Topology I | Humboldt-Universität zu Berlin
C. Wendl / F. Schmäschke | Summer Semester 2017

PRACTICE EXAM (PROBEKLAUSUR)

Instructions

You have three hours in total, though the exam is designed to be doable in significantly less time than that. For reference, you may use any notes or books that you bring with you, but nothing electronic, i.e. no calculators or smartphones.

Answers can be written in German or English, and all answers require justification (within reason) in order to receive full credit. You may use results that were proved in the lectures or on problem sets without reproving them, but state clearly which results you are using. If you would like to use a result that you've found in a book but it was not covered in the class, then you need to explain the proof.

Keep in mind that if you get stuck on one part of a problem, it may sometimes be possible to skip it and do the next part.

***Disclaimer**: Since this practice exam tries to touch upon a relatively broad spectrum of topics, it is intentionally longer than the actual exam will be. The exam will be designed to be doable within two hours—this one is not.*

Problems [100 pts total]

1. Endowing each of the following sets with the subspace topology as a subset of $\mathbb{R}^2$, determine whether or not each is (i) locally compact, (ii) locally connected, (iii) locally path-connected.

   (a) [15 pts] $Y = \mathbb{R} \times (\mathbb{R} \setminus \mathbb{Q})$

   (b) [15 pts] $X = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; x = 0 \text{ or } xy = \cos x \right\}$

2. [10 pts] Given a path-connected space X and an integer $n \geq 1$, consider the space X' obtained by attaching an n-cell to X along a continuous map $f : S^{n-1} \to X$, i.e.

   $$X' := X \cup_f \mathbb{D}^n := (X \sqcup \mathbb{D}^n)/\sim,$$

   where $x \sim f(x)$ for every $x \in \partial \mathbb{D}^n = S^{n-1}$. Let

   $$i : X \to X'$$

   denote the injective map defined by composing the inclusion $X \hookrightarrow X \sqcup \mathbb{D}^n$ with the quotient projection $X \sqcup \mathbb{D}^n \to X'$. Use the Seifert-van Kampen theorem to show that for $n \geq 3$, the induced map

   $$i_* : \pi_1(X) \to \pi_1(X')$$

   is always an isomorphism.

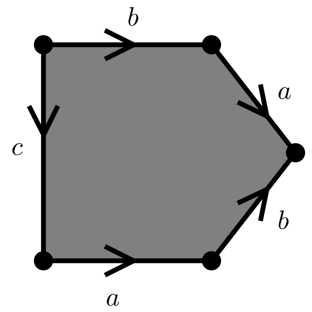
   Comment: This shows that for any path-connected finite cell complex $X = X^0 \cup \ldots \cup X^n$, the fundamental group of X matches that of its 2-skeleton X^2.

3. [10 pts] Prove that for an arbitrary simply connected and locally path-connected space X, every continuous map $f : X \to S^1$ is homotopic to a constant map.
   Hint: Think about covering spaces of S^1. What does the lifting theorem tell you?

4. [10 pts] Suppose M, N and Q are closed, connected and oriented smooth manifolds, all of the same dimension. Prove that for any continuous maps $f : M \to N$ and $g : N \to Q$,

   $$\deg(g \circ f) = \deg(f) \cdot \deg(g).$$

5. The following picture defines a topological space X, with a cell decomposition consisting of one 0-cell (represented by every vertex, all of them identified), three 1-cells (the edges labeled a, b and c, with pairs of edges identified whenever their labels match), and one 2-cell (the interior of the polygon, attached to its edges).



   (a) [10 pts] Is X a manifold (without boundary), a manifold with boundary, or neither?

   (b) [10 pts] Compute the cellular homology groups $H_k^{\mathrm{CW}}(X)$ with respect to the given cell decomposition, for $k = 0, 1, 2$. In particular, show that $H_2^{\mathrm{CW}}(X)$ is trivial and $H_1^{\mathrm{CW}}(X)$ is not.

   (c) [10 pts] Denoting the singular homology of X by $H_*(X)$, compute $H_1(X)$.
   Note: Various theorems we have proved about $H_1(X)$ in lecture or on homework might be useful here, e.g. the fact that it is invariant under homotopy equivalence. The most useful theorem would of course be the one that implies $H_*(X) \cong H_*^{\mathrm{CW}}(X)$, but you may not use this, since we haven't proved it.

   (d) [10 pts] Let $A \subset X$ denote the circle in X represented by edge c in the picture. Use the long exact sequence of the pair (X, A) to prove[1]

$$H_2(X, A) \cong \mathbb{Z}.$$

   You may use without proof the fact that $H_2(X) = 0$, which follows from part (b) and the theorem that $H_2(X) \cong H_2^{\mathrm{CW}}(X)$.
   Hint: It may be helpful to notice that the map $i_* : H_1(A) \to H_1(X)$ induced by the inclusion $i : A \to X$ is trivial. (Why?)

[1] The relative homology groups $H_*(X, A)$ and long exact sequences are topics to be discussed in the lecture on 19.07.2017, so if you're reading this before then, don't panic.