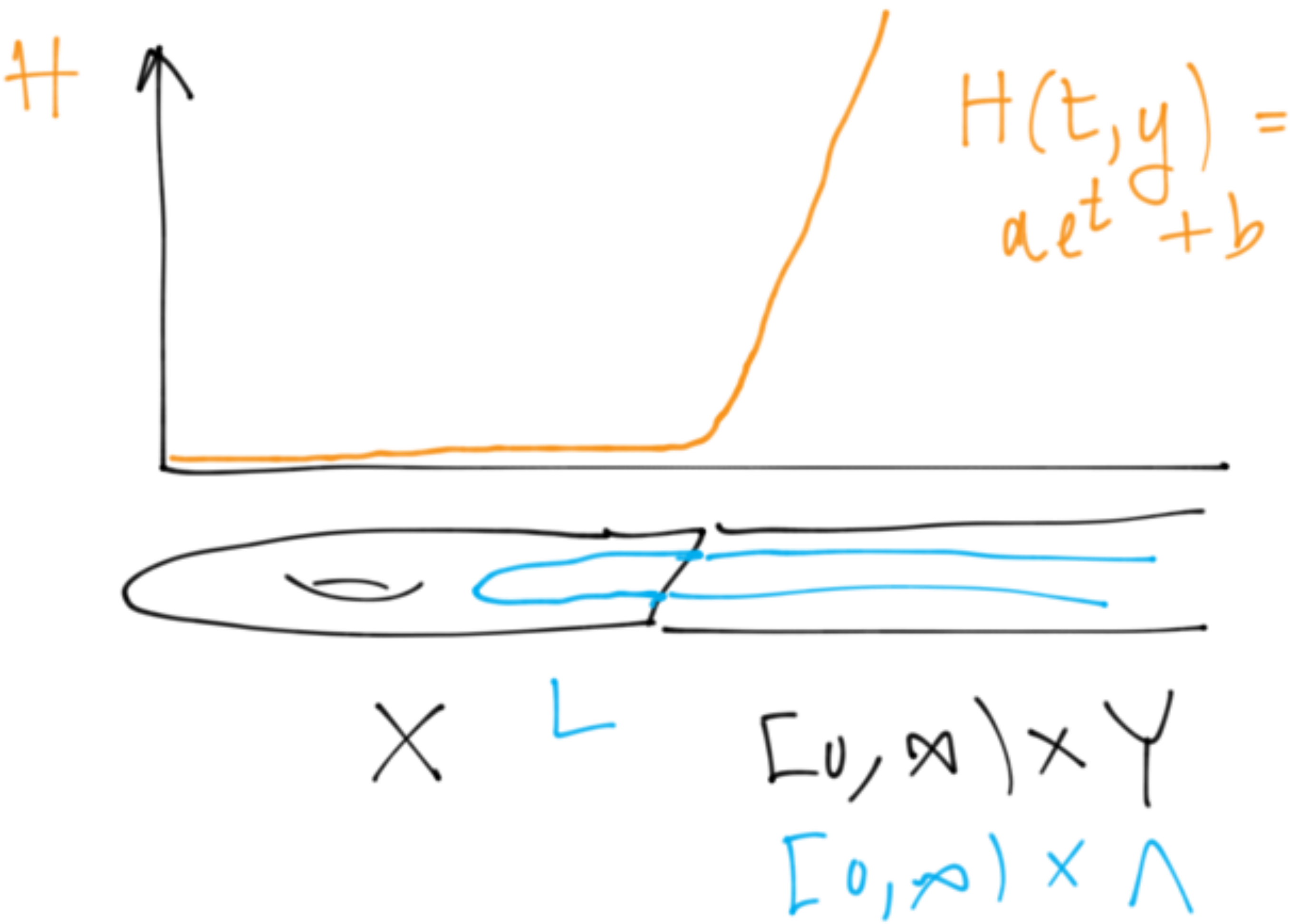
Wrapped Floer homology - Lagrangian SH



CW generated by Hamiltonian chords

$L \to L$

Differential counts
Floer holomorphic strips

$$u : \underset{s}{\mathbb{R}} \times \underset{t}{[0,1]} \longrightarrow X$$

$$\left(du + X_H \otimes dt \right)^{0,1} = 0 .$$

$$\frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - X_H \right) = 0$$

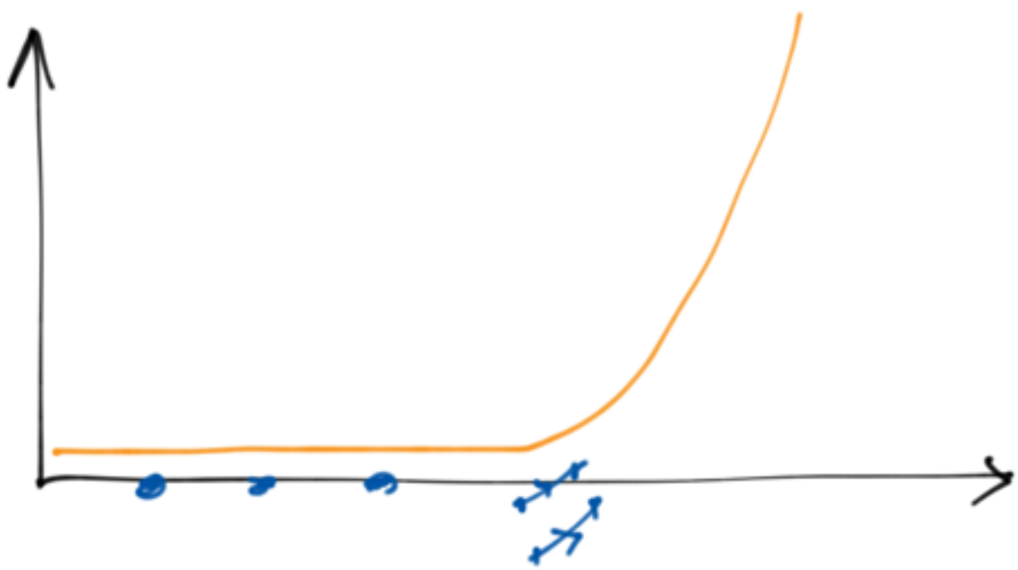
As in SH we find that d = 0 and that the homology is invariant under exact deformations of (X,L).

$$WH(L) = \ker d / \operatorname{im} d .$$

The generators of CW are in natural 1-1 correspondence with

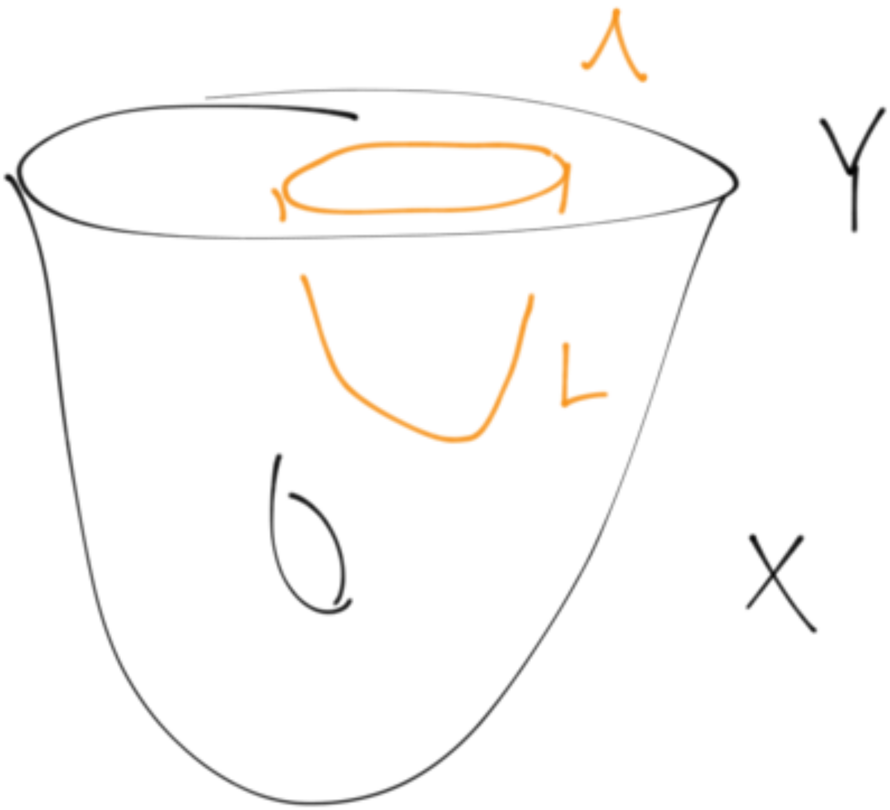
Reeb chords of Λ

Critical points of $H|_L$



The grading in CW is again given by a Maslov index.

Wrapped Floer homology and linearized Legendrian homology.



Let $F : L \to \mathbb{R}$ be a Morse function with

$$F(t,y) = \varepsilon \cdot t \quad \text{in } [0,\infty) \times \Lambda .$$

Take $\quad L_0 := L \;;\; L_1 = \Gamma_{dF}$

Let

$$C(L_1, L_0) = C_{10} \oplus \mathbb{Q}(L_1 \cap L_0)$$

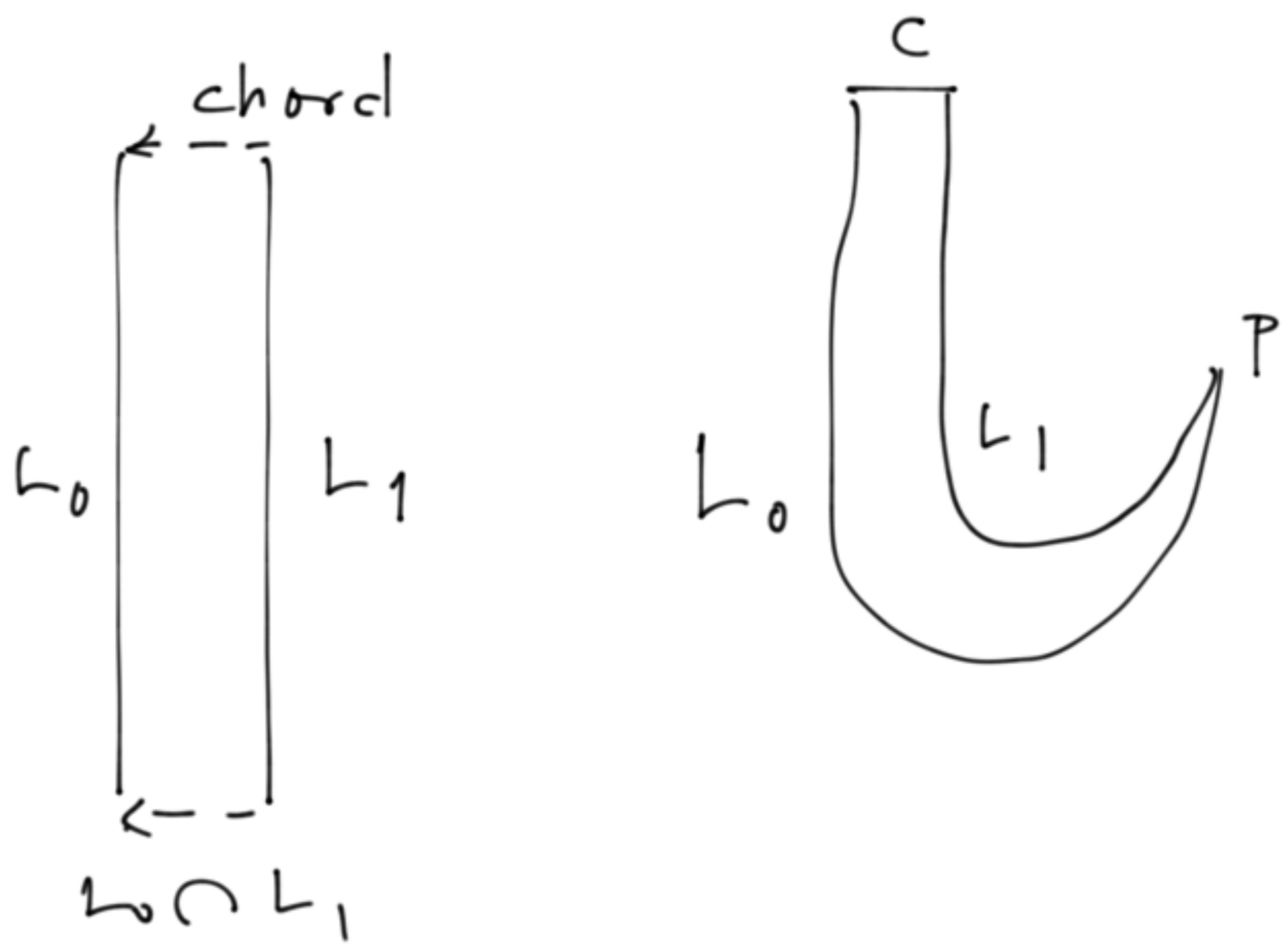
$$= C_{10} \oplus \mathrm{Morse}(F).$$

and define $d : C(L_1, L_0) \circlearrowleft$

$$d = \begin{bmatrix} d_{10} & 0 \\ \delta & d_F \end{bmatrix}$$

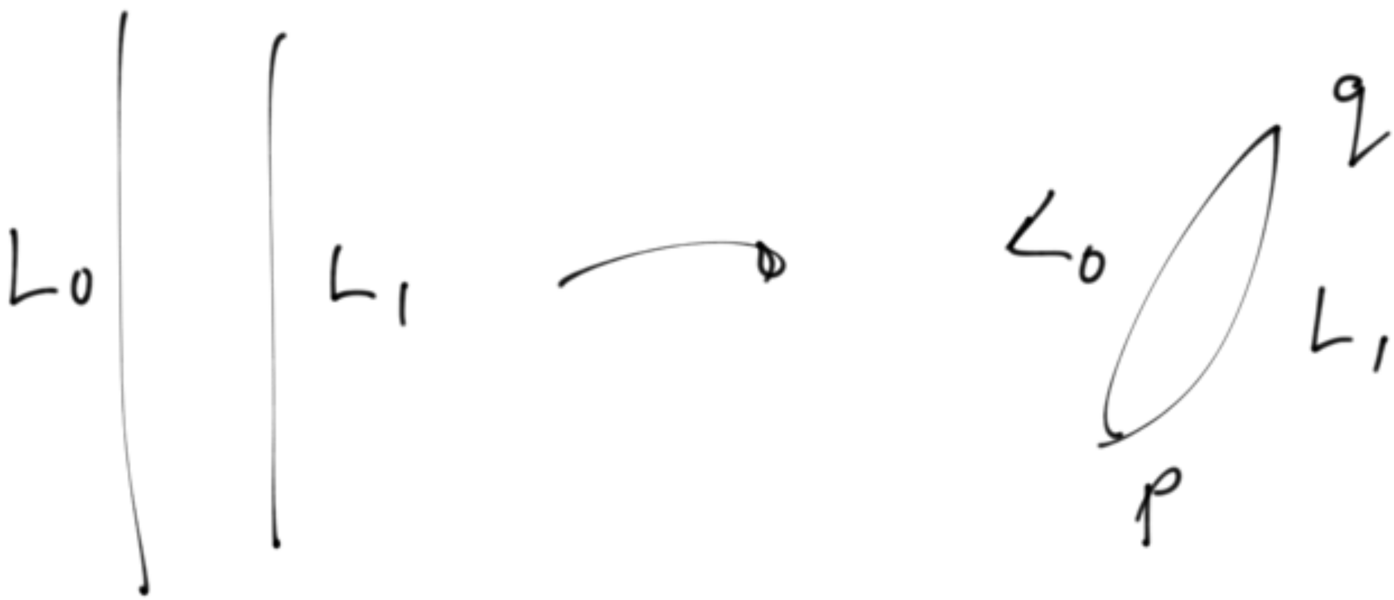
Here $\delta : C_{10} \longrightarrow \mathbb{Q}(L_1 \cap L_0)$

counts holomorphic strips

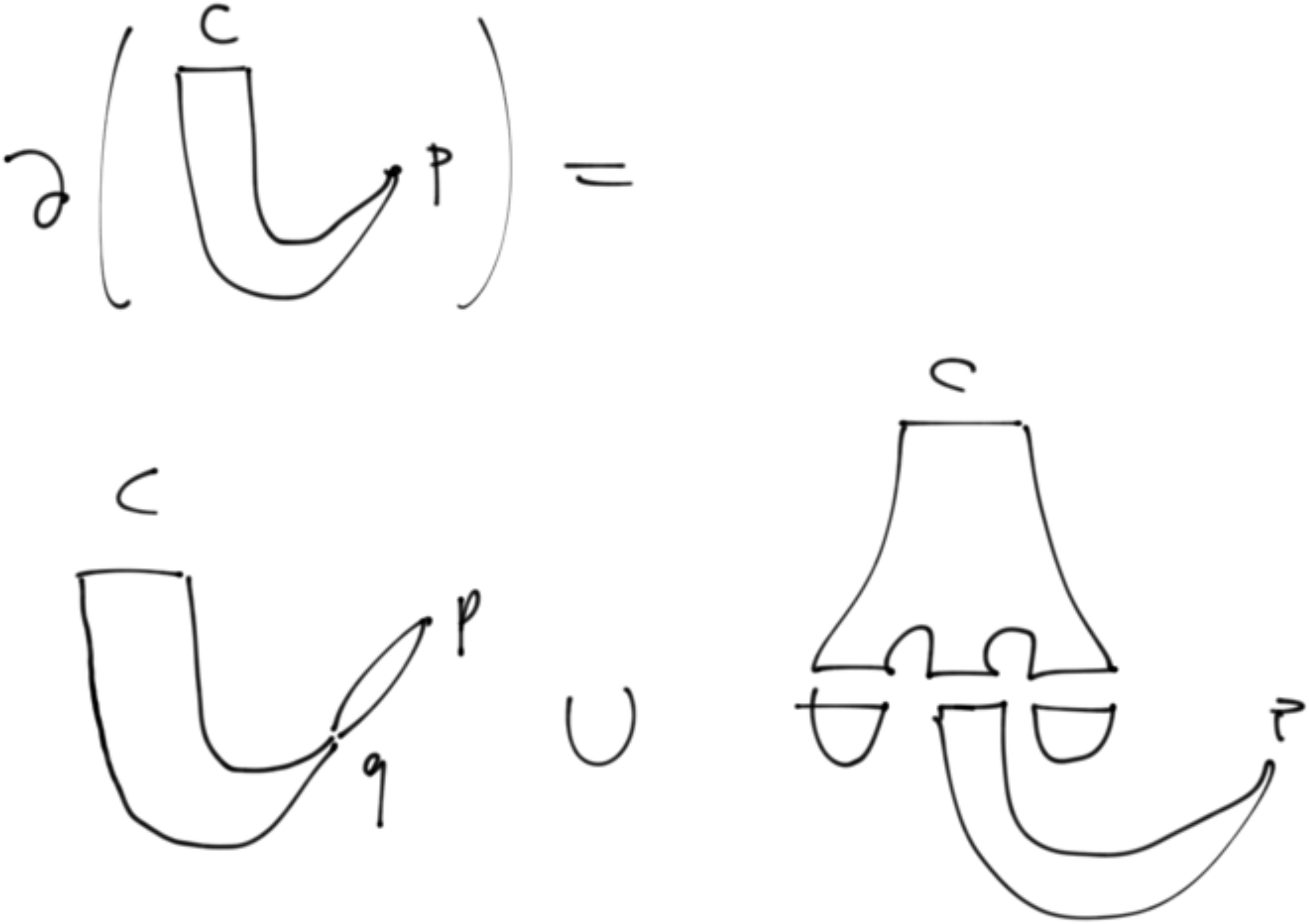


and $\quad d_F : \mathbb{Q}(L_1 \cap L_0) \circlearrowleft$

is the Floer differential



Looking at the boundary of 1-dimensional moduli spaces we find that d is a differential.



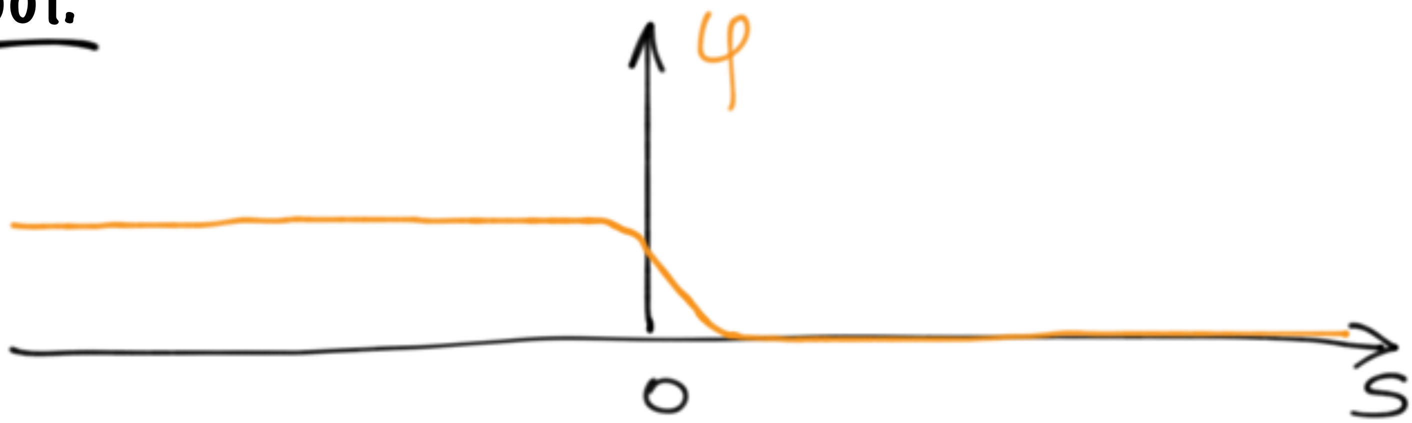
$$\Rightarrow \quad d^2 = 0 .$$

Theorem. There is a natural chain map

$$\phi : C(L_1, L_0) \longrightarrow \underset{\displaystyle WH(L)}{\underset{\cong}{WH(L_1, L_0)}}$$

which is a chain isomorphism.

Proof.



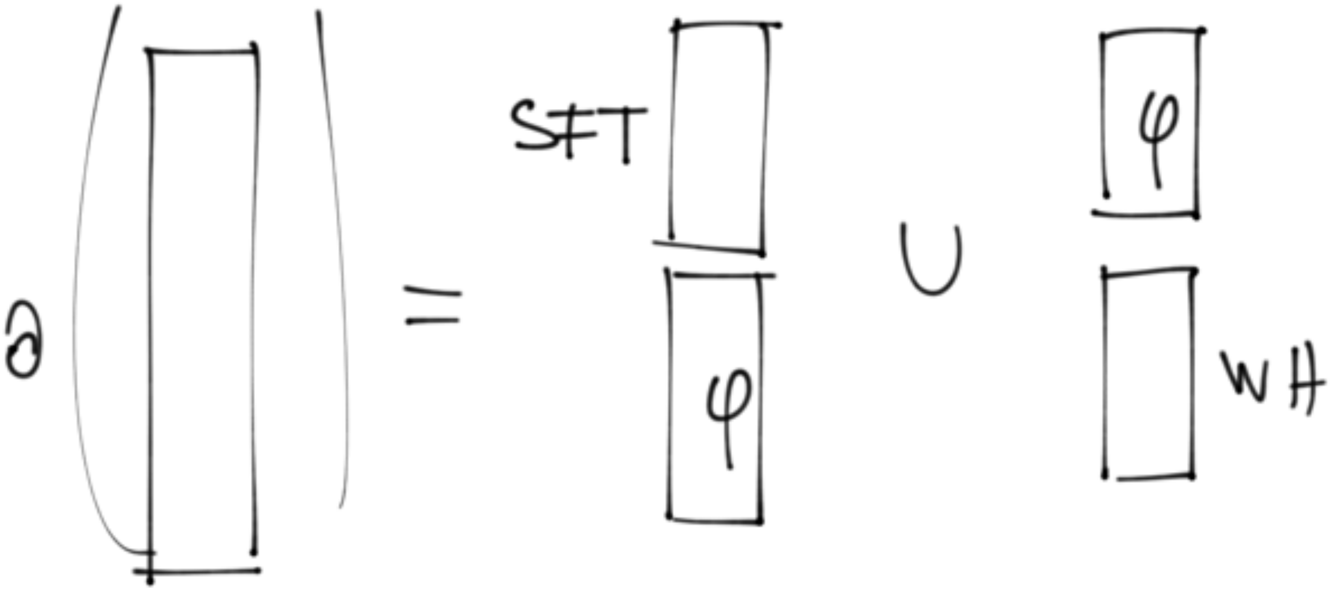
ϕ counts solutions of

$$\left(du + X_H \otimes \varphi(s)\,dt\right)^{0,1} = 0$$

or $$\frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - \varphi(s) X_H\right)$$

on $\mathbb{R} \times [0,1]$.

The chain map equ follows from identifying ∂(1-dim)



For Reeb chords and intersection pts one can construct explicit sol's from trivial strips

$$\Rightarrow \quad \Phi = \begin{array}{cc} & \xrightarrow{\text{action}} \\ \text{action}\Big\downarrow & \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \end{array}$$

$\Rightarrow$ Chain isomorphism. $\square$

This chain isomorphism also respects the action filtration and we find

$$C_{10} \approx WH^{+}$$

and $\quad C_{10} \longrightarrow$ Morse

corresponds to $\quad WH^{+} \to WH^{0}$.

Examples

L - Lagrangian filling of $\Lambda \subset \mathbb{R}^{2n-1}_{st}$

$$HW(L) = 0.$$

$$F = T^*_q M \subset T^* M$$

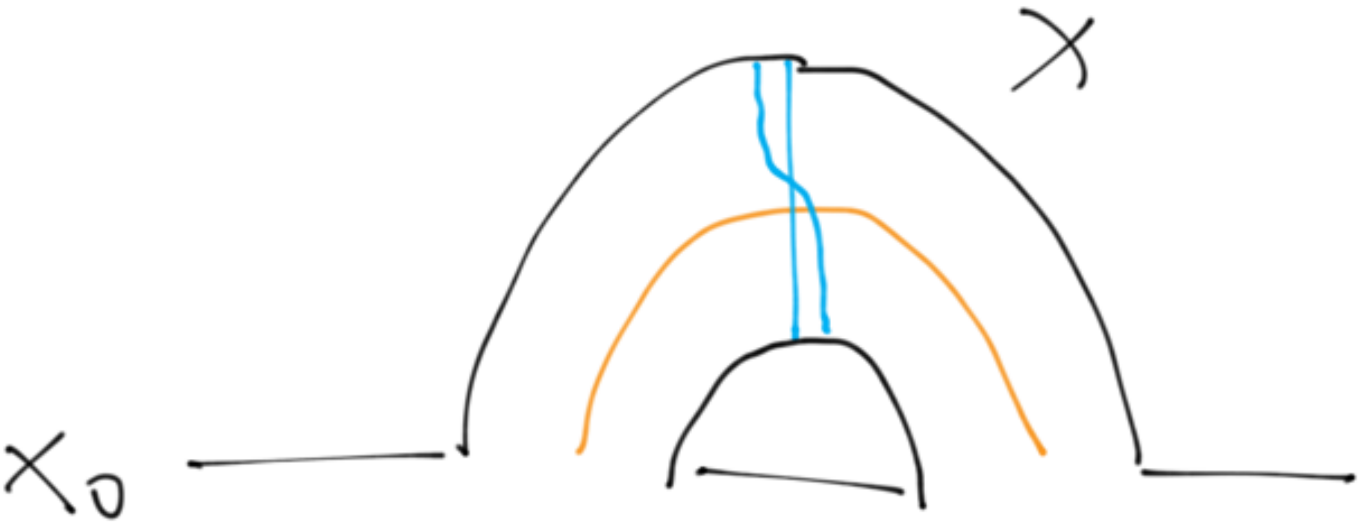
$$HW(F) = H_*(\Omega(M))$$

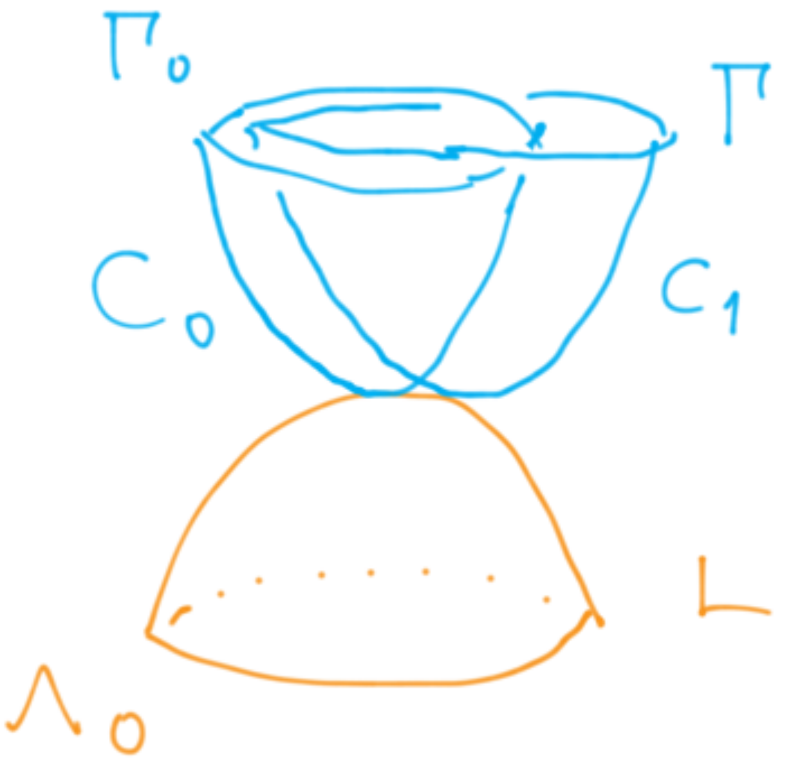
$$\Psi : CW(F) \to C_*(\Omega(M))$$

chord

chain of loops

The surgery map for wrapped Floer homology of co-core disks.



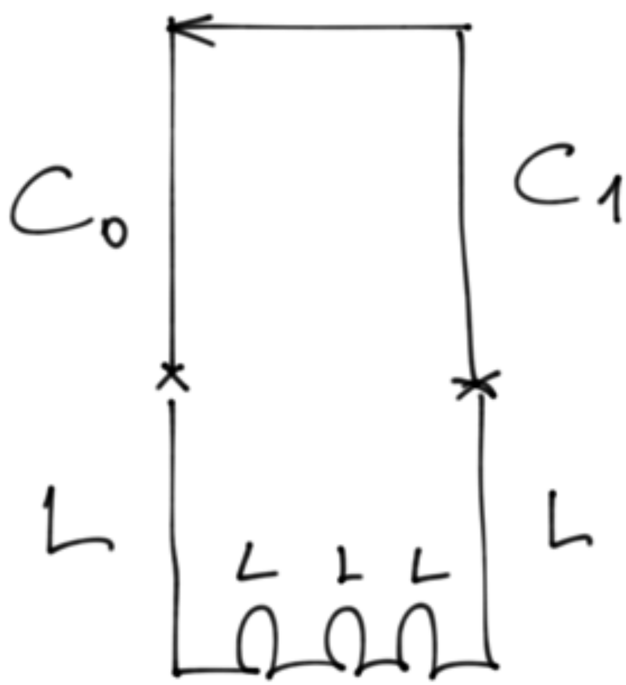


The surgery map

$$\Phi : C(L_1, L_0) \to A(\Lambda)$$

Counts holomorphic disks with one positive puncture, two Lagrangian intersection punctures and several negative punctures:

On C_{10}

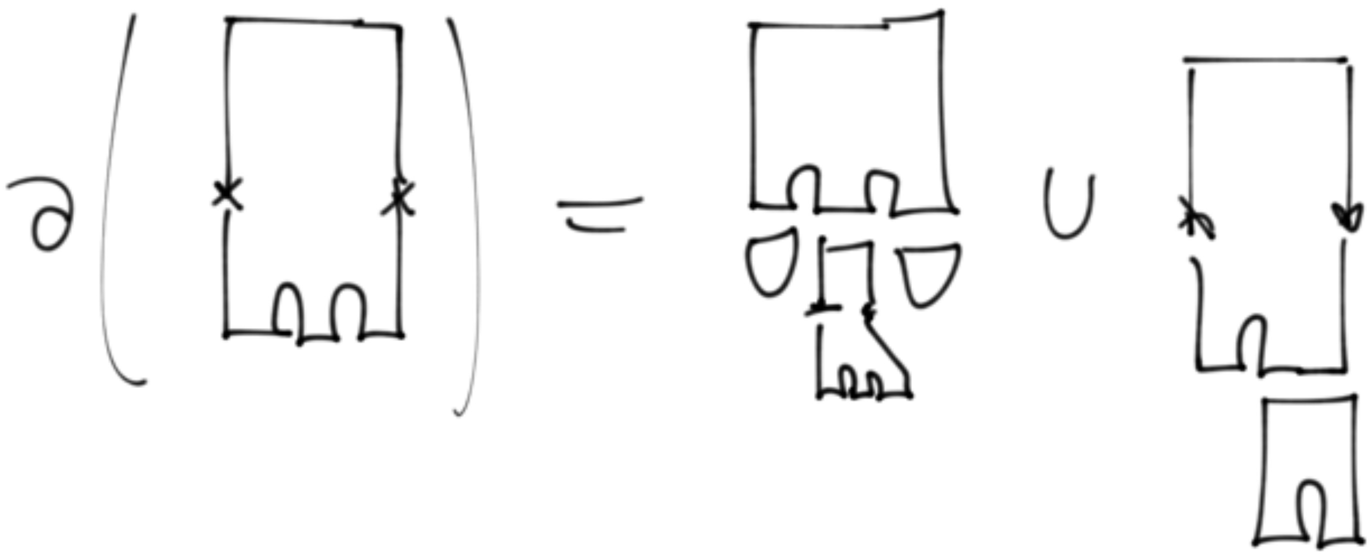


On $L_1 \cap L_0 = \{z\}$

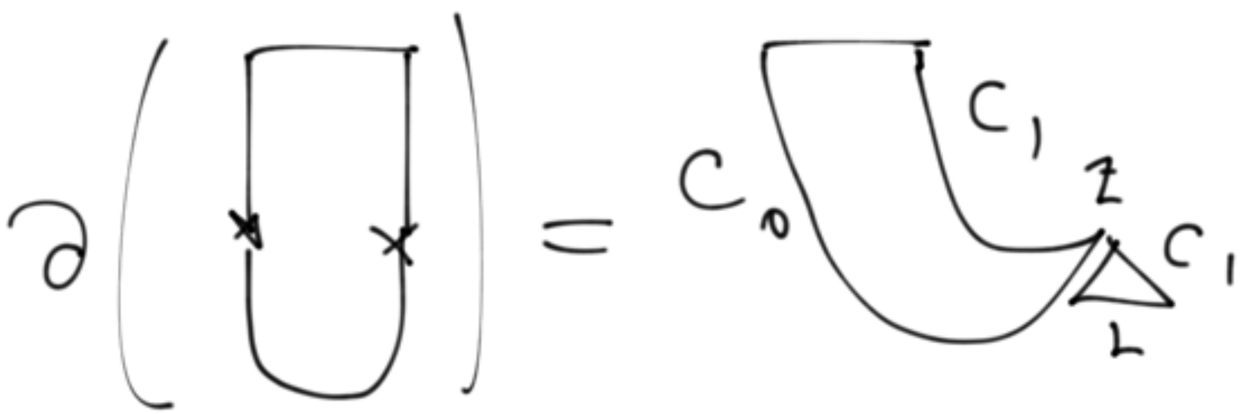
$$\Phi(z) = 1 \in A(\Lambda).$$

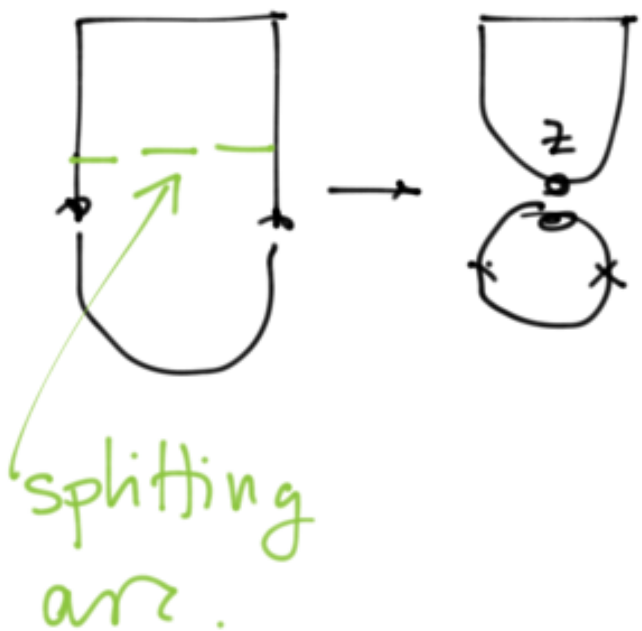
Theorem. The map Φ is a chain map.

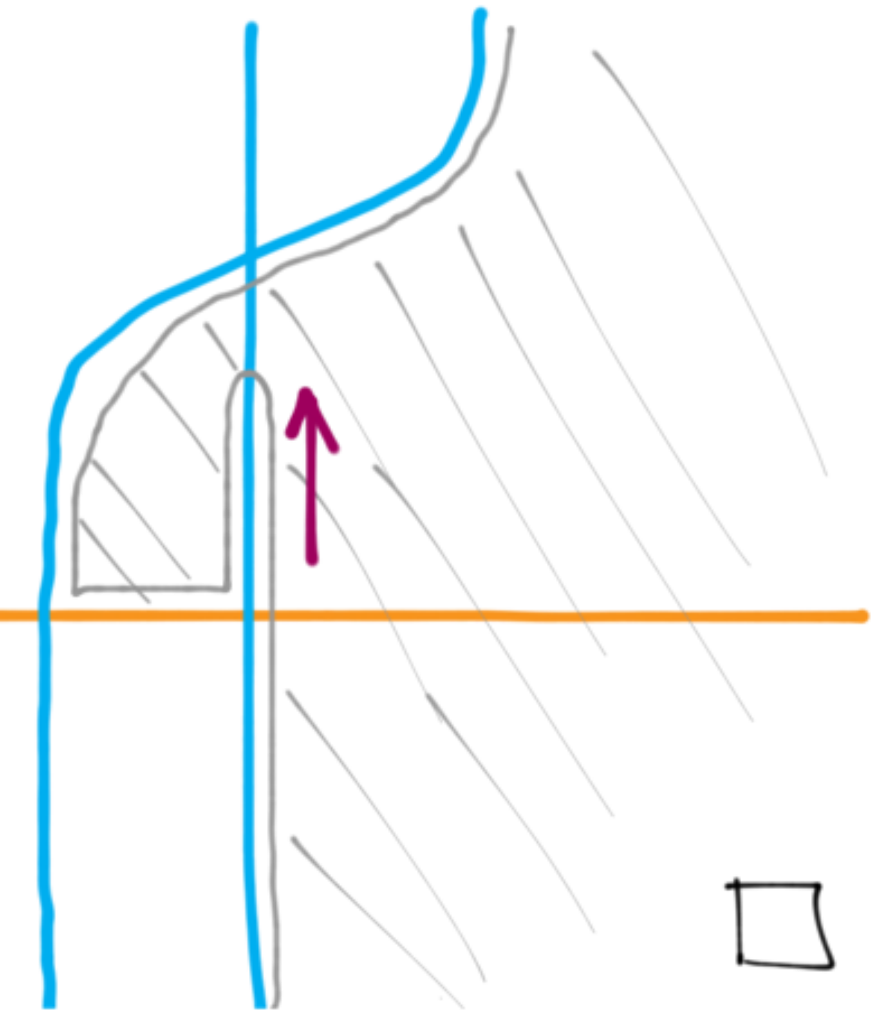
Proof.



and

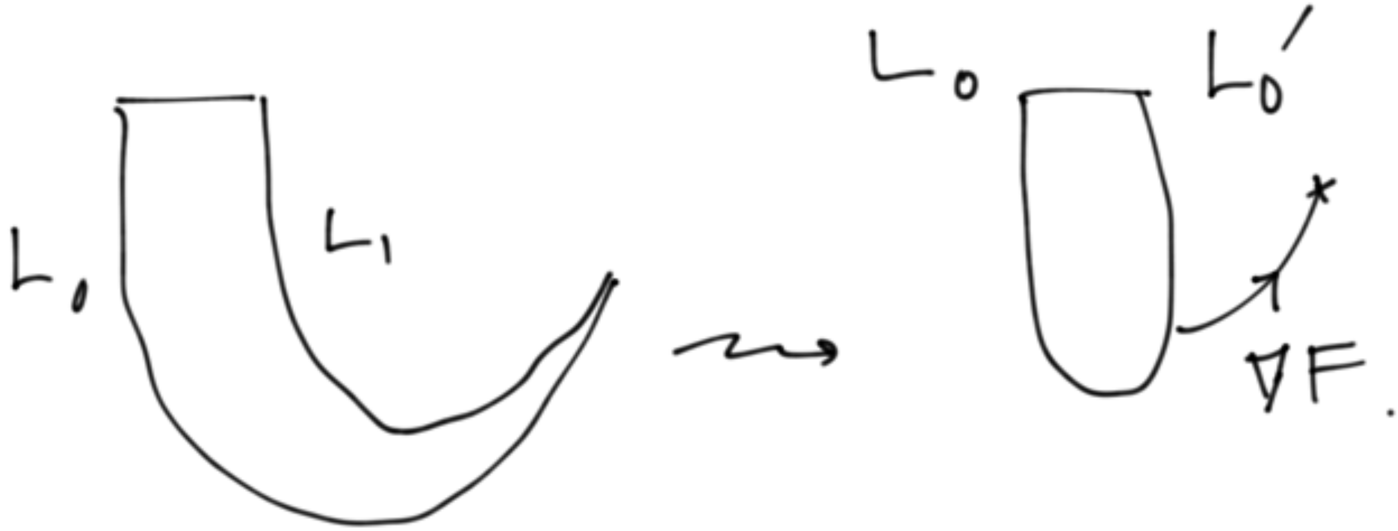






□

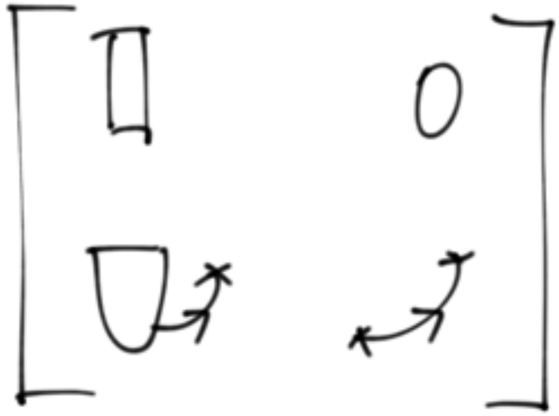
As we push L_1 towards L_0 we get an interpretation of the low energy part of the holomorphic disks as Morse flow lines.



We then get the following version of the two copy complex:

$$C(L_0, L_1) = C^{lin}(\Lambda) \oplus \mathrm{Morse}(L)$$

$d =$



We now turn to the proof that Φ is an isomorphism.

The proof has two basic steps.

1) Show that there is a natural 1-1 correspondence between the generators of $C(L_1, L_0)$ and generators of $A(\Lambda)$ (as a $\mathbb{Q}$-vector space).

2) Show that the chain map is upper triangular with 1's on the diagonal with respect to the energy filtration.

The proof of both 1) and 2) utilizes an explicit model of the handle that we turn to next.

Model of the handle.

We construct the model in two steps.

Take $\varepsilon > 0$ and let $p \gg 0$ be an integer, and s an integer with $\frac{1}{10}p < s < \frac{1}{5}p$

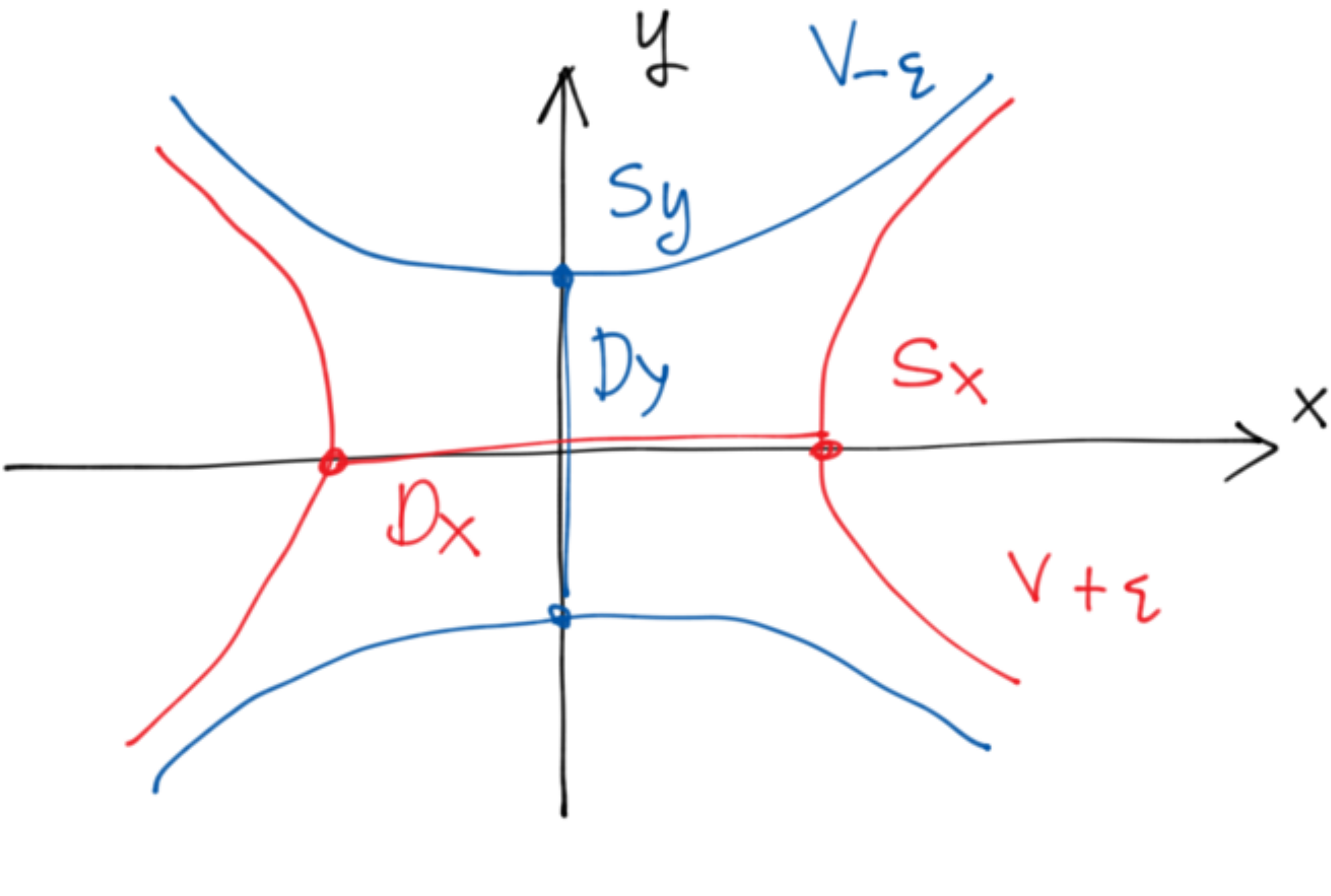
Write coordinates on $\mathbb{C}^n$ as $x + iy = (x_1, x_2) + i(y_1, y_2)$

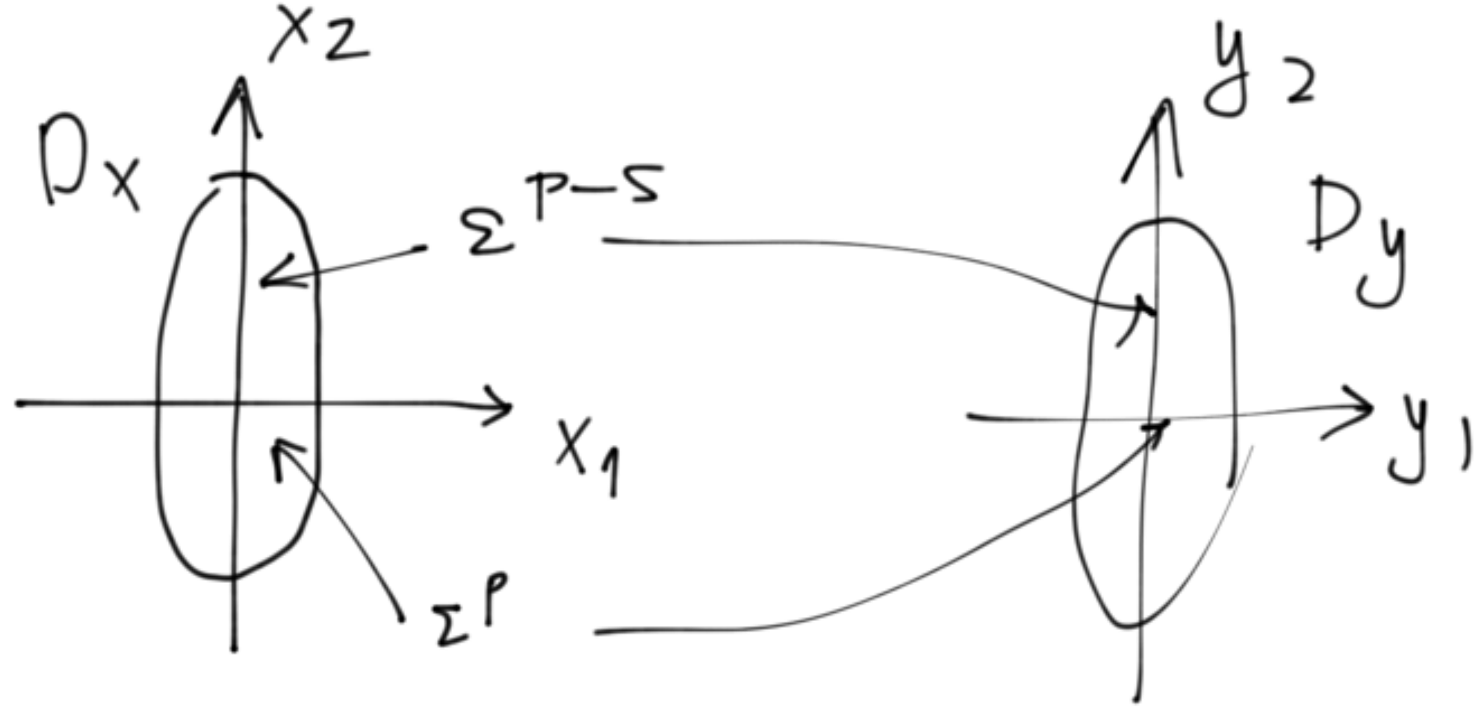
$x_1 + iy_1 \in \mathbb{C}$, $x_2 + iy_2 \in \mathbb{C}^{n-1}$

The basic handle:

$$H_\varepsilon =$$

$$\left\{(x,y) : \left|2x_1^2 - y_1^2 + \varepsilon^{2s}\left(2x_2^2 - y_2^2\right)\right| \leq \varepsilon^{2p}\right\}$$





Liouville v.f.

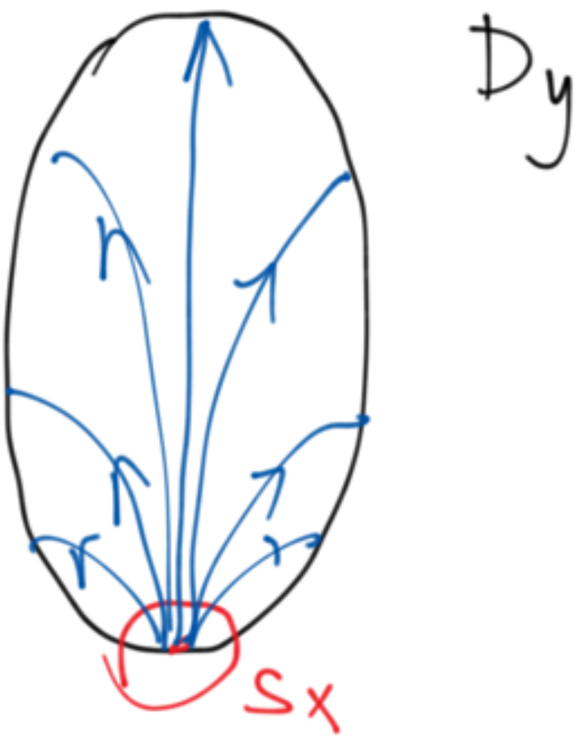
$$V = 2x\cdot\partial_x - y\cdot\partial_y$$

$$\alpha = 2x\cdot dy + y\cdot dx$$

$$R_\alpha = N(x,y)\cdot \left(2x_1\partial_{x_1} + y_1\partial_{y_1} + \varepsilon^{2s}(2x_2\partial_{x_2} + y_2\partial_{y_2})\right)$$

So up to normalization the Reeb flow is

$$\begin{cases} \dot{x}_1 = y_1 \\ \dot{y}_1 = 2x_1 \\ \dot{x}_2 = \varepsilon^{2s} y_2 \\ \dot{y}_2 = \varepsilon^{2s} 2x_2 \end{cases}$$

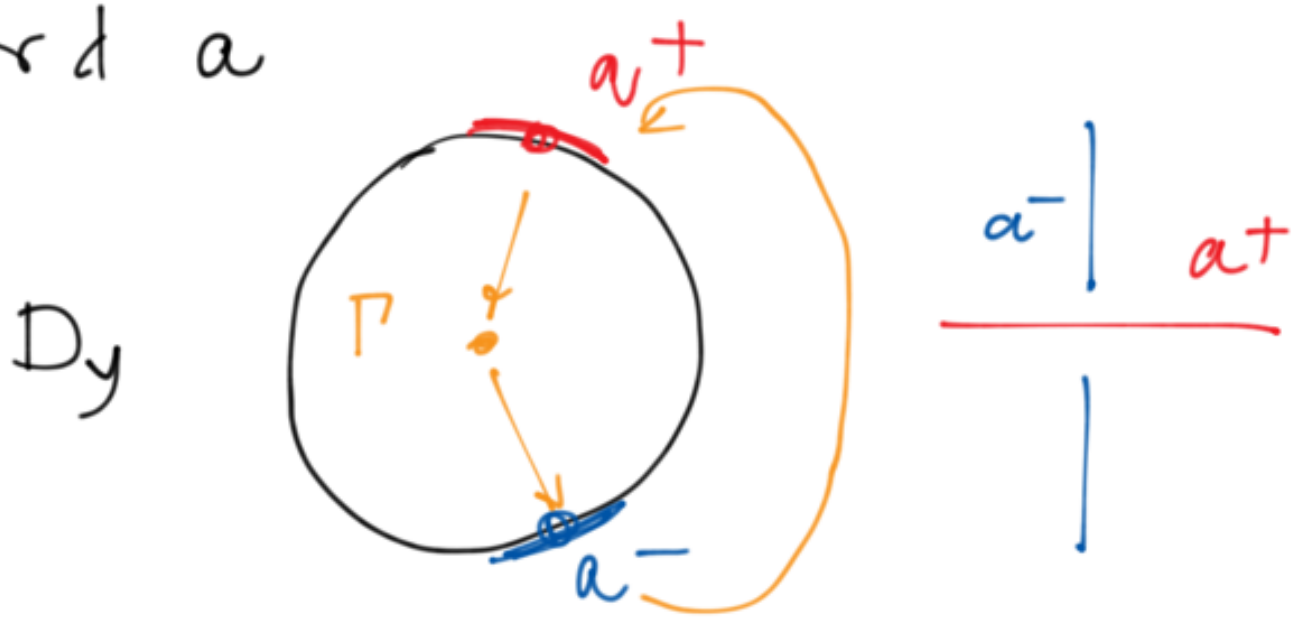


Theorem. For any $A>0$ there is $\varepsilon_A>0$ such that for all handles of size $\varepsilon<\varepsilon_A$ there is a 1-1 correspondence between chords of Γ and words of chords of Λ of action less than A.

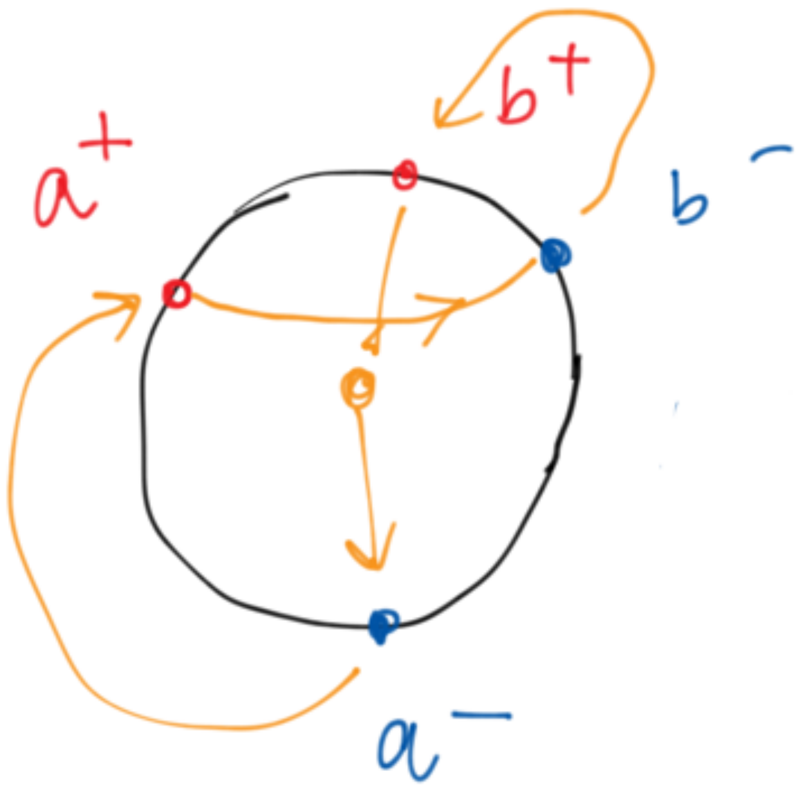
Proof.

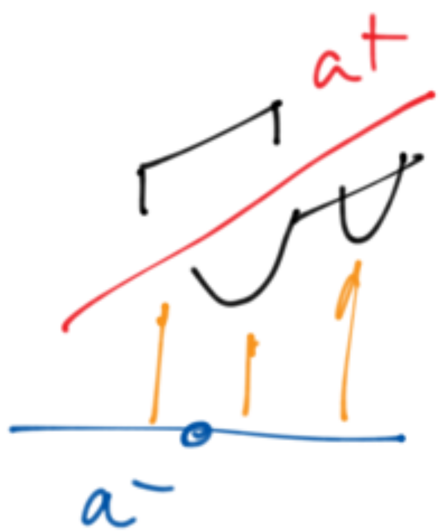
As $\varepsilon \to 0$ it is clear that chords of $\Gamma \longrightarrow$ words.

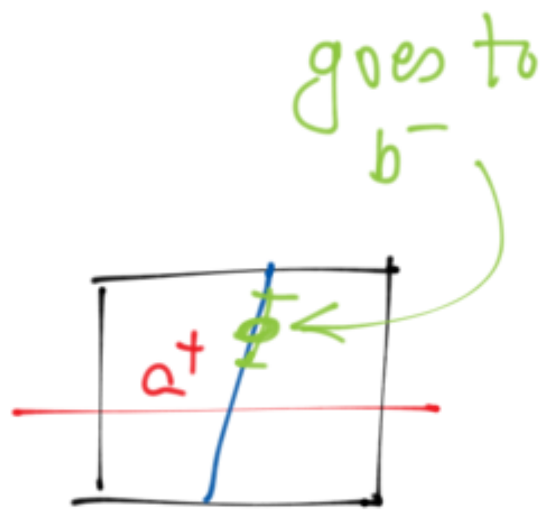
Consider first a 1-letter word a

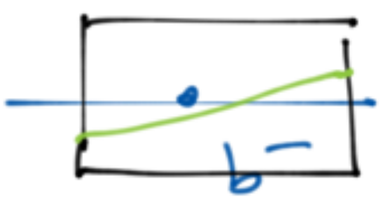


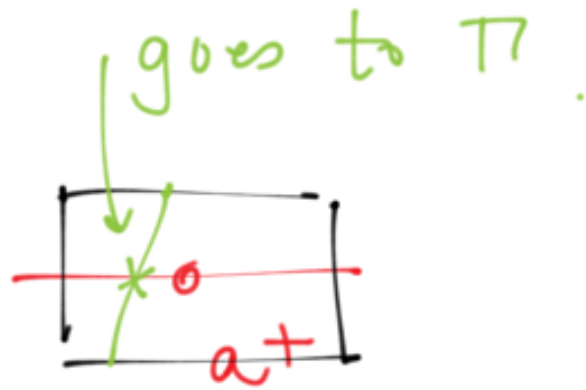
2 letter word. ab



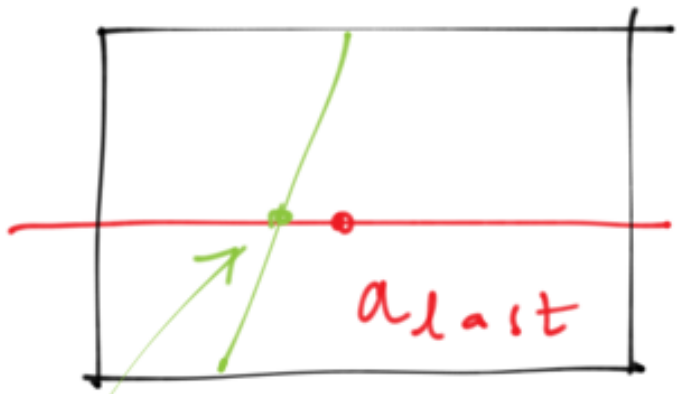








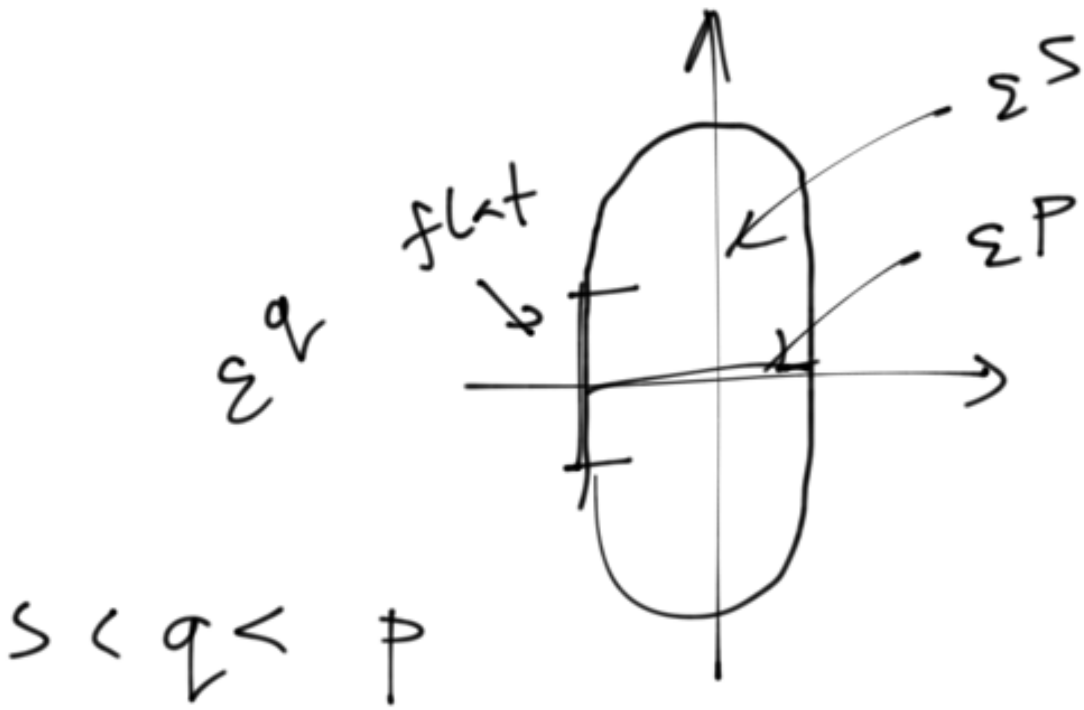
Longer words are similar
in the last step one
always finds



the unique inters
gives the chord.

$\square$

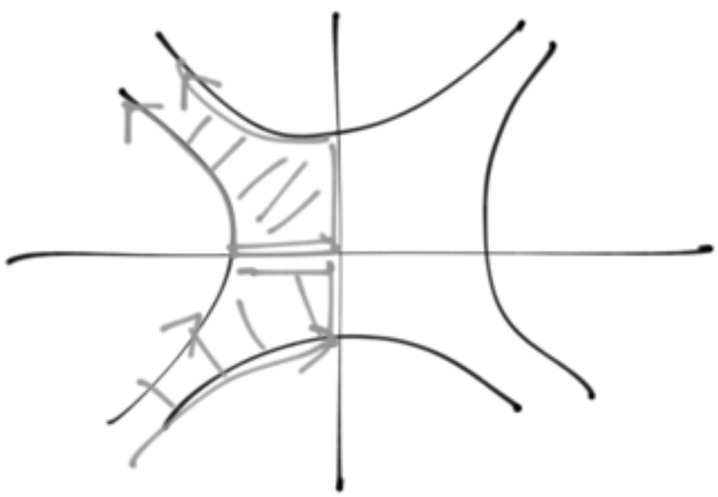
To prepare for the disk counting we first flatten the handle making it a product near the first coordinate plane



Observe next that the result on Reeb chords is independent of the attaching map and consider a chord with endpoints at (+1,0) and (-1,0).

We take an almost complex structure in a neighborhood of the corresponding Reeb chord so that holomorphic disks project to holomorphic disks in the complementary $\mathbb{C}^{n-1}$.

It is now easy to construct the required disk



An explicit check shows that this disk is uniformly transverse for variations of (sup)-norm $\leq 2^q$.

It follows that the disk is unique in a

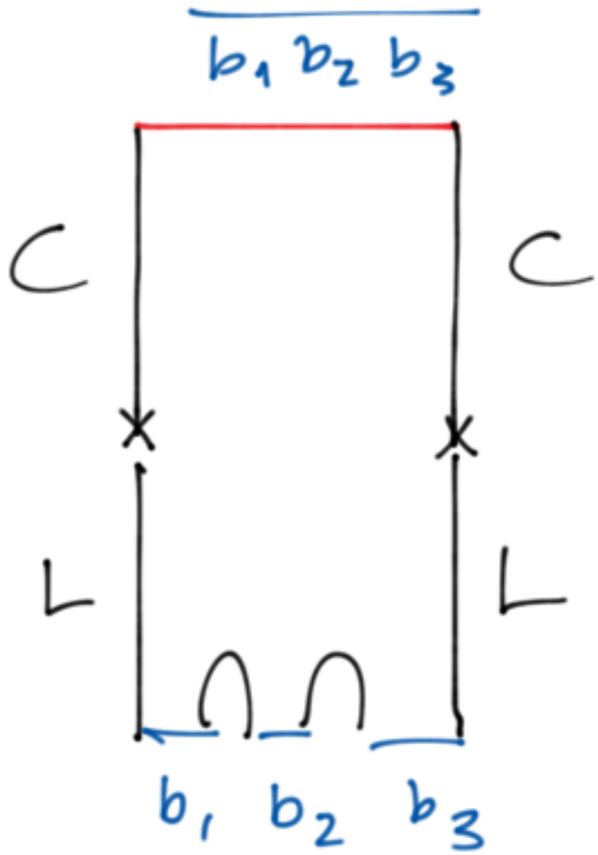
$\delta \cdot 2^q$ -neighborhood.

An action argument using projection to $\mathbb{C}^{n-1}$ then shows that any disk with given asymptotics must lie inside a $\delta\varepsilon^{q}$-nbhd.

We conclude that for $\varepsilon > 0$ small enough the disk is unique.

Theorem.

For any $A > 0$ there is $\varepsilon\ > 0$ such that for all $\varepsilon < \varepsilon$ there is algebraically 1 disk with a chord of Γ at its positive puncture and with the corresponding word of chords at its negative punctures.

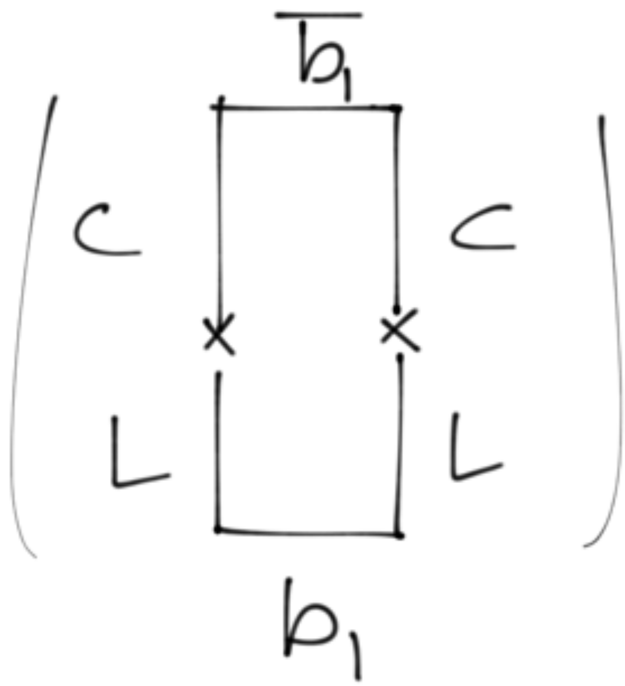


Proof.

Order chords of Λ by action

$$b_1 < b_2 < \ldots$$

Choose attaching map with $b_1^{\pm}$ at $(\pm 1, 0)$ as above

$\#$ 

$= n(\overline{b_1}; b_1) = \pm 1$

Let $\mathcal{M}(\bar{b}_1 ; b_1)$ denote the corresponding moduli-space.

Deform the attaching map in $I = [0,1]$ so that $b_2^{\pm} = (\pm 1, 0)$.

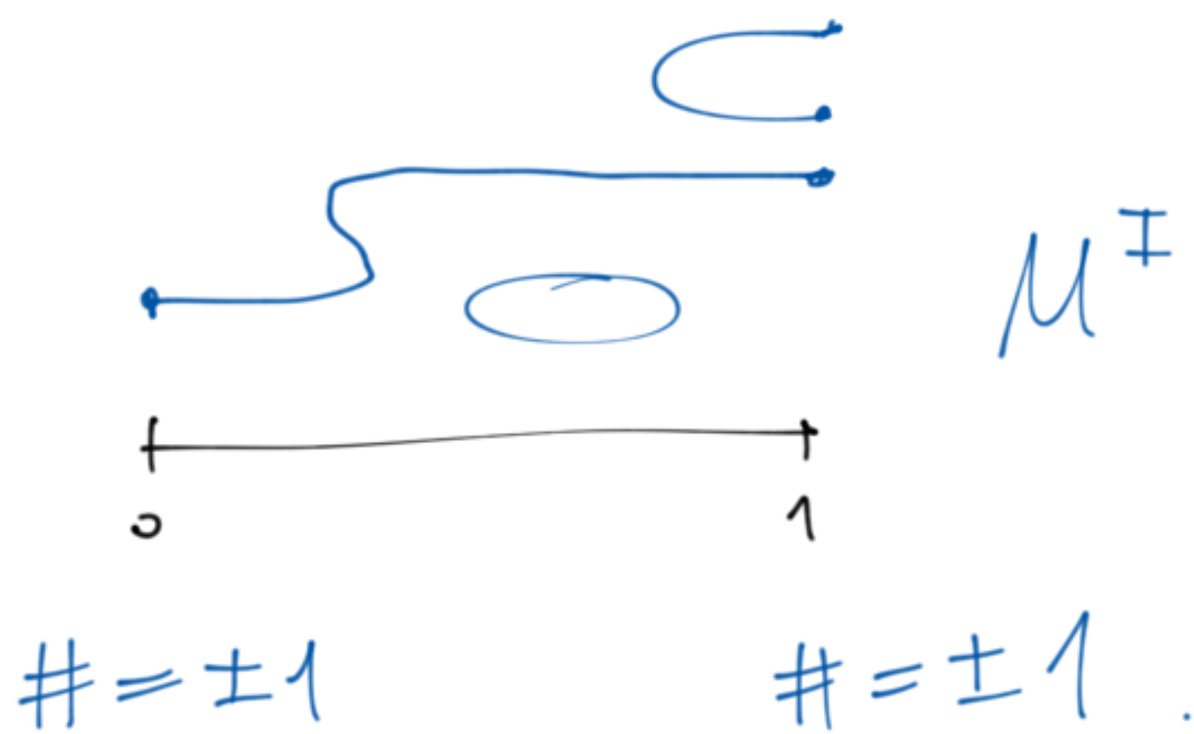
$$\mathcal{M}^I(\bar{b}_1, b_1) = \bigcup_{t \in I} \mathcal{M}^t(\bar{b}_1, b_1)$$

$$\dim(\mathcal{M}^I(\bar{b}_1, b_1)) = 1$$

and compact up to splitting

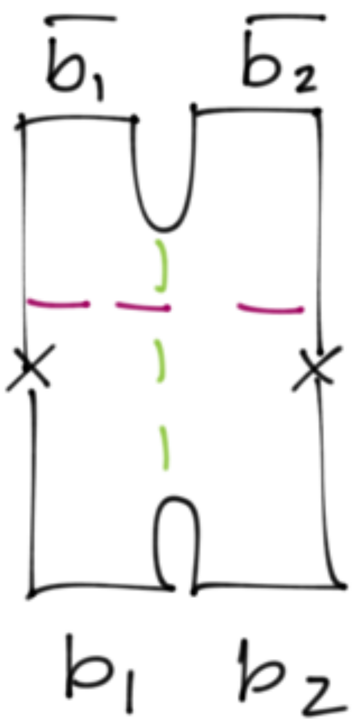
However, there is no Reeb chord / word of Reeb chords of action between $\bar{b}_1$ and b_1

so $M^{\mp}$ is a cobordism

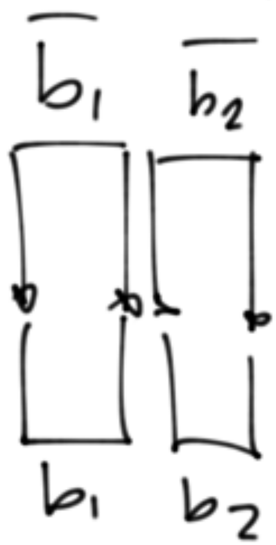


At $t=1$ also $n(\bar{b}_2, b_2)=1$

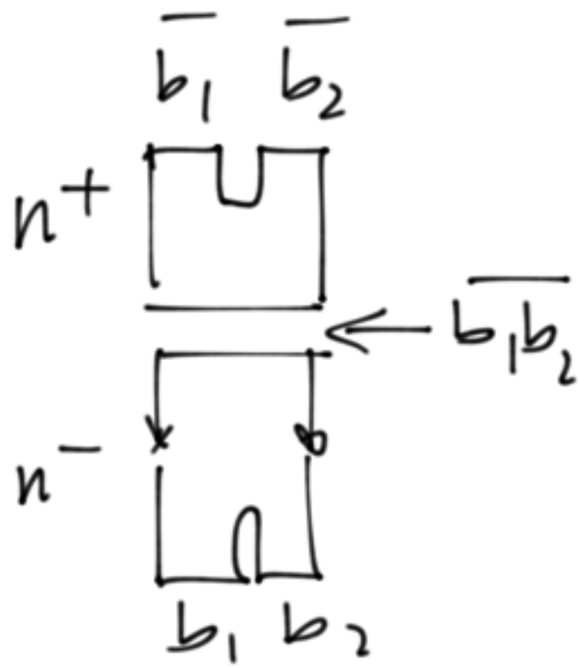
Consider $\mathcal{M}(\bar{b}_1, \bar{b}_2; b_1, b_2)$



$\partial \mathcal{M} =$



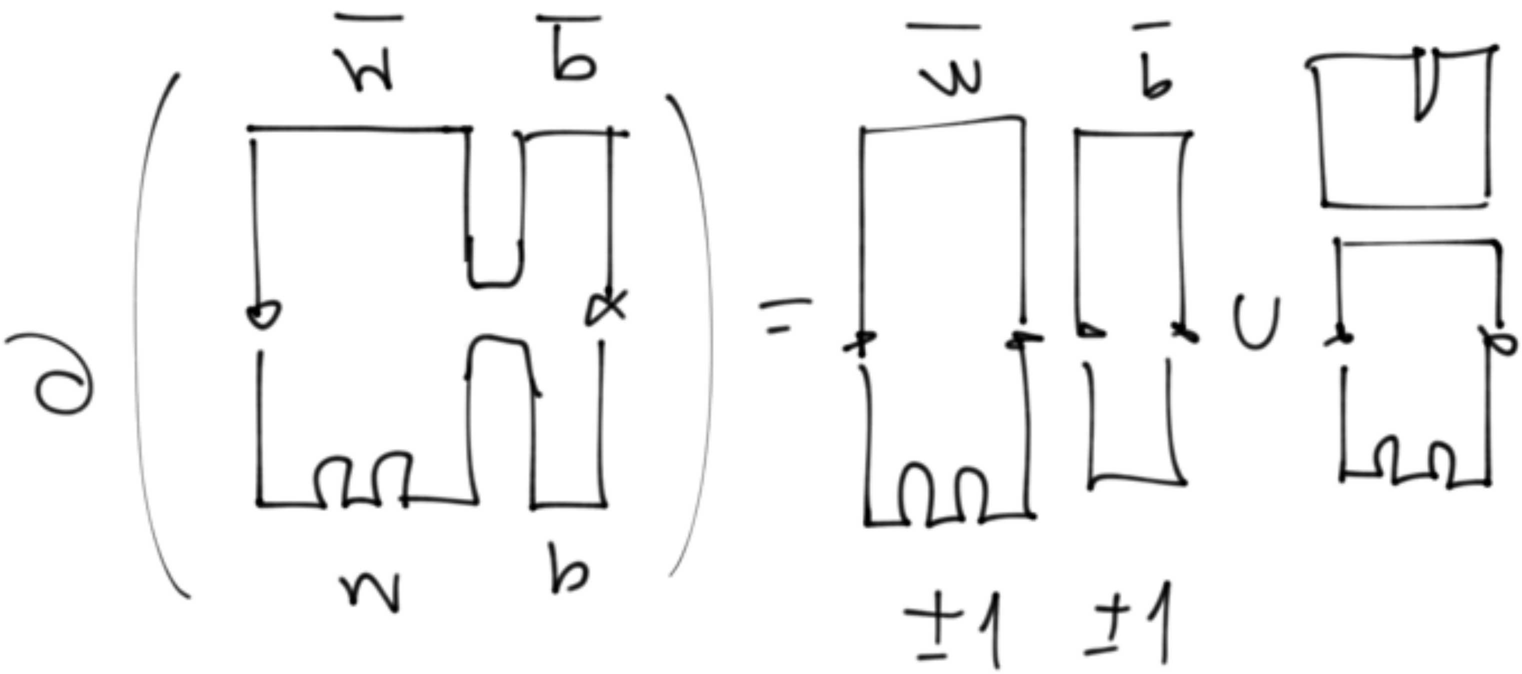
$\cup$



We find $n^+ \cdot n^- = \pm 1$

$$\Longrightarrow \quad n\left(\overline{b_1 b_2}\,;\, b_1 b_2\right) = \pm 1$$

It is now clear how the induction works:



$\square$

Example.

$$\Lambda \subset \mathbb{R}^{2n-1} \subset S^{2n-1}$$

Surgery gives T^*S^n.

$$C = T^*_q S^n .$$

$$A(\Lambda \subset S^{2n-1}) \cong A(\Lambda \subset \mathbb{R}^{2n-1})$$

HA :	1	a	a^2	a^3	$\dots$
deg	0	$(n-1)$	$2(n-1)$	$3(n-1)$	

$H(\Omega)$:  . .

$0 \quad (n-1) \quad 2(n-1) \quad 3(n-1) \quad 4(n-1) \quad \dots$

Symplectic homology without Hamiltonian

As in the case of wrapped Floer homology we can also express the symplectic homology through holomorphic curves at infinity and Morse theory in the inside.

$P(X)$ Reeb orbits in Y.

$\mathrm{Morse}(X)$ Critical pts of $H : X \to \mathbb{R}$

$C(X)$ is generated by two copies $\check{P}(X)$ and $\hat{P}(X)$ of $P(X)$ and $\mathrm{Morse}(X)$

Inspired by the Morse-Bott description of SH for time independent Hamiltonian we fix a point p on each geometric Reeb orbit.

$$d: C(X) = \hat{P} \oplus \check{P} \oplus M_0 \circlearrowleft$$

$$d = \begin{bmatrix} d_{\wedge}^{\wedge} & d_{\wedge}^{\vee} & 0 \\ d_{\vee}^{\wedge} & d_{\vee}^{\vee} & 0 \\ 0 & d_{M_0}^{\vee} & d_{M_0}^{M_0} \end{bmatrix}$$

$$d_{\vee}^{\wedge}(\hat{\gamma}) = \begin{cases} 0 & \gamma \text{ good} \\ 2\check{\gamma} & \gamma \text{ bad} \end{cases}$$

$d^{\wedge}_{\wedge}$ (and $d^{\vee}_{\vee}$) counts

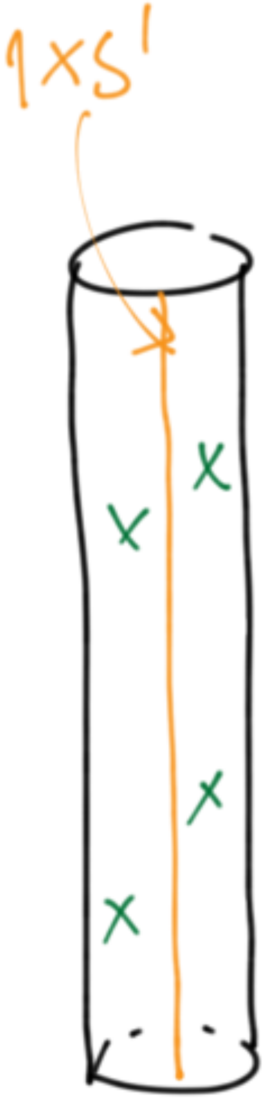
augmented holomorphic cylinders constrained by p at $-\infty$ (and $+\infty$)

$d^{\vee}_{\wedge}$ counts augmented

holomorphic Morse-Bott

buildings constrained

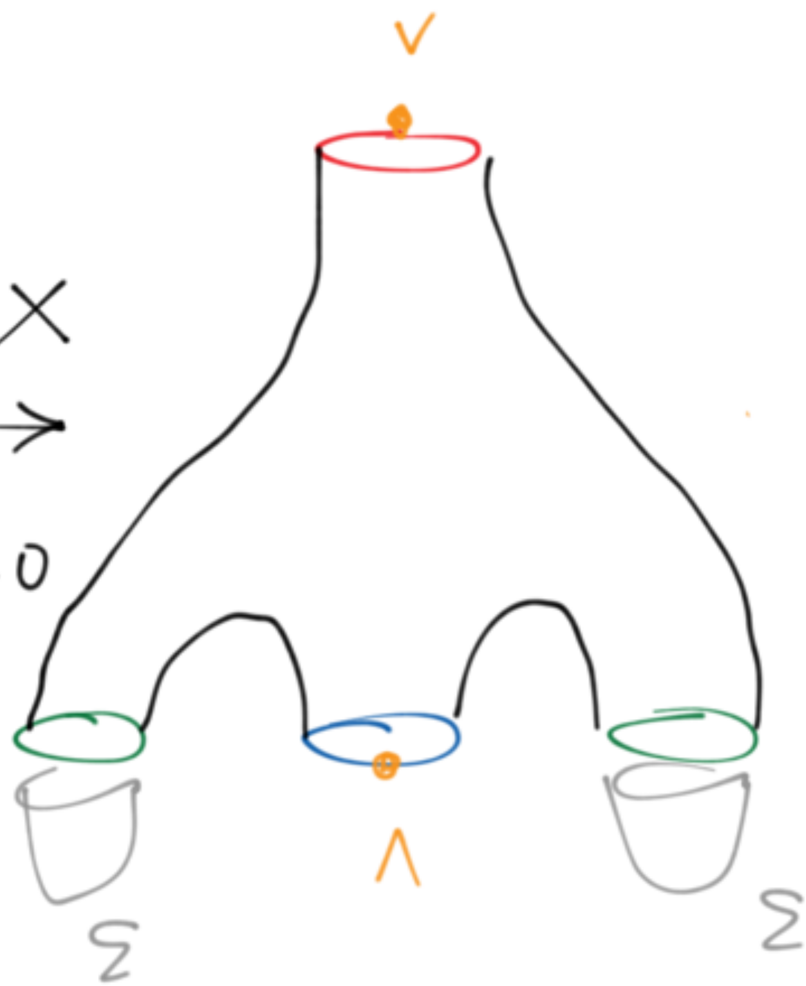
at both $-\infty$ and $+\infty$



$\mathbb{R}\times S^1$

$u : \mathbb{R}\times S^1 \to X$

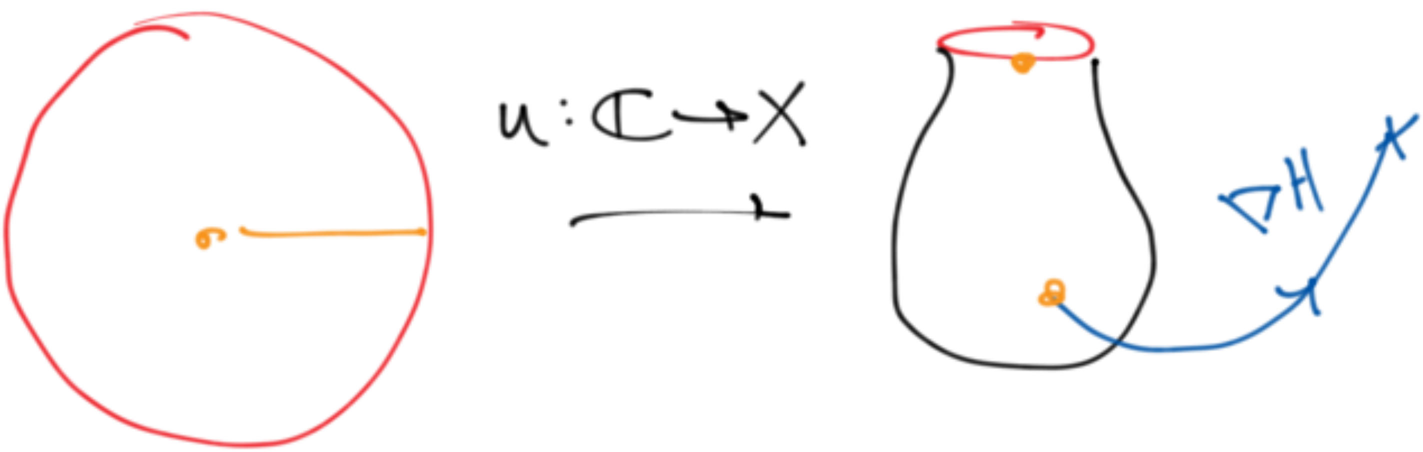
$du + J\, du\, i = 0$



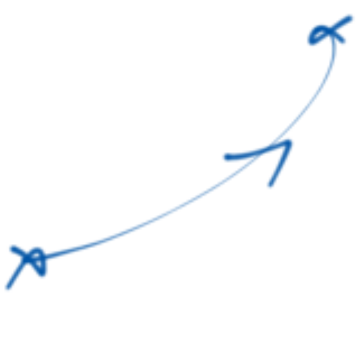
For $d^{\vee}_{\wedge}$ also

contribute.

Finally $d^{\vee}_{M_0}$ counts constrained holomorphic spheres



and $d^{M_0}_{M_0}$ counts flow lines



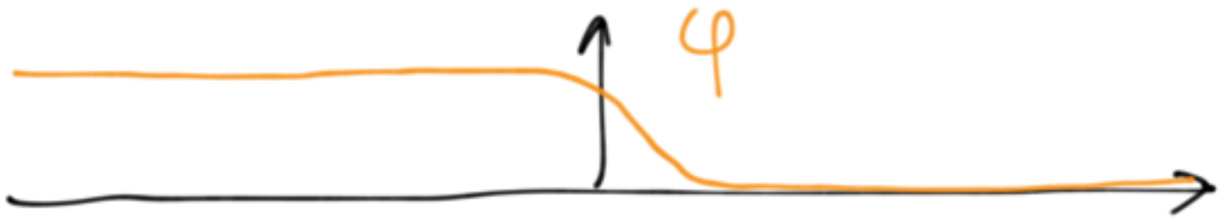
Looking at the boundary of 1-dim moduli spaces one finds that $d^2 = 0$

Theorem. There is a natural chain map

$$\Phi : C(X) \longrightarrow SC(X)$$

which is a chain isomorphism.

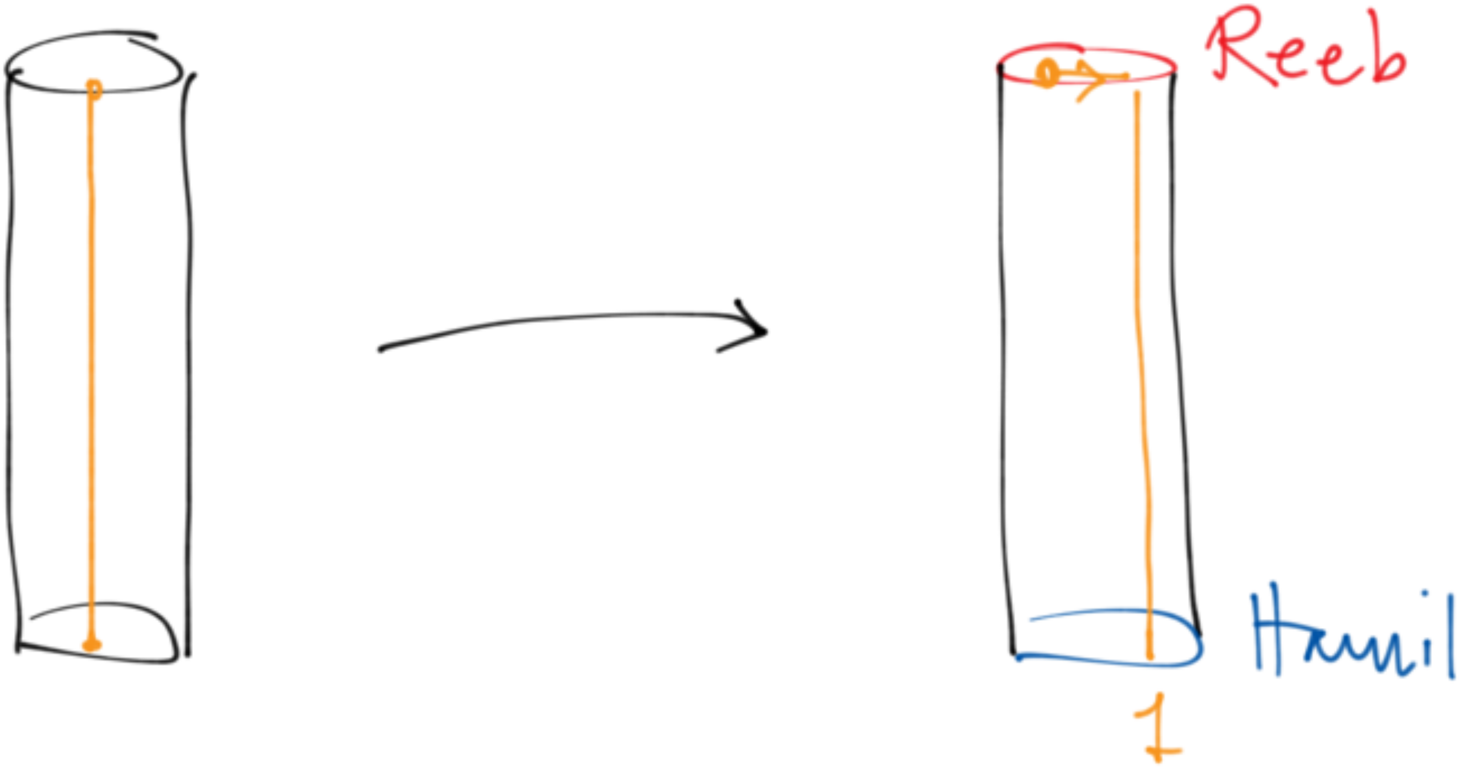
Proof. Recall



$\Phi(\tilde{\gamma})$ counts curves

of the form

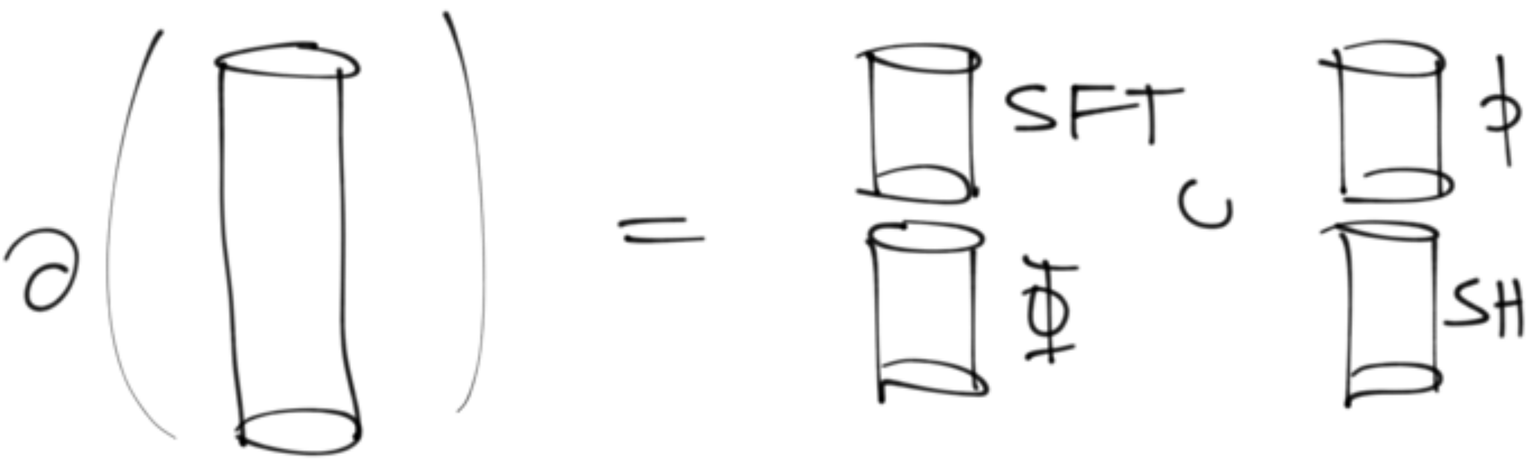
$u: \mathbb{R} \times S^1 \longrightarrow X$



$\vee$ constrains at $+\infty$

$\wedge$ does not

$$d\Phi - \Phi d = 0.$$



Morse part requires extra consideration.

In analogy with the Lagrangian case

one shows that the SFT split off curve is holomorphic with a flow-line attached

$$\left(\frac{\partial u}{\partial s} + \nabla X_H = 0 \right).$$