Convergence in the box topology (Problem Set 2, #5)

Given a topological space X and a (potentially infinite or uncountable) set I, the **box topology** on the set X^I of arbitrary functions $f : I \to X$ is generated by all sets of the form

$$\mathcal{U} = \left\{ g \in X^I \;\middle|\; g(\alpha) \in \mathcal{U}_\alpha \text{ for each } \alpha \in I \right\} \tag{1}$$

for arbitrary collections of open subsets $\mathcal{U}_\alpha \subset X$ associated to each $\alpha \in I$. The difference between this and the usual product topology is that in the set $\mathcal{U} \subset X^I$ above, we do not require $\mathcal{U}_\alpha$ to be X for all but finitely many $\alpha \in I$, thus the box topology contains many subsets that are not in the product topology and is thus a *stronger* topology. To see how much stronger, we consider what convergence in the box topology means.

Proposition 1. *A sequence $f_n \in X^I$ converges in the box topology to $f \in X^I$ if and only if there exists a finite subset $J \subset I$ and an index $n_0 \in \mathbb{N}$ such that $f_n|_J : J \to X$ converges pointwise to $f|_J : J \to X$ and, for every $n > n_0$ and $\alpha \in I \setminus J$, $f_n(\alpha)$ lies in every neighborhood of $f(\alpha)$.*

Notice the order of quantifiers for the part of this statement that refers to convergence on $I \setminus J$, and what it means in particular if X is any kind of "nice" space such as $\mathbb{R}$ or $\mathbb{R}^n$, or more generally, anything that is Hausdorff: then $f_n(\alpha)$ can only lie in every neighborhood of $f(\alpha)$ if in fact $f_n(\alpha) = f(\alpha)$. We therefore have:

Corollary 2. *If X is Hausdorff, then a sequence $f_n \in X^I$ converges to $f \in X^I$ in the box topology if and only if there exists a finite subset $J \subset I$ such that f_n converges to f pointwise on J while, outside of J, f_n is* identically equal *to f for all n sufficiently large.*

If for instance $I = \mathbb{R}$, this means that $f_n : \mathbb{R} \to X$ can only converge to $f : \mathbb{R} \to X$ if for all n sufficiently large, $f_n = f$ almost everywhere (in the sense of measure theory). So the box topology isn't just strong—it is so strong that it prevents almost all reasonable sequences from converging. In that sense, speaking informally, it is nearly as strong as the discrete topology (in which only sequences that are eventually constant can converge). This is one reason why the box topology is not considered to be the "natural" topology on X^I.

Proof of Proposition 1. It is easy to see that if I has a finite subset $J \subset I$ satisfying the stated properties for given $f_n, f \in X^I$, then $f_n \to f$ in the box topology, as f_n will belong to every "box neighborhood" of f of the type indicated above (see (1)) for sufficiently large n.

The converse is similarly easy to prove if I is finite (in which case the box topology is the same as the product topology), so let us assume I is infinite. Arguing by contradiction, assume $f_n \to f$ in the box topology but there is no subset $J \subset I$ with the stated properties. This implies that for every finite subset $J \subset I$ and every $n_0 \in \mathbb{N}$, there exists $n > n_0$, $\alpha \in I \setminus J$ and a neighborhood $\mathcal{U}_\alpha \subset X$ of $f(\alpha)$ with $f_n(\alpha) \notin \mathcal{U}_\alpha$.

We claim that there exists a sequence $\alpha_n \in I$, whose elements are all distinct from one another, with neighborhoods $\mathcal{U}_{\alpha_n} \subset X$ of $f(\alpha_n)$ and a subsequence f_{k_n} of f_n such that $f_{k_n}(\alpha_n) \notin \mathcal{U}_{\alpha_n}$ for every $n \in \mathbb{N}$. Indeed, for $n = 1$, we can simply choose any $k_1 \in \mathbb{N}$ and $\alpha_1 \in I$ with a neighborhood $\mathcal{U}_{\alpha_1} \subset X$ of $f(\alpha_1)$ such that $f_{k_1}(\alpha_1) \notin \mathcal{U}_{\alpha_1}$. Now if we assume the desired sequence exists for $n = 1, \ldots, N$, then by the previous paragraph, there exists $k_{N+1} > k_N$ and $\alpha_{N+1} \in I \setminus \{\alpha_1, \ldots, \alpha_N\}$ with a neighborhood $\mathcal{U}_{\alpha_{N+1}} \subset X$ of $f(\alpha_{N+1})$ such that $f_{k_{N+1}}(\alpha_{N+1}) \notin \mathcal{U}_{\alpha_{N+1}}$. This proves the claim by induction.

Now given the sequence above, the set

$$\{ g \in X^I \mid g(\alpha_n) \in \mathcal{U}_{\alpha_n} \text{ for all } n = 1, 2, 3, \ldots \}$$

is an open subset of X^I in the box topology[1] that contains f, but by construction, f_{k_n} does not belong to it for any n, so f_n cannot converge to f. □

[1] A crucial detail here is that since the set $\{\alpha_1, \alpha_2, \ldots\}$ is infinite, the set of functions we are defining is not open in the product topology. Otherwise Proposition 1 would also hold for convergence in the product topology, but for infinite products, it does not.