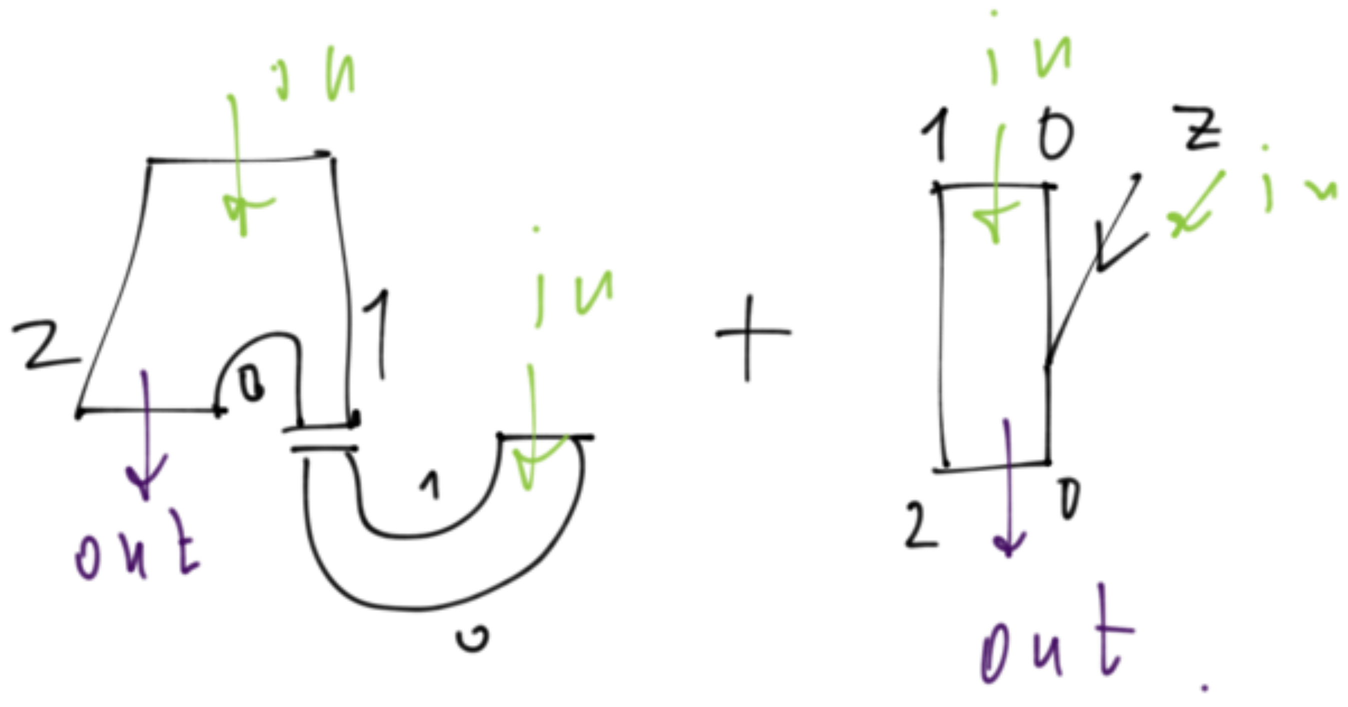
Conclusion.

The product on $A^{H_0}(\Lambda)$ is given by



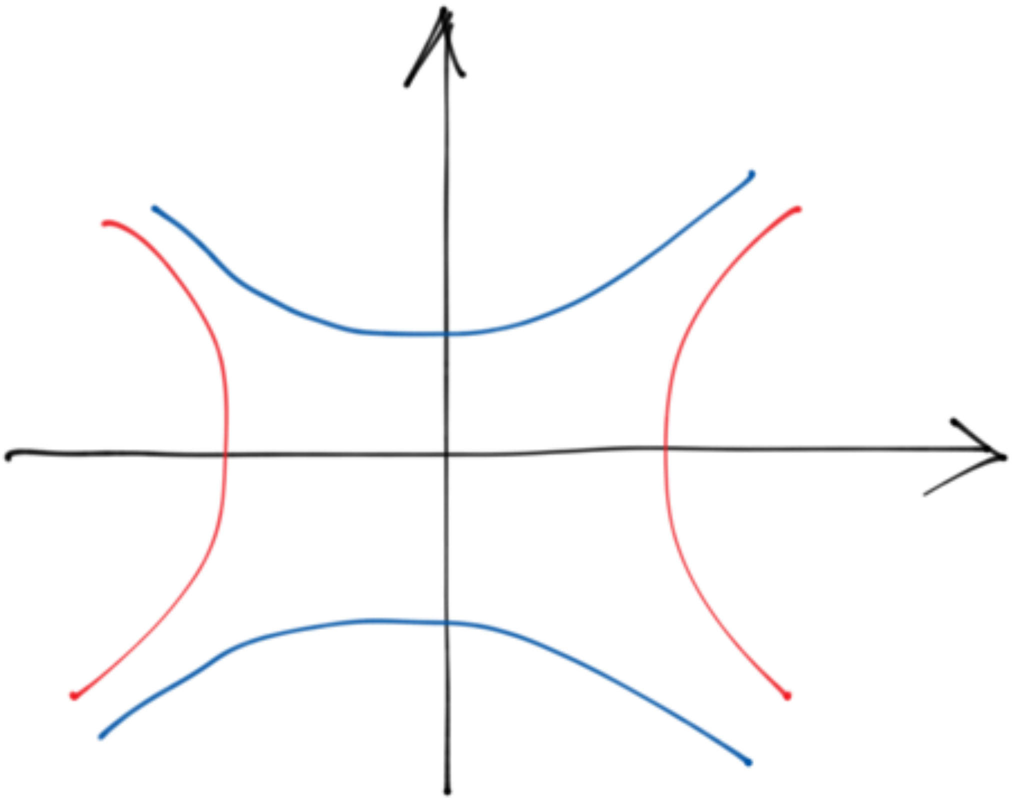
Example

For T^*S^n the unit is

$$\hat{a} - z$$

Note $\partial \hat{a} = y$.

Upside down surgery



The handle does not see the difference between pos and neg

$\Longrightarrow$ There is a contact form such that

$$\mathrm{Chords}(\Lambda) = \mathrm{Words}(\mathrm{Ch}(\Gamma))$$

$$\mathrm{Orbits}(X_0) = \mathrm{Cycword}(\mathrm{Ch}(\Gamma))$$

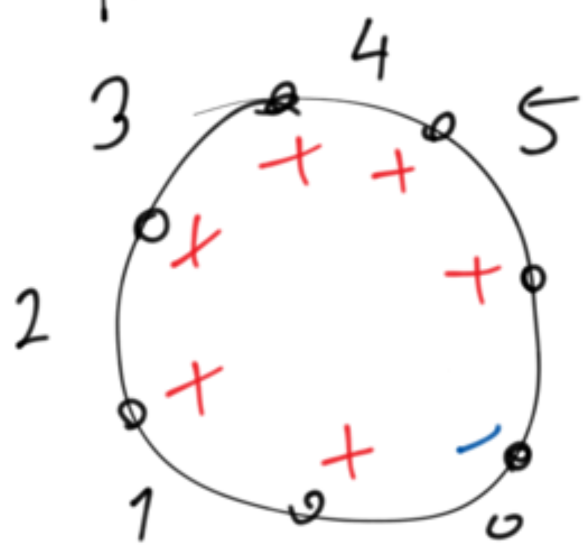
Moreover, basic curves can be constructed as before. This leads to an upside down surgery isomorphism

$HH_*(C)$ generated by cyclic words of chords and one distinguished.

Non-dist $\quad C_k \to C_j \quad j<k$

dist $\quad C_k \to C_j \quad j>k$.

Differential counts

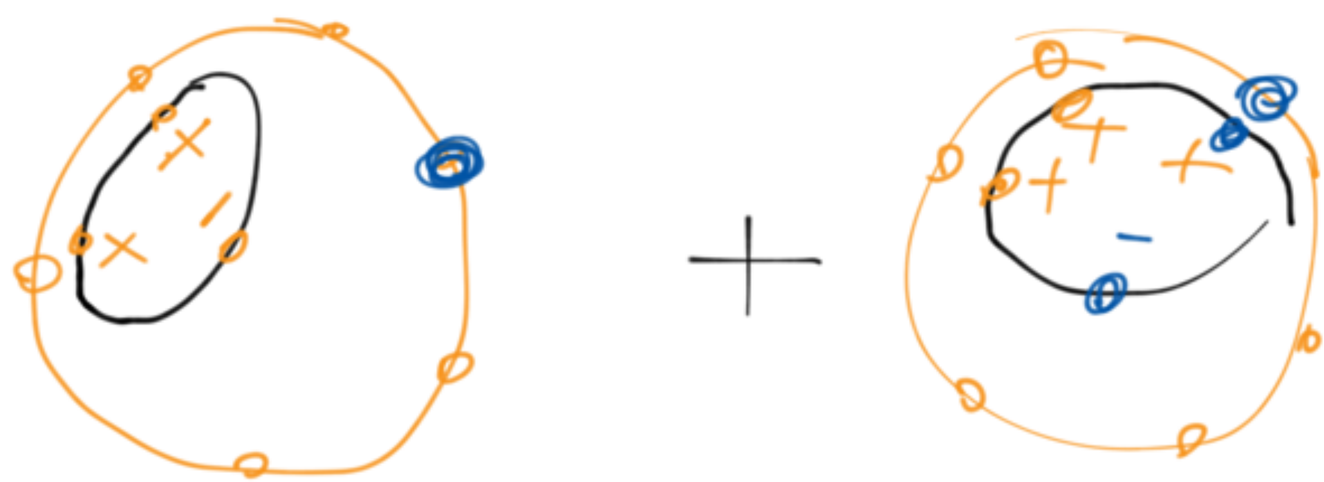


__Note__: at one puncture we jump __up__

When negative jumps up
no distinguished

When positive jumps up
that and neg distinguished.

Differential on HH_*



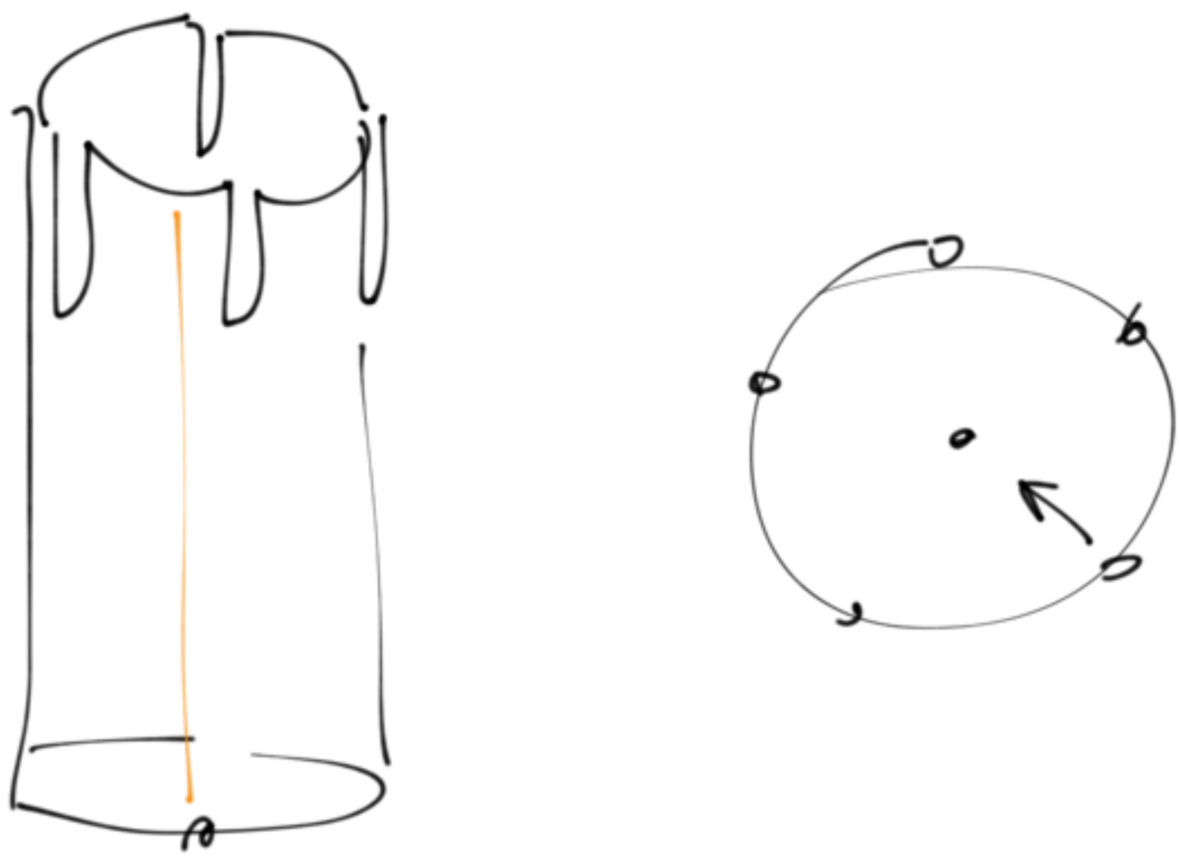
Consider $HH_*(C) \oplus SH(X)$

with differential

$$d = \begin{bmatrix} d_{HH_*} & 0 \\ \delta & d_{SH} \end{bmatrix}$$

Here $\delta(\underset{\sim}{c}_1 c_2 \dots c_m)$

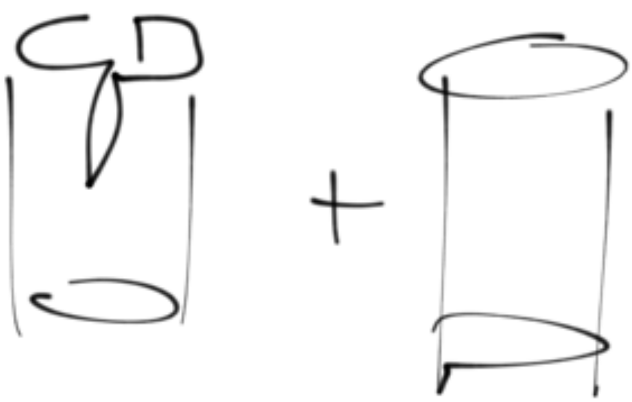
counts curves



As before $d^2 = 0$.

The same type of curves with negative end in Y_0 gives a chain map

$$\Phi : HH_*(C) \oplus SH(X) \longrightarrow SH(X_0)$$



which is a chain-isomorphism.

Now $SH(X_0) \cong 0$ if X_0 subcrit so

$$\delta : HH_*(C) \longrightarrow SH(X)$$

is an isomorphism.

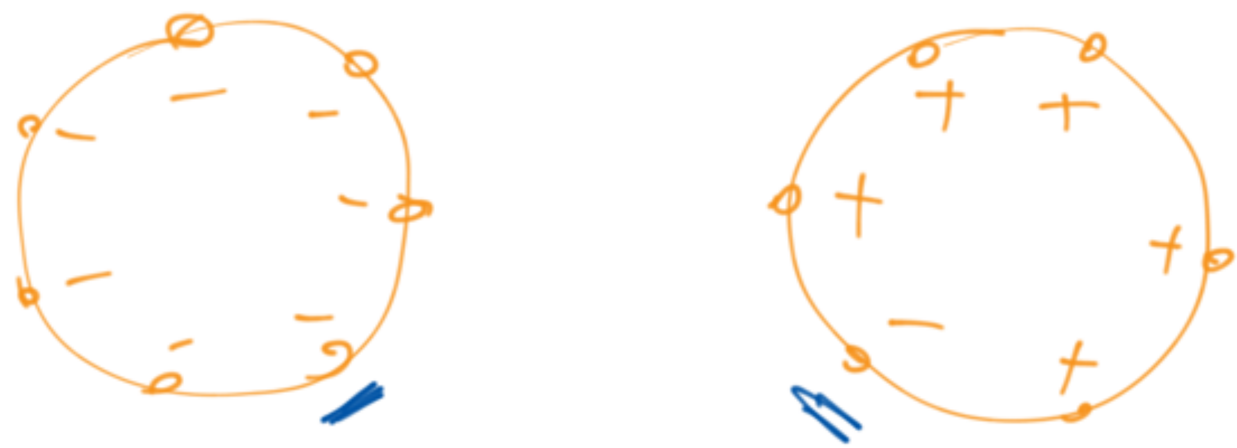
<u>Cor</u> (Abouzaid) C generates $\mathcal{F}uk(X)$.

We also consider the Hochschild cohomology.

$HH^q(C)$ with generators

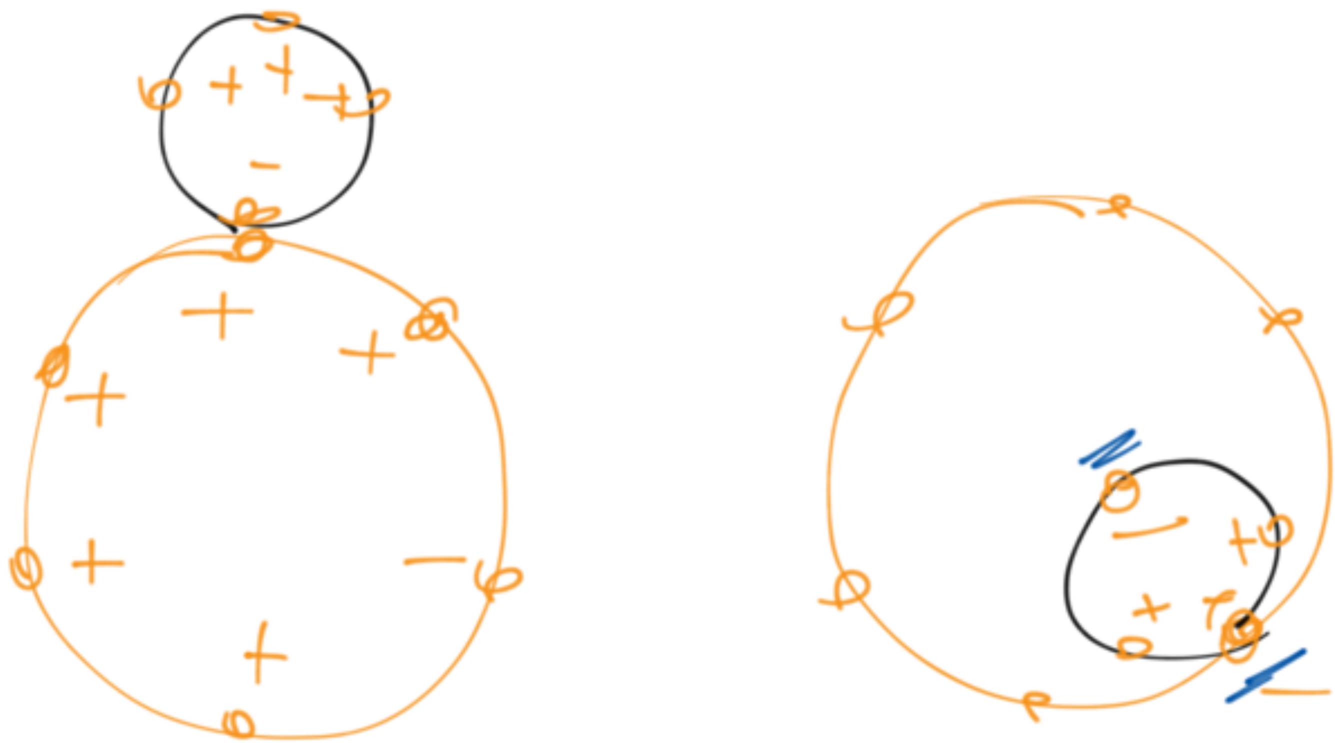
HH_*:

HH^*



The differential on $HH^*(C)$ attaches the disks

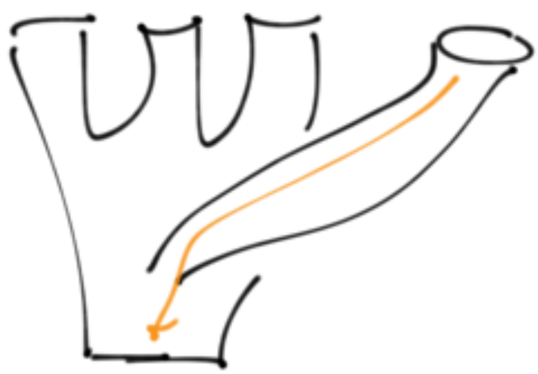
that we studied as



There is a natural chain map

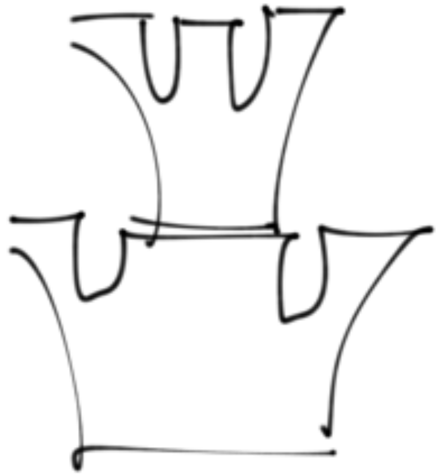
$$Ab : SH(X) \longrightarrow HH^*(C)$$

counting

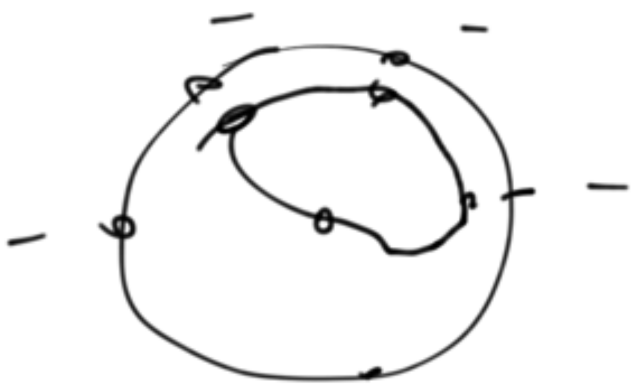


which is also an isomorphism

$HH^*(C)$ is a ring



$HH_*(C)$ is an $HH^*(C)$ module

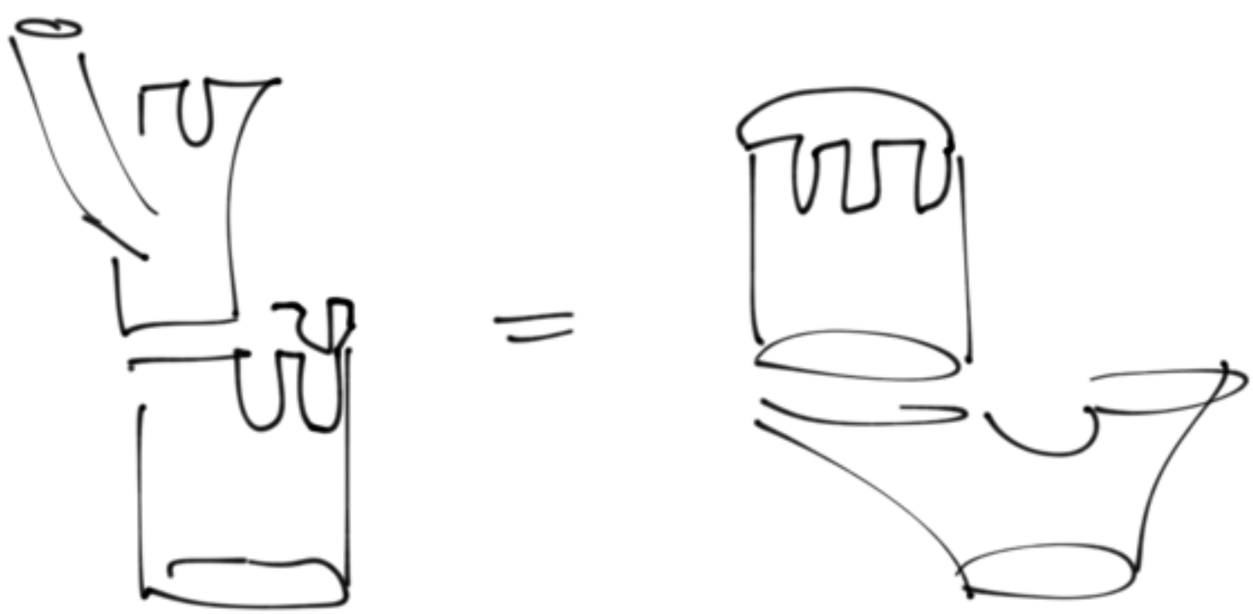


The map $Ab : SH \to HH^{*}$ is a map of rings.

Via Ab, $HH_*(C)$ becomes an $SH(X)$-module and the map

$$HH_*(C) \to SH(C)$$

a map of modules:



We then have

$$\begin{tikzcd} HH_*(C) \arrow[rr, "\text{surj.}"] \arrow[dr] & & HH^*(C) \\ & SH(X) \arrow[ur] & \end{tikzcd}$$

$$u \mapsto 1 \mapsto 1$$

For $r \in HH^*$ take $r \cdot u \in HH_*$

$$r \cdot u \mapsto r \cdot 1 = r$$

So $SH(X) \to HH^*$

<u>surj.</u>

Consider $s \in SH(X)$.

Then,

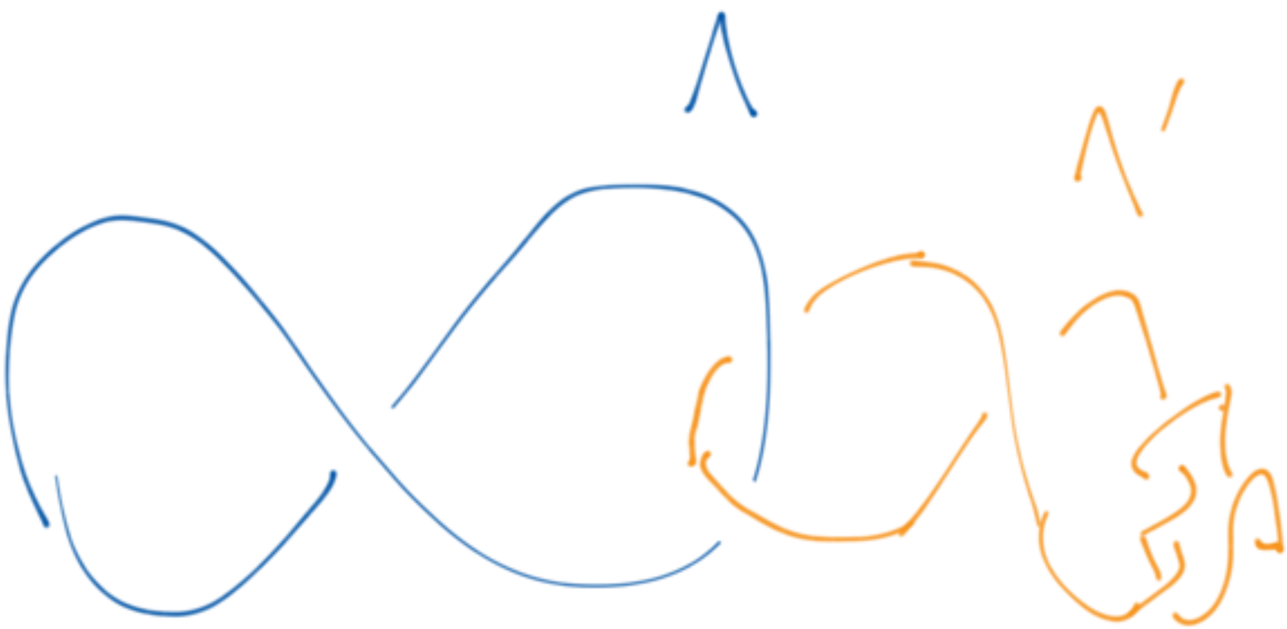
$\delta(Ab(s) \cdot u) = s \cdot \delta(u) =$

$= s \cdot 1 = s$

$\Longrightarrow Ab(s)$ injective

$\Longrightarrow Ab$ iso.

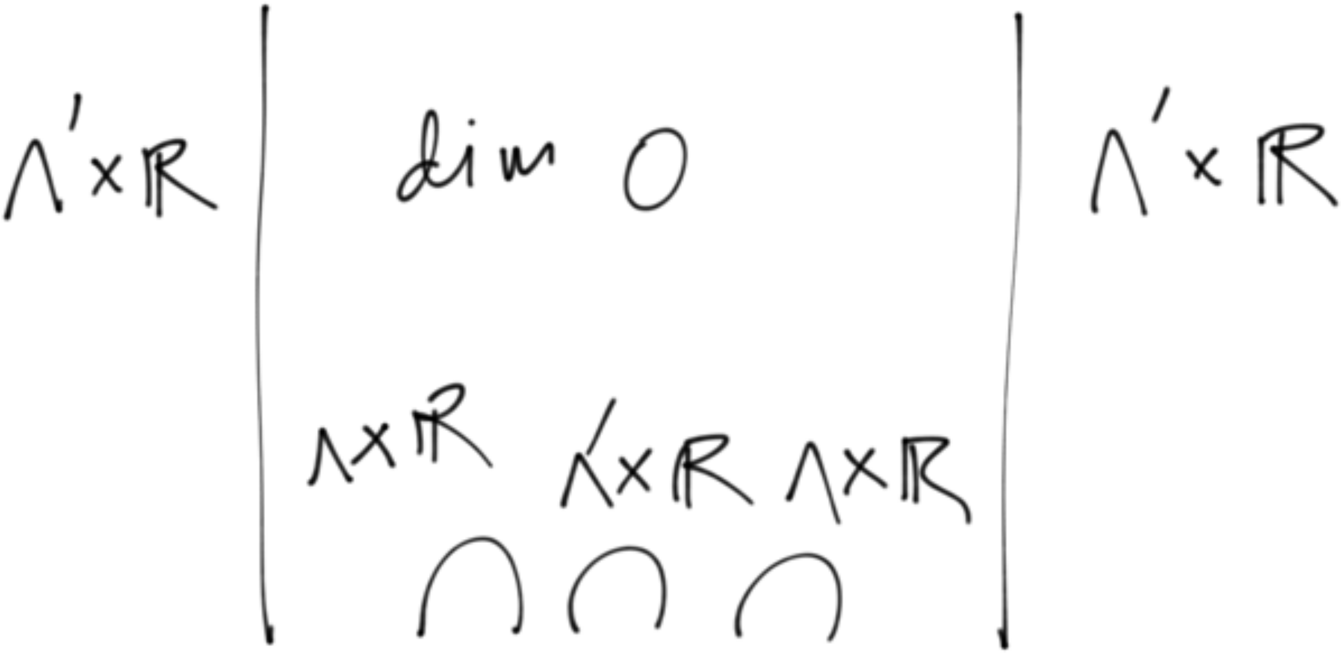
Computing Legendrian DGA after surgery.



Chords (Λ') =

$\Lambda' \to \Lambda \to \Lambda \to \Lambda \to \Lambda \to \Lambda'$

The map



$$\Phi : A_+(\Lambda') \longrightarrow A_-(\Lambda', \Lambda)$$

is a chain iso.

Note. Subcritical handles do not change

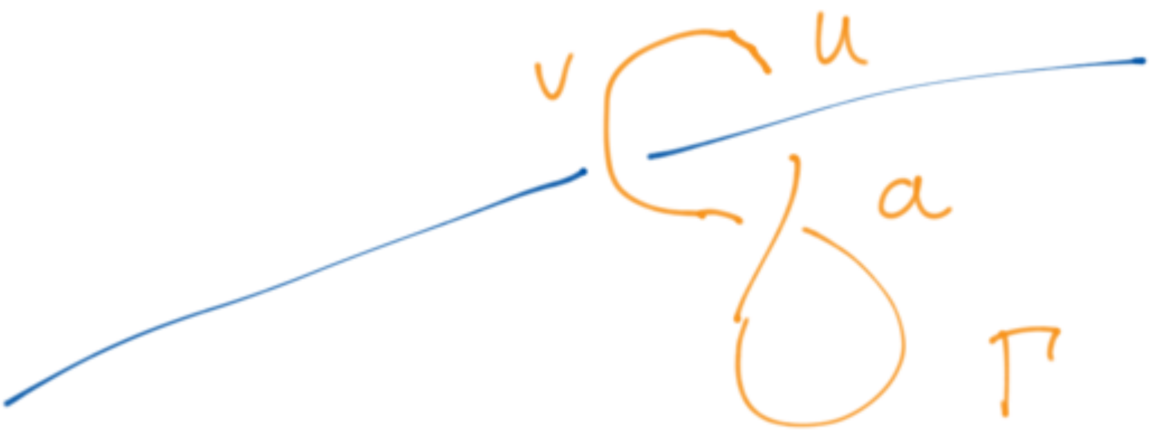
$A(\Lambda)$

Surgery along loose spheres in the complement does not change Λ.

Example

L is non-flexible.

DaAs of co-core spheres



Λ

$\overbrace{v\, c_m \cdots c_1\, u}^{\text{word}}\ ,\ a$

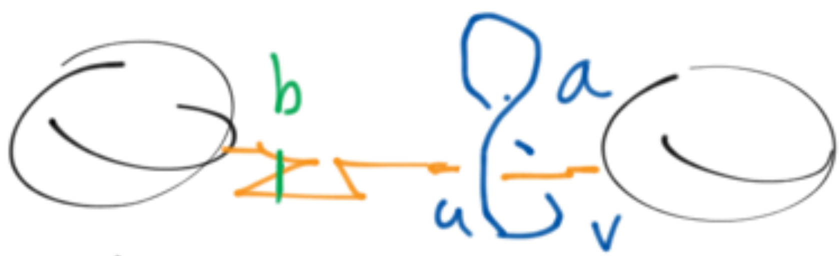
Example

$$\partial(\mathcal{U}^* S^n)$$

$1 \quad a[1] \quad \ldots \quad a[j]$

$$\partial a[j] = \sum_{j=i+k+1} a[i]\, a[k].$$

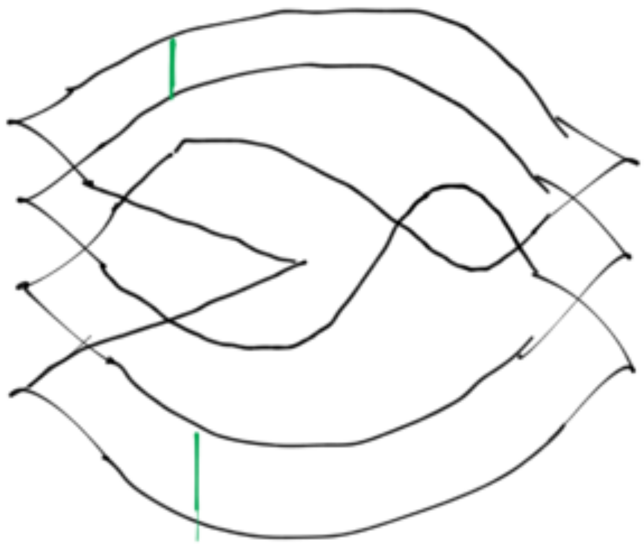
Example Cancelling handle



$d(a - v\, b\, n) = 0 \implies$ unknot dga

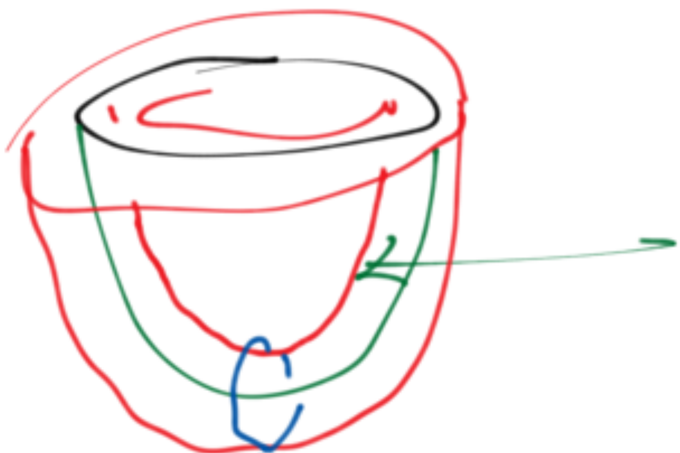
Example

Take



Λ

$$A(\Lambda) \neq A(\text{unknot})$$



Lag disk $D \subset B^4$

S

$$A(S) = WH(D) = 0$$

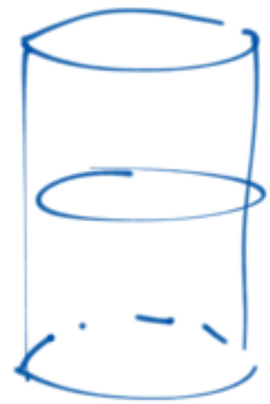
$$A(S, pt) = A(\Lambda)$$

$$\partial b = 1 + w \cdot pt \Rightarrow |w| < 0$$

Knot contact homology and Legendrian surgery

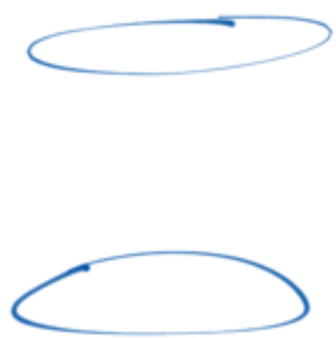
M - smooth manifold

T^*M - symplectic manifold



$$\omega = dq \wedge dp$$

ST^*M - contact manifold

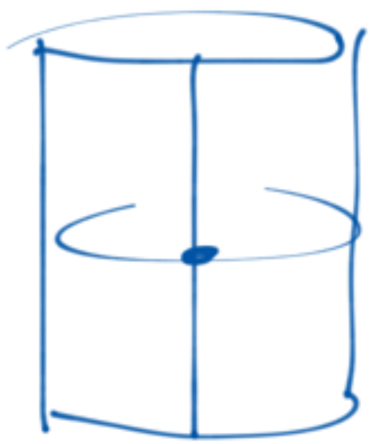


$$\alpha = p\, dq$$

$K \subset M$ - submanifold

$L_K \subset T^*M$ — Lagrangian submanifold

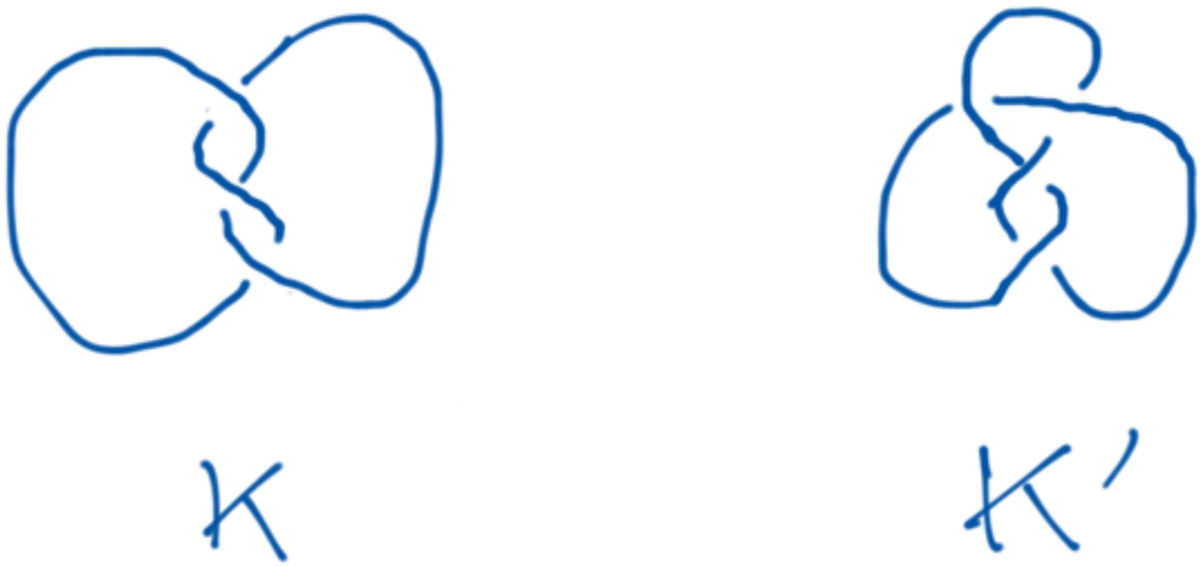
$$L_K = \{(q,p) : q \in K,\ p|_{T_qK} = 0\}$$



$\Lambda_K = L_K \cap ST^*M$

— Legendrian submanifold

$K, K' \subset \mathbb{R}^3$ — smooth knots



Theorem (Shende, E-Shende-Ng)

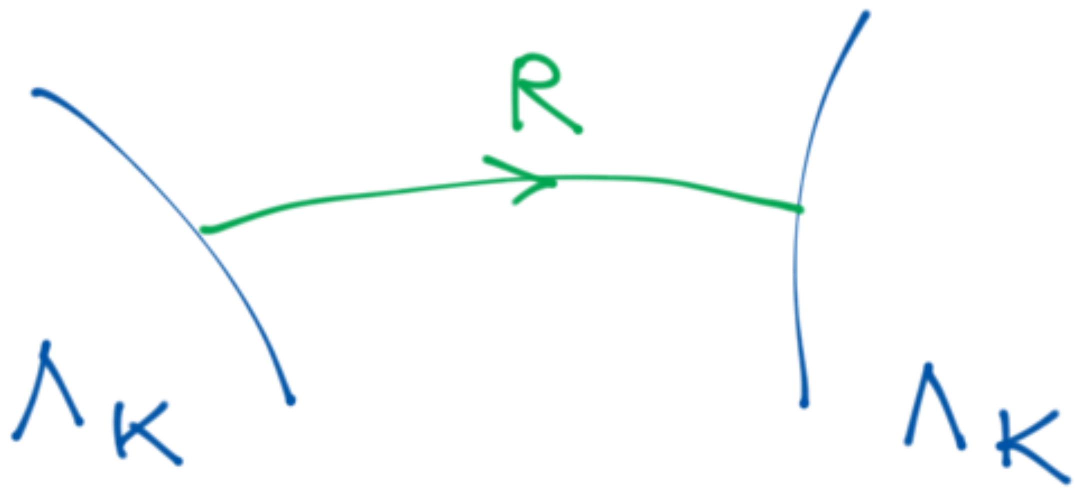
Two knots are smoothly isotopic if and only if their Legendrian conormals are parametrized Legendrian isotopic. In fact the latter are distinguished by a version of knot contact homology with a certain product.

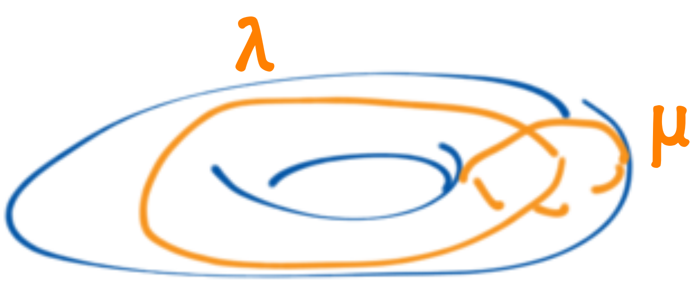
Knot contact homology

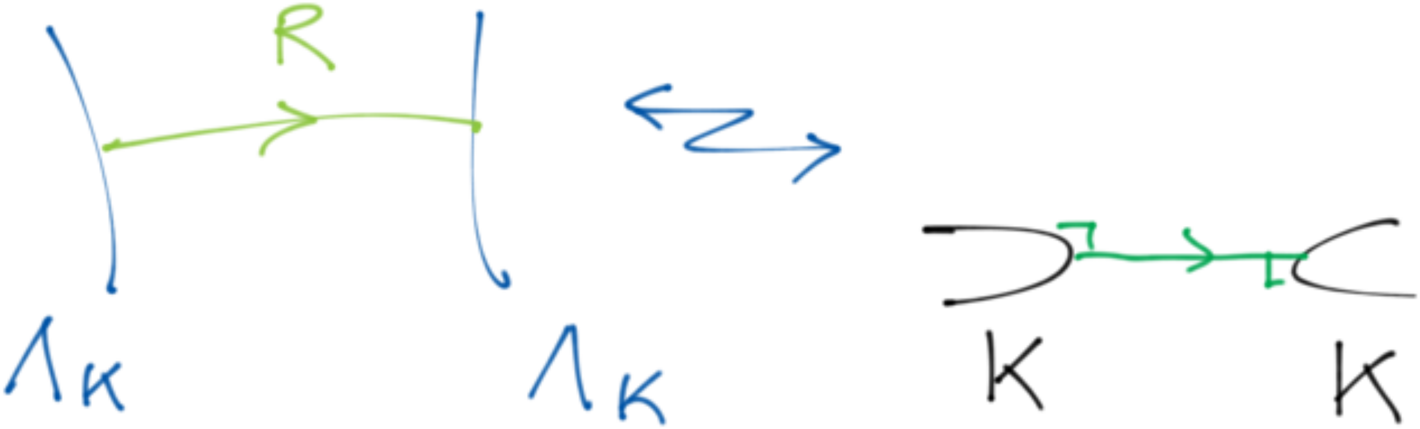
$$\mathcal{A}(\Lambda_K) = \mathbb{Z}\langle \text{Reeb chords}, \lambda^{\pm 1}, \mu^{\pm 1} \rangle$$

$$\alpha = p\,dq \quad , \quad d\alpha(R, \cdot) = 0$$

$$\alpha(R) = 1 .$$

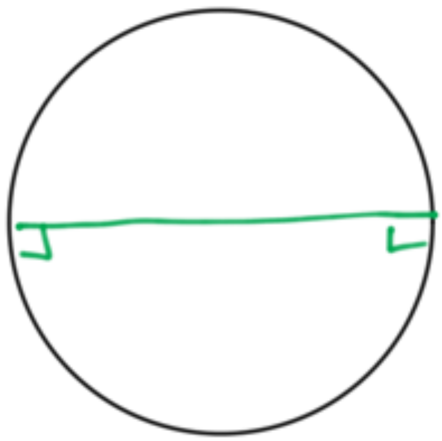






Grading = Morse grading of geodesic

Example, the unknot

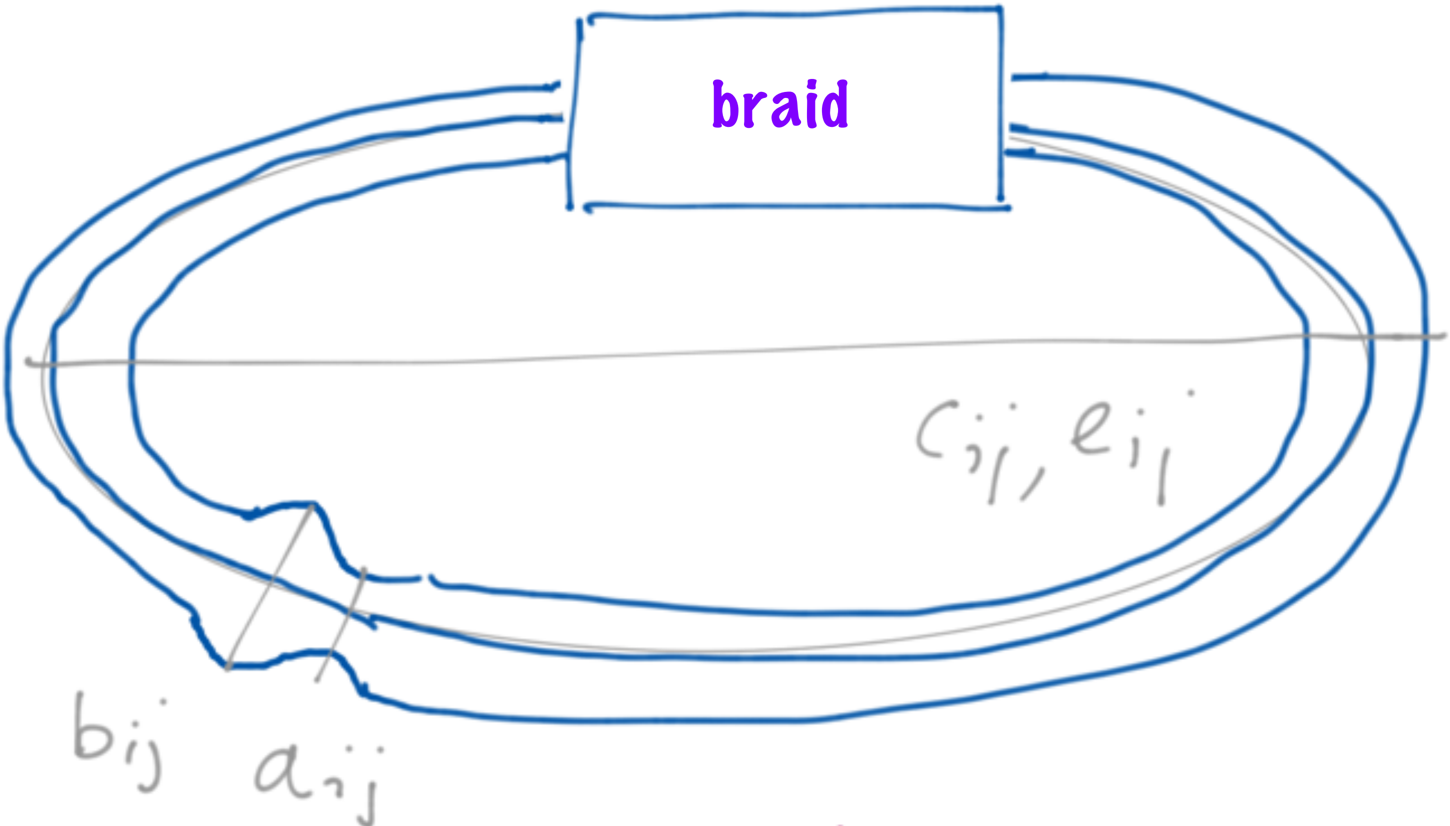


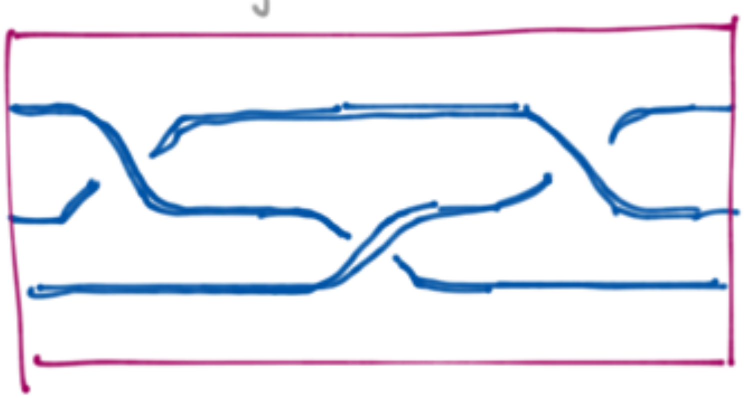
deg 0

$$\mathcal{A}_U = \mathbb{Z}\langle c, e, \lambda^{\pm 1}, \mu^{\pm 1}\rangle$$

deg 1 deg 2 .

Example, any other knot





$$\mathcal{A}_K = \mathbb{Z}\langle \overset{0}{a_{ij}},\ \overset{1}{b_{ij}},\ \overset{1}{c_{ij}},\ \overset{2}{e_{ij}},\ \overset{0}{\lambda^{\pm 1}},\ \overset{0}{\mu^{\pm 1}} \rangle$$

The differential can be computed in an "adiabatic limit" where the holomorphic disks limit to Morse flow trees.

Example, the unknot.

$$\mathcal{A}_u = \mathbb{Z}\langle \overset{2}{e}, \overset{1}{c}, \overset{0}{\lambda^{\pm 1}}, \overset{0}{\mu^{\pm 1}} \rangle$$

$$\partial e = c - c = 0$$

$$\partial c = 1 - \lambda - \mu + \lambda\mu =$$

$$= (1 - x)(1 - \mu).$$

Enhanced knot contact homology: add the conormal of a point.

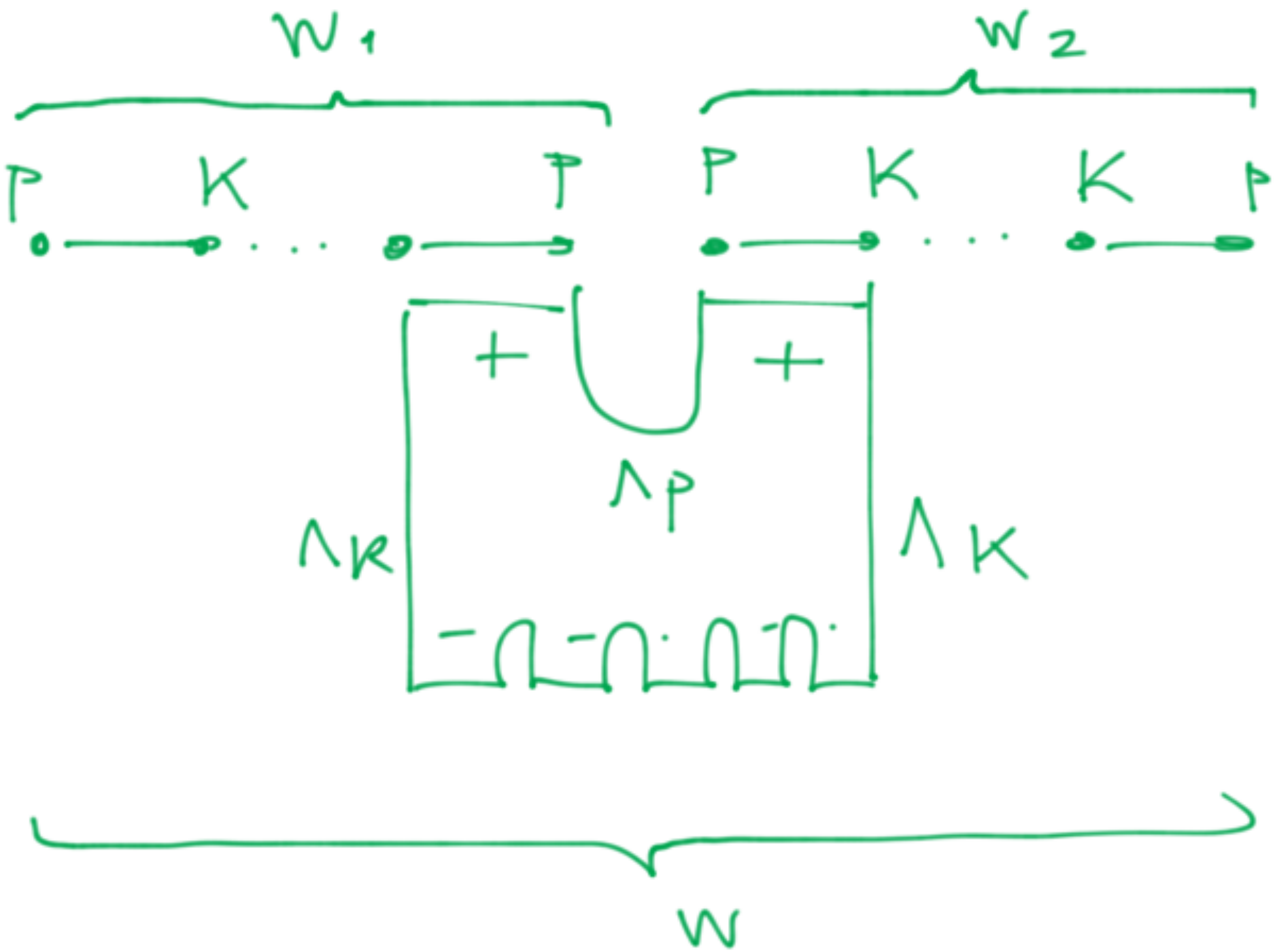
$$\Lambda_p = ST^*_p\,\mathbb{R}^3 \approx S^2.$$

$\mathcal{A}_{pKp}$ is generated by words of chords

The differential is as before but neglects outputs with more than one mixed chord.

Holomorphic disks with two mixed positive punctures induce a product

$$\mathcal{A}_{pKp} \otimes \mathcal{A}_{pKp} \xrightarrow{\mu} \mathcal{A}_{pKp}$$



$$\mu(w_1, w_2) = w.$$

Theorem

The degree 0 part of enhanced knot contact homology with multiplication μ is ring isomorphic to the group ring of the fundamental group of the knot complement, preserving longitude and meridian.

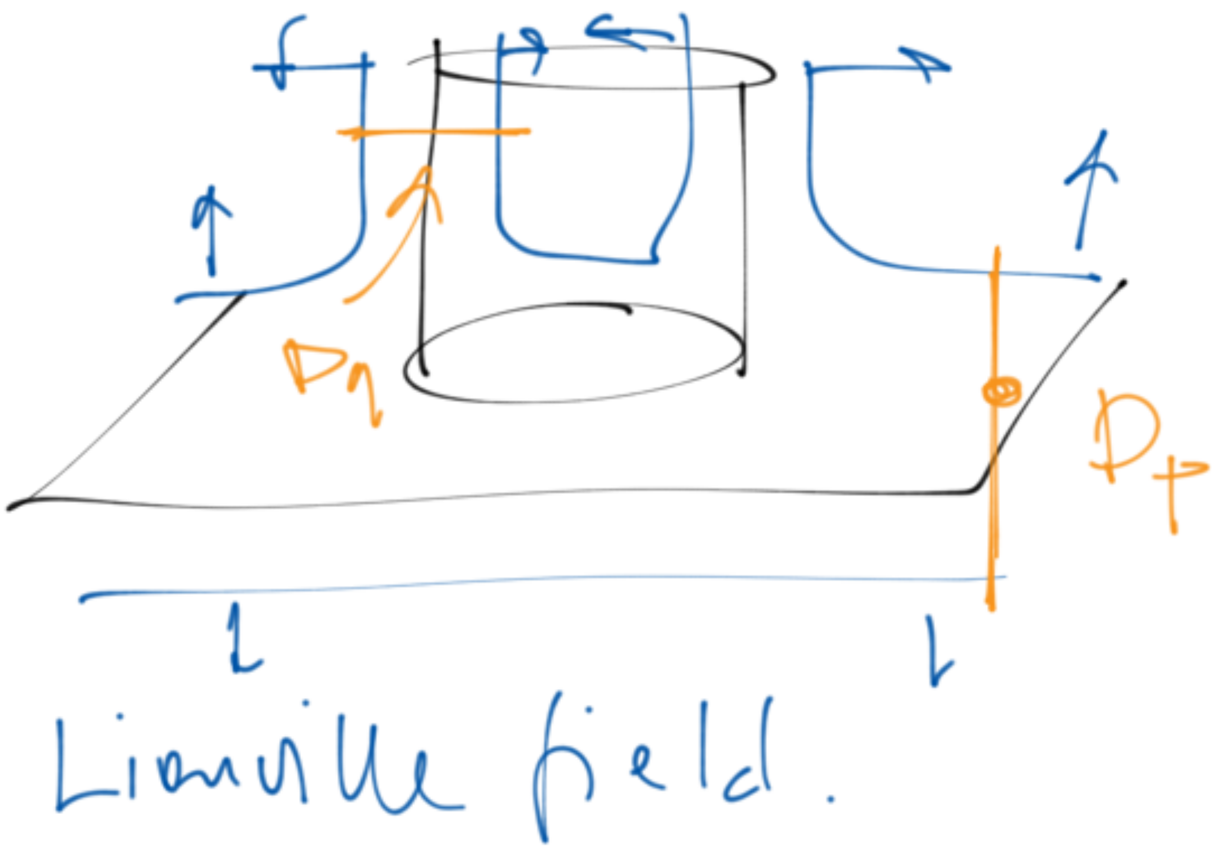
$$(\mathcal{A}^{0}_{pK_p}, \mu) \approx \mathbb{Z}[\pi_1]$$

Since knot groups are left orderable this gives a complete knot invariant by Waldhausen's classical theorem.

Surgery interpretation

Consider the manifold with Lagrange skeleton

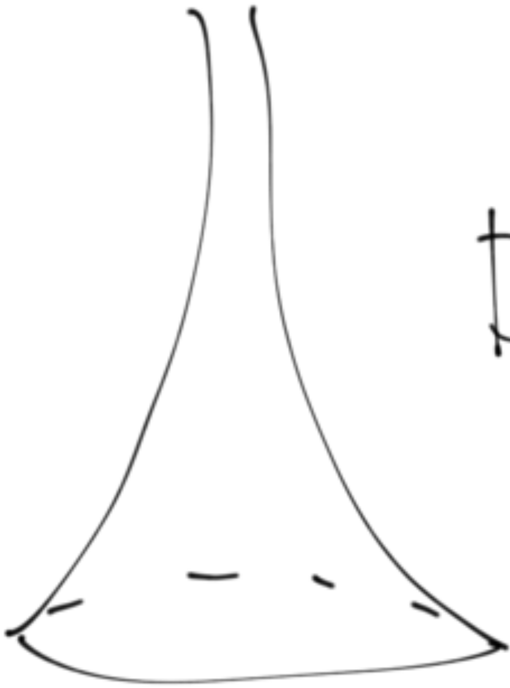
$\mathbb{R}^3 \cup L_K$



We build this space by attaching

$T^*(S^1 \times \mathring{D}^2)$ along Λ_K

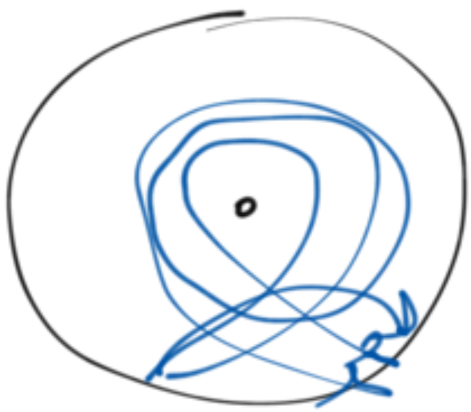
We use the Reeb flow from $ds^2 + g$ where g is the complete hyperbolic metric on $\mathring{D}^2$



D^2.

Consider Reeb-ch

of Dq



etc

Chords = words

$$c_1 \lambda^{k_1} \mu^{l_1} c_2 \lambda^{k_2 l_2} c_3$$

$$\mathcal{A}(\Lambda_K) \approx HW(D_q)$$

$$\mathcal{A}(\Lambda_K, \Lambda_p) = HW(D_p)$$

product  on $HW(D_p)$