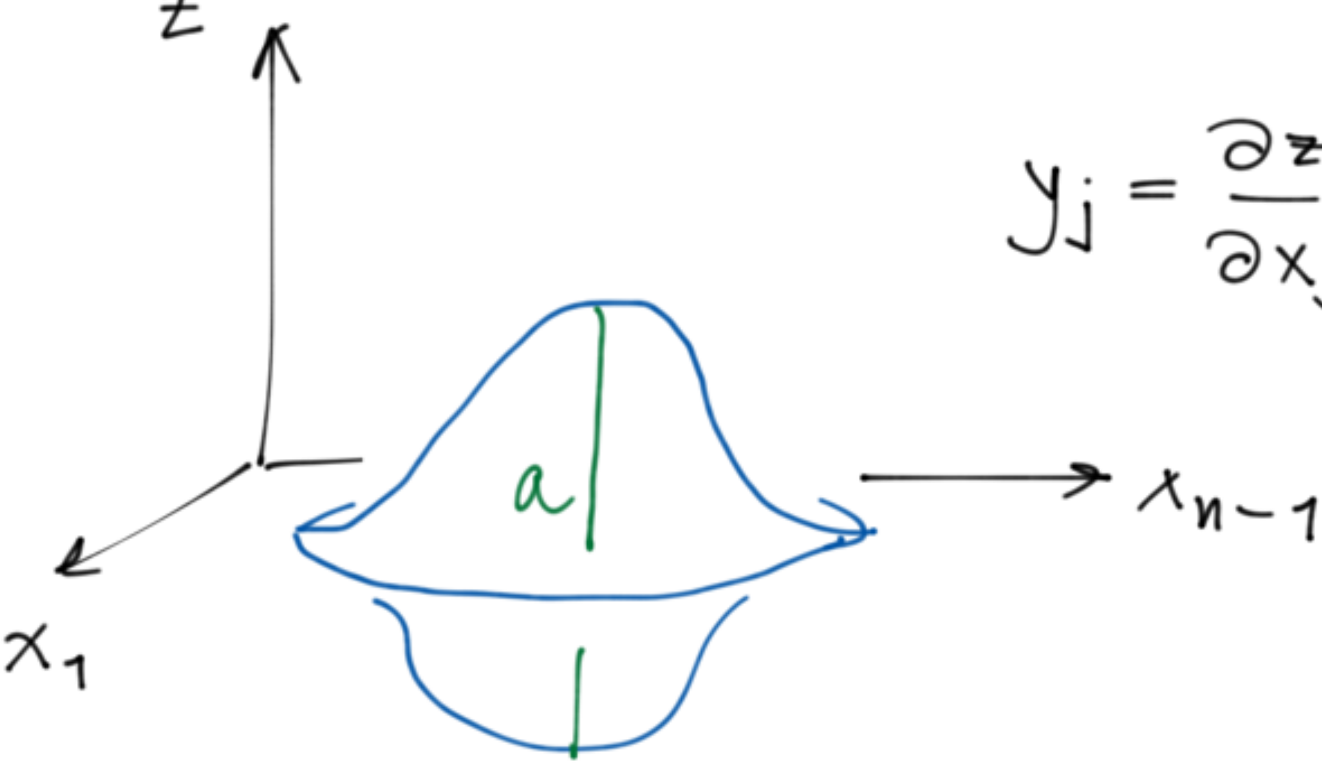
$$HA = \mathbb{Q}[b_1, b_2, b_3] \big/ 1 + b_1 + b_3 + b_1 b_2 b_3$$

(Note that b_j's commute in homology.)

Legendrian unknot of dimension n-1

S^{n-1} in $\mathbb{R}^{2n-1}_{st}$ $\quad \alpha = dz - y_j dx_j$

$$y_j = \frac{\partial z}{\partial x_j}$$



$|a| = n-1$

$HA(\Lambda) = \mathbb{Q}[a]$.

Augmentations

$$\varepsilon : Q(Y) \longrightarrow \mathbb{Q} \ , \ \varepsilon \circ \partial = 0 \ .$$

Linearized contact homology

Change variables in Q(Y):

$$\psi(\gamma) = \gamma - \varepsilon(\gamma) \ ; \ d = \psi^{-1} \circ \partial \circ \psi$$

$$Q^0 \subset Q^1 \subset Q^2 \subset \dots \subset Q^j \subset \dots Q(Y)$$

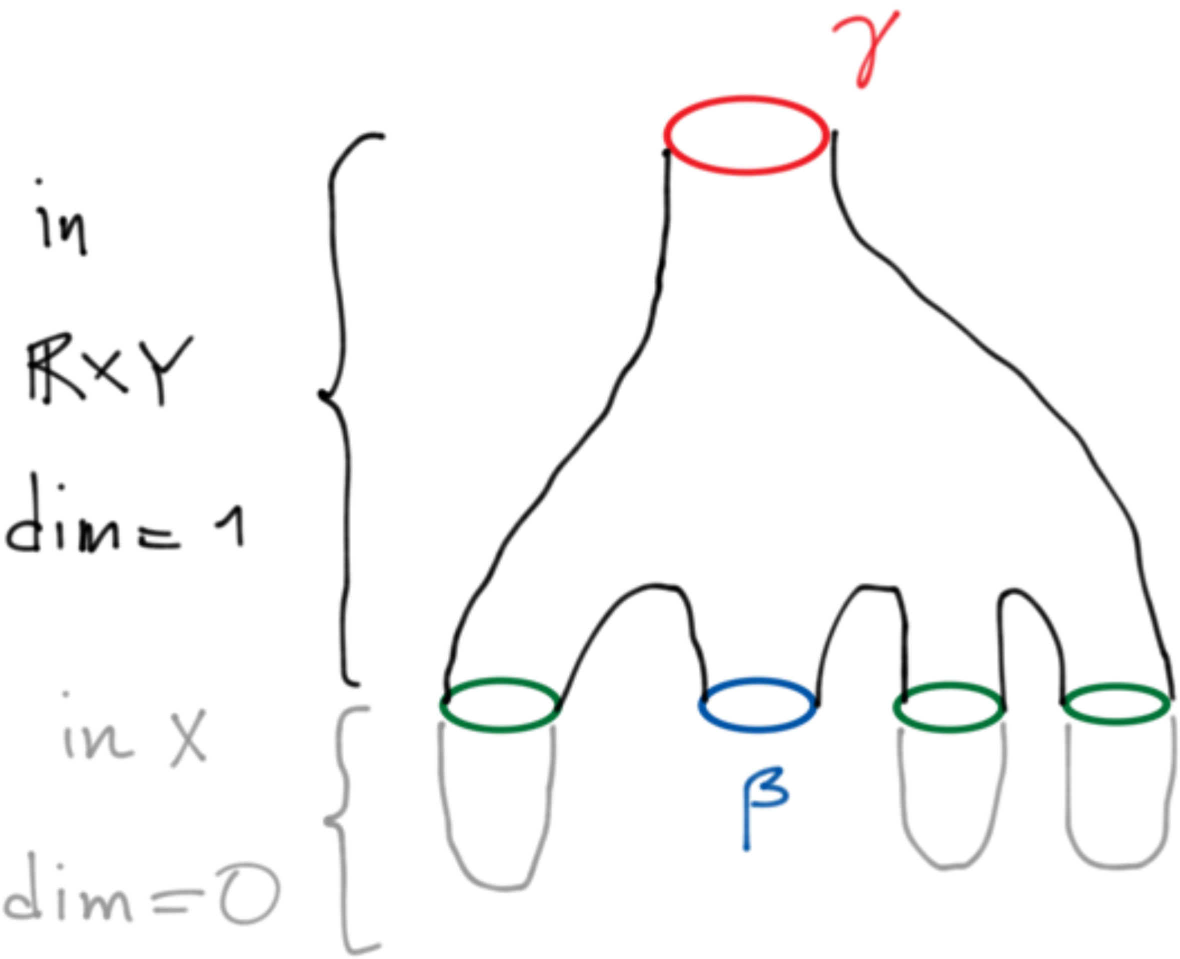
Q^j - polynomials of degree $\leq$ j.

The differential d respects the degree filtration and the homology of the induced differential on Q^1 / Q^0 is called ε-linearized contact homology.

$$\left(C^{lin}(Y), d^{lin}\right) := \left(Q^1/Q_0, d_1\right)$$

When $\varepsilon = \varepsilon_X$ the linearized differential has a geometric interpretation as a count of (partial) two-level curves:

d^{lin}_{γ} counts



Similarly, ε allows us to filter the Legendrian DGA by orbit degree and get a Legendrian DGA involving chords only.

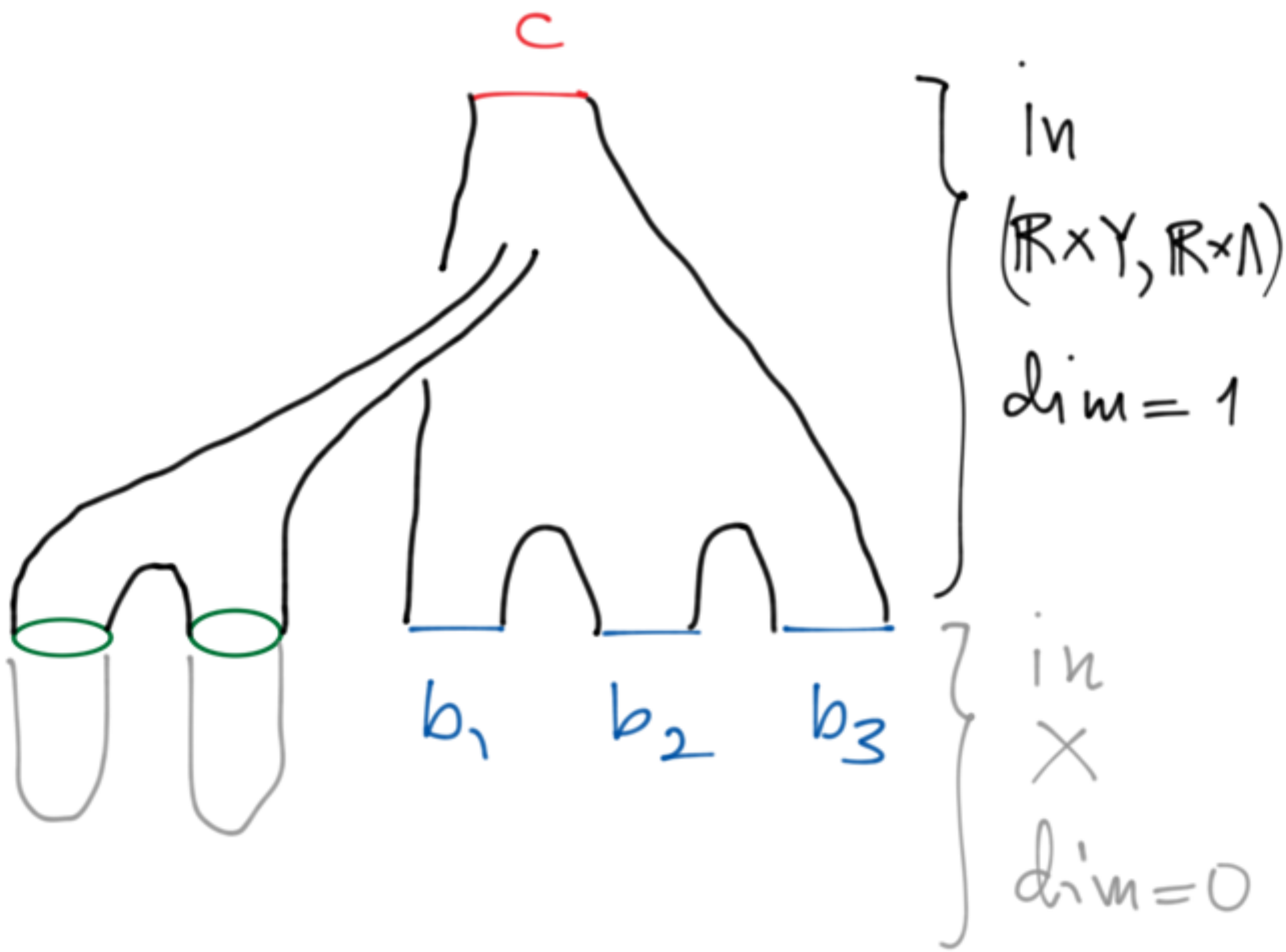
$$A(\Lambda) = \mathbb{Q}\langle \text{Reeb chords} \rangle.$$

$$d(c) =$$

$$\sum_{|c|-|\underline{b}|=1} |\mathcal{M}^{\mathbb{R}\times Y}(c, \underline{b}; \beta)| \frac{1}{k_1! \ldots k_s!} \frac{1}{\kappa(\beta_1)} \cdots \frac{1}{\kappa(\beta_n)} \varepsilon(\beta)\, \underline{b}$$

When $\varepsilon = \varepsilon_X$ there is the following geometric interpretation:

dc counts (partial) α-level curves



If the Legendrian DGA admits an augmentation ε then we similarly get a linearized Legendrian DGA as the first page of a spectral sequence calculating the Legendrian contact homology.

$$\psi(c) = c - \varepsilon(c) \quad ; \quad d = \psi^{-1} \partial \psi$$

$$A^0 \subset A^1 \subset \dots \subset A^j \subset \dots \subset A(\Lambda)$$

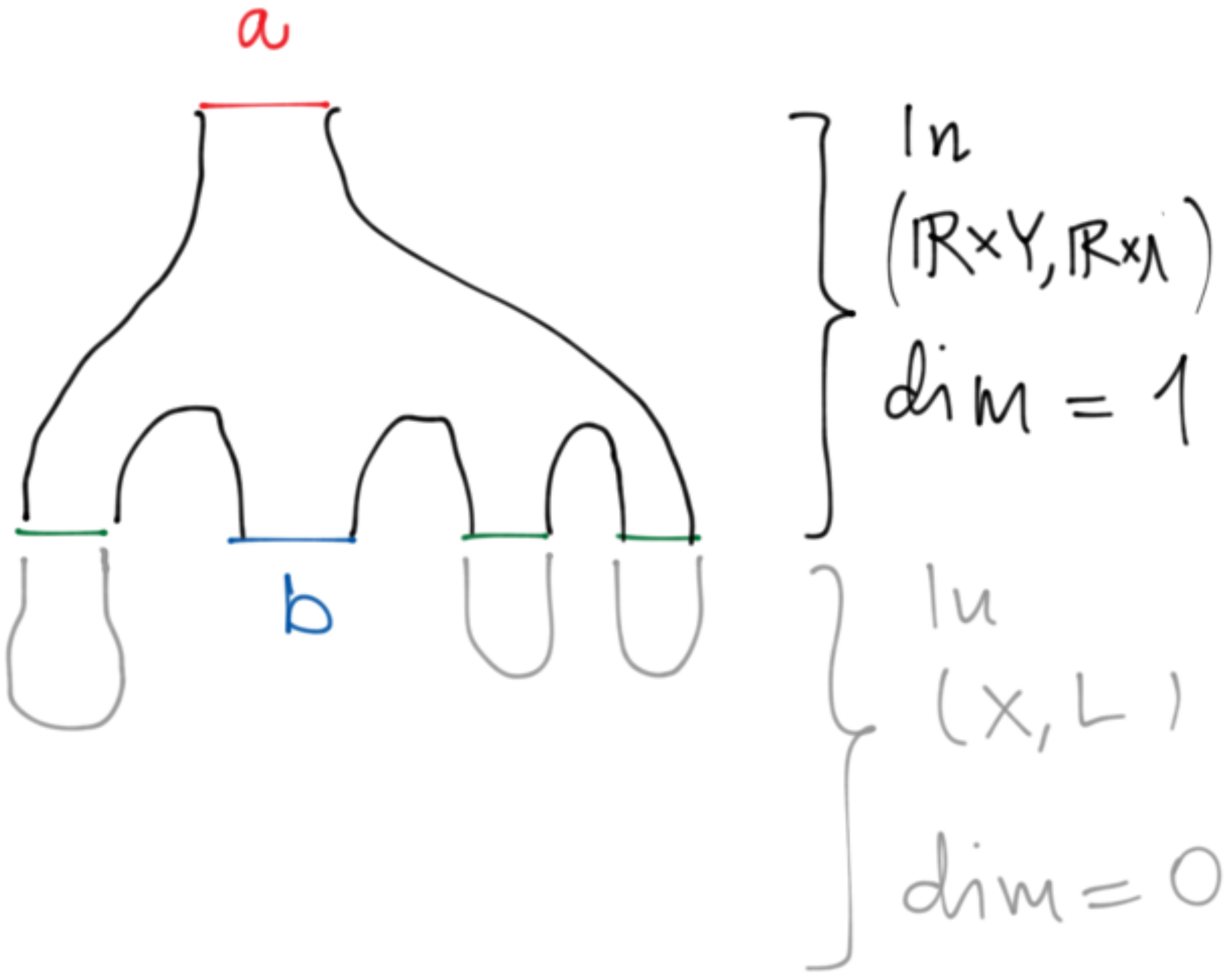
$$\left(C^{lin}(\Lambda), d^{lin}\right) = \left(A^1 / A_0, d_1\right)$$

$$d^{lin}(c) =$$

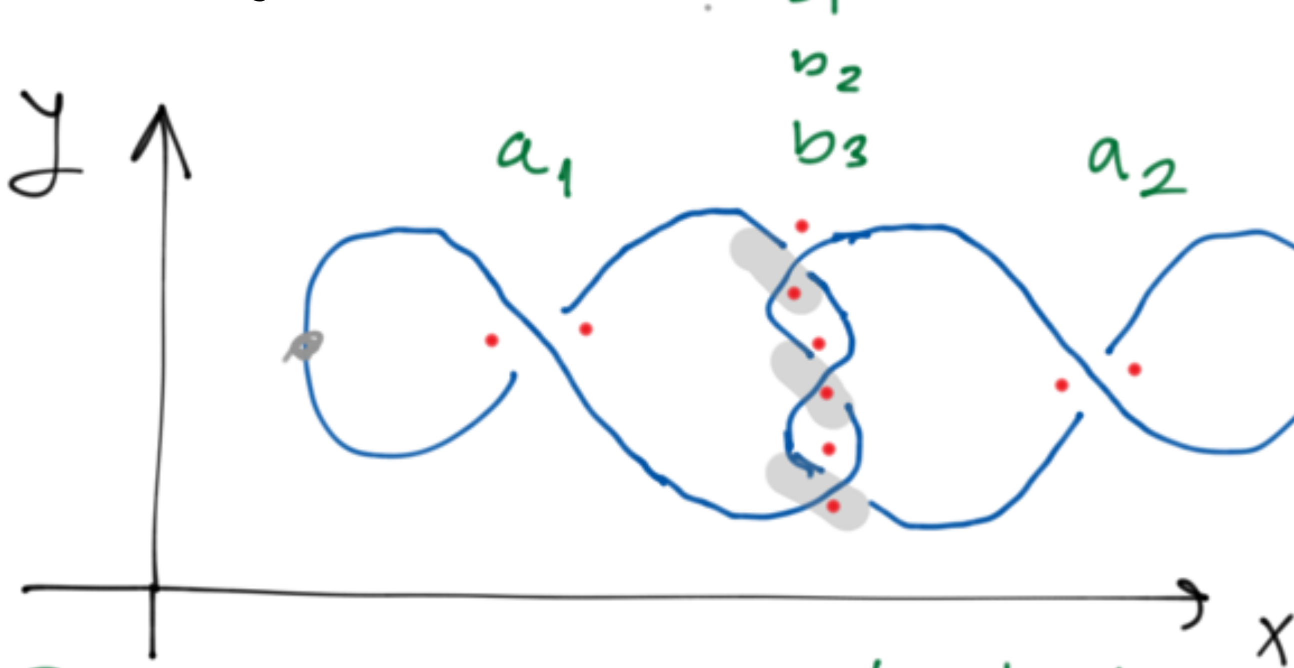
$$\sum_{|c|-|b|=1} |\mathcal{M}^{(\mathbb{R}\times Y, \mathbb{R}\times\Lambda)}(c, \underline{b}^l b \underline{b}^r)| \left(\varepsilon(\underline{b}^l)\varepsilon(\underline{b}^r)\right) b$$

If the augmentation $\varepsilon = \varepsilon_L$ is induced by an exact Lagrangian we get the following geometric interpretation:

$d^{lin}(c)$ counts



Recall Legendrian trefoil:



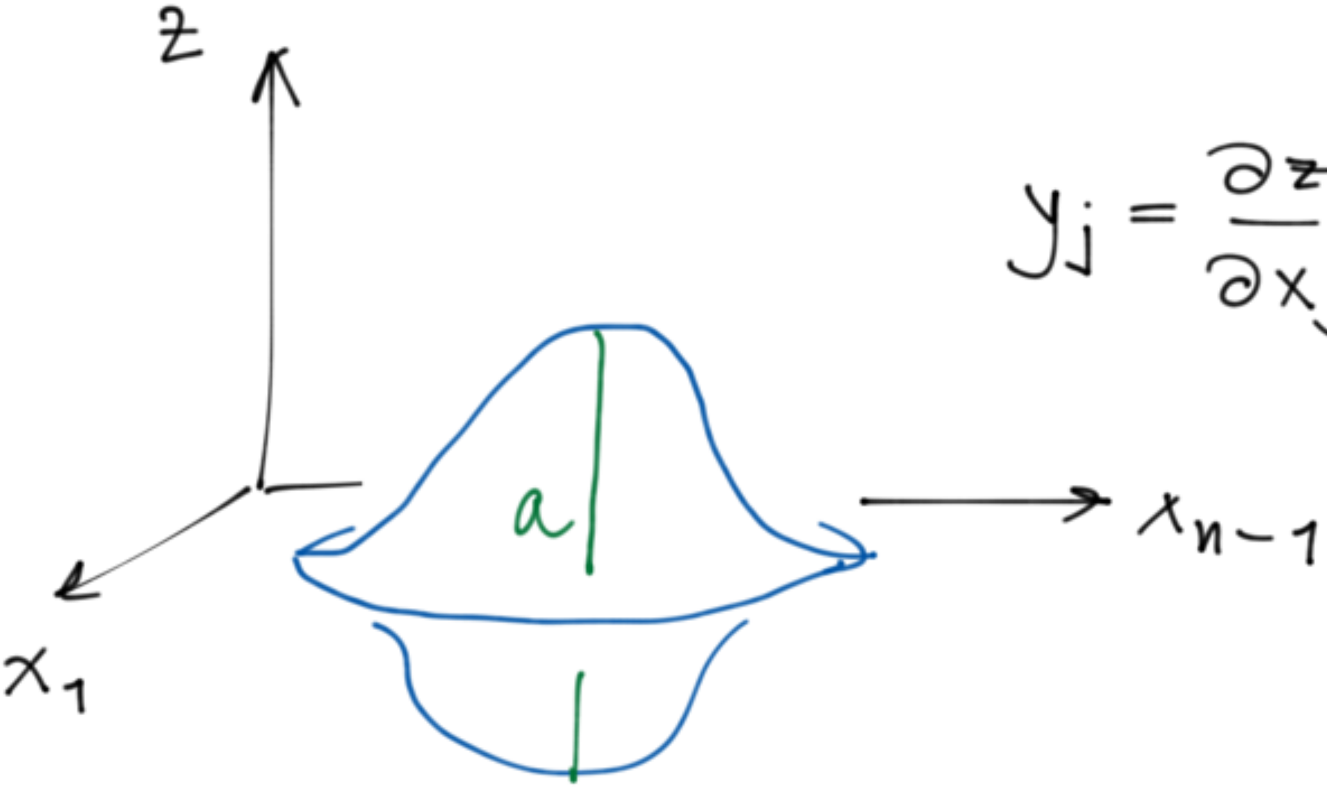
$$\partial a_1 = -1 - b_1 - b_3 - b_3 b_2 b_1$$

$$\partial a_2 = 1 + b_1 + b_3 + b_1 b_2 b_3$$

	ε_1	ε_2	ε_3	ε_4	ε_5
b_1	1	0	1	0	1
b_2	0	0	1	1	1
b_3	0	1	0	1	1

$$CH^{lin}_* = \begin{cases} \mathbb{Q}, & * = 1 \\ \mathbb{Q}^2, & * = 0 \\ 0, & \text{otherwise} \end{cases}$$

Recall Legendrian unknot of dimension n-1:



$$y_j = \frac{\partial z}{\partial x_j}$$

$$|a| = n-1, \quad \partial a = 0$$

$$CH_*^{lin} = \begin{cases} \mathbb{Q}, & * = n-1 \\ 0, & \text{otherwise} \end{cases}$$

Linearized Legendrian homology and 2-copies

$\Lambda \subset Y$, neighborhood

$N(\Lambda) \approx U \subset J^1(\Lambda)$

$\alpha \longleftrightarrow dz - p\,dq$

$\Lambda_0 := \Lambda$

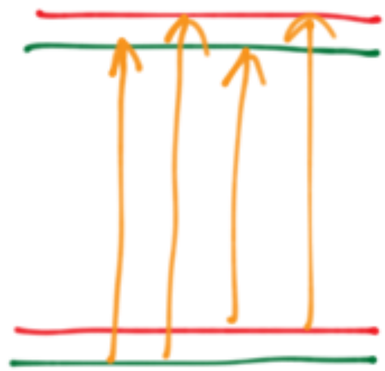
$\Lambda_1 := j^1(f)$

$f : \Lambda \to \mathbb{R}$ small positive Morse fctn.

$$Ch(\Lambda_0 \cup \Lambda_1) =$$

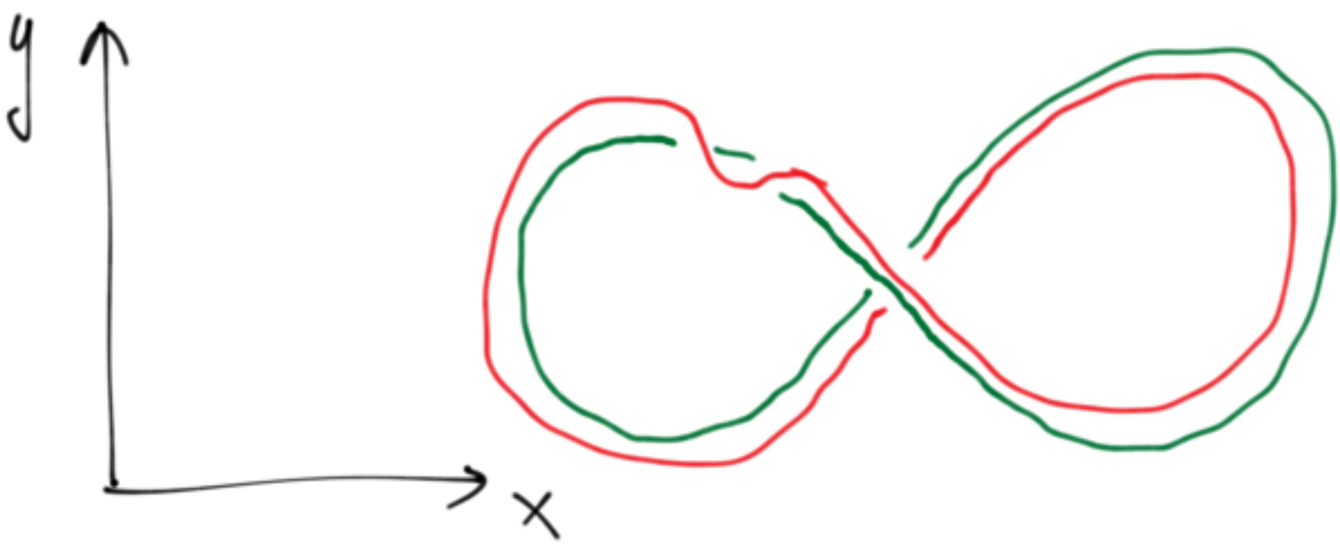
$$C_{00} \cup C_{11} \cup C_{01} \cup C_{10}$$

$$\cup \; Crit(f) .$$



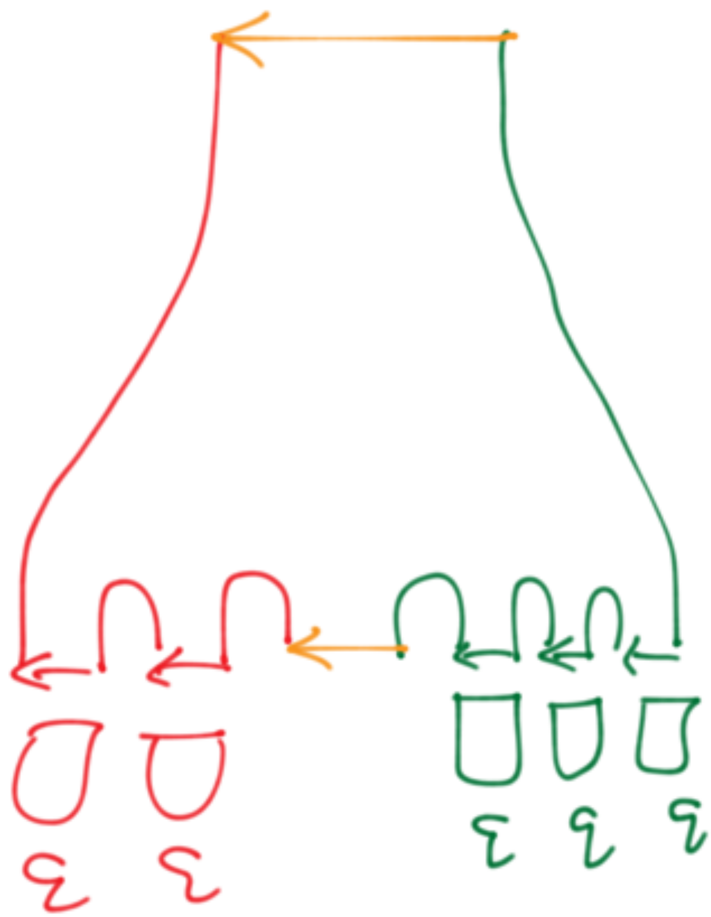
$$C_{ij} \approx Ch(\Lambda)$$

Example: $\alpha = dz - y\,dx$

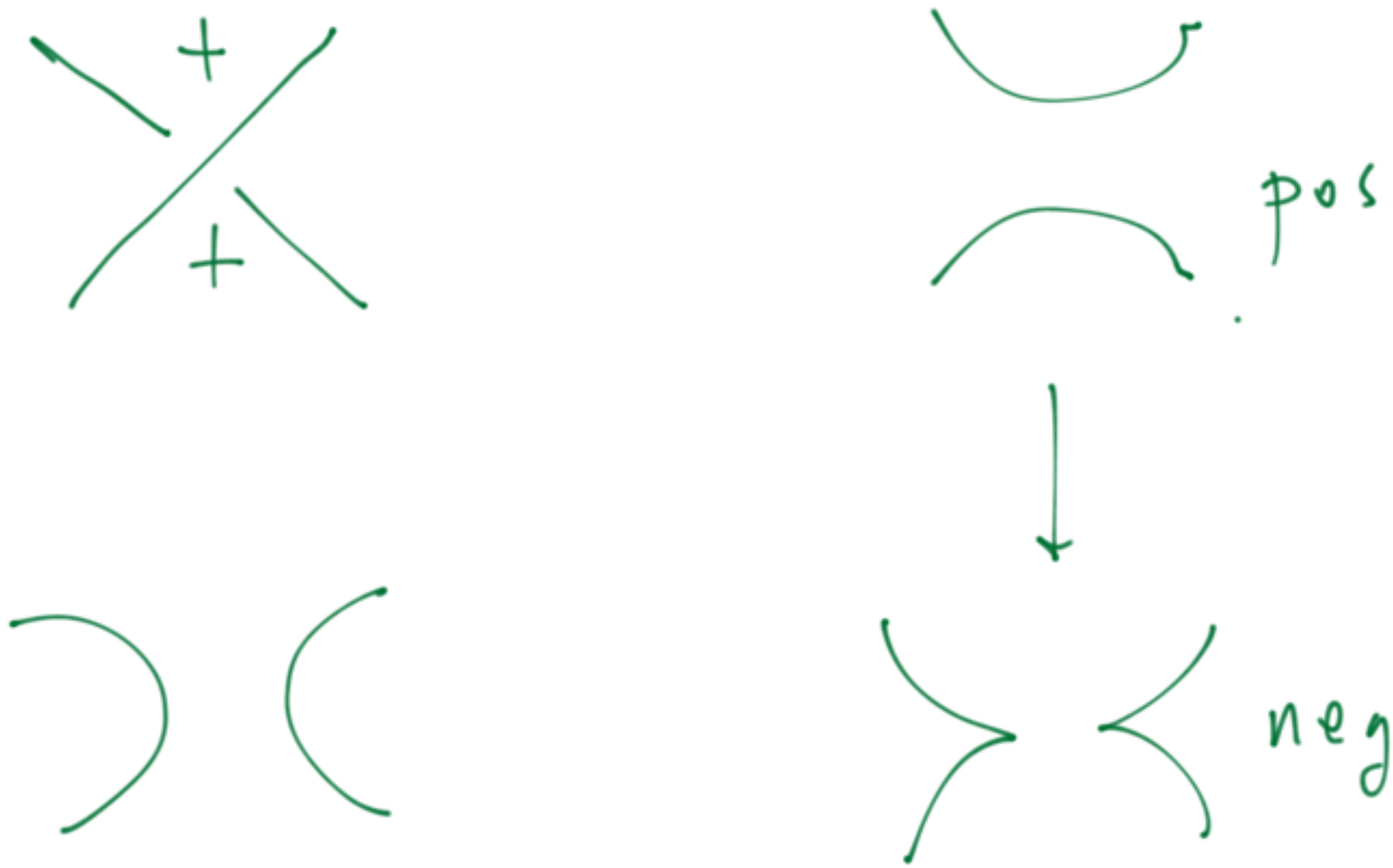


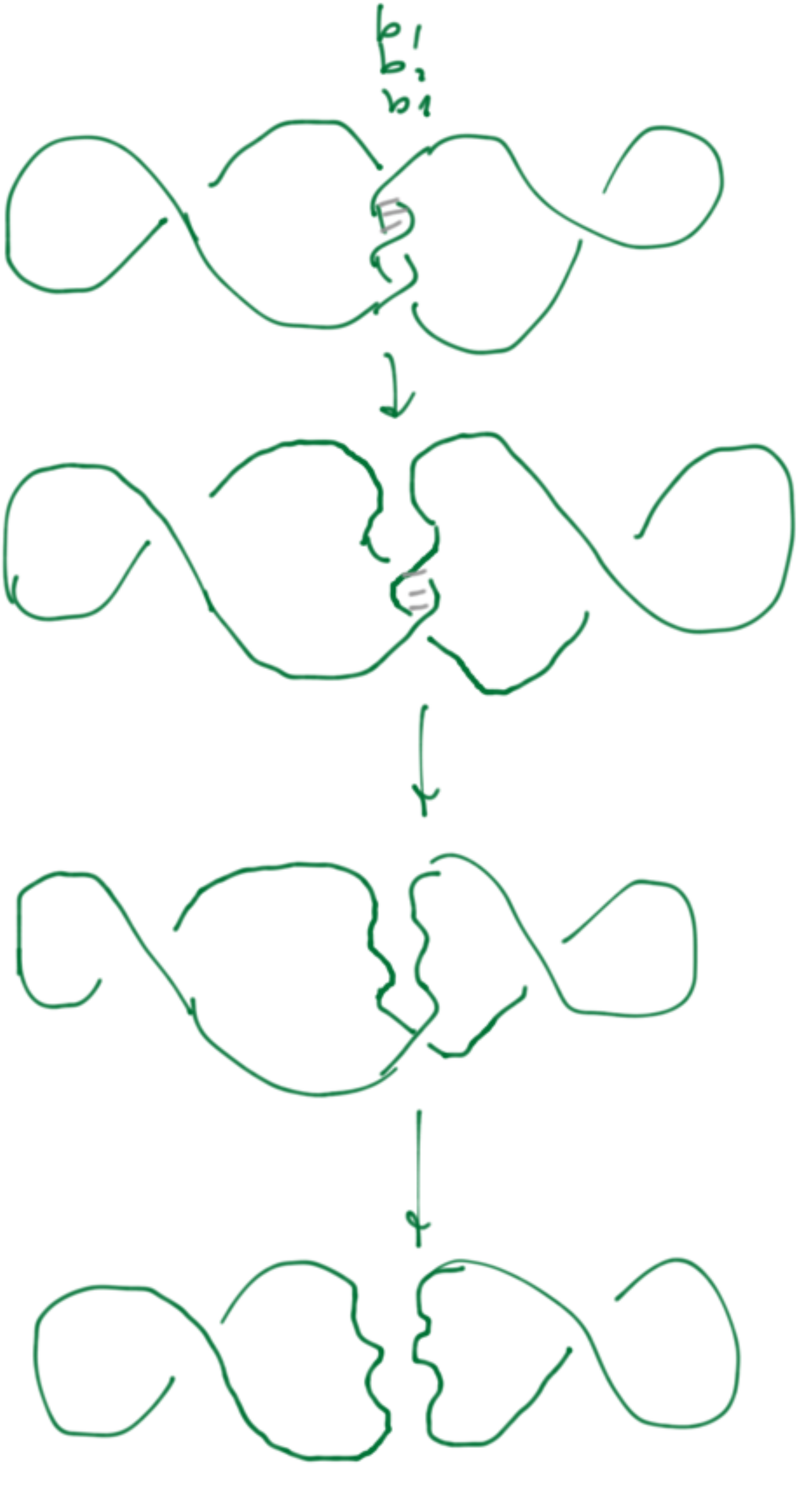
The differential of the 2-copy respects the filtration given by degree in mixed chords. To reproduce the linearized Legendrian homology we count disks with one mixed 10 chord:

$d_{10} : C^{10} \longrightarrow C^{10}$ counts



The tori can be drawn explicitly and their chain maps computed via flow trees.





$$
\begin{array}{ccc}
b_1 & b_2 & b_3 \\
\downarrow & \downarrow & \downarrow \\
1 & b_2+1 & b_3 \\
\downarrow & \downarrow & \downarrow \\
1 & 0 & b_3+1 \\
 & \downarrow & \\
1 & 0 & 0
\end{array}
$$