Topology I
C. Wendl / F. Schmäschke

Humboldt-Universität zu Berlin
Summer Semester 2017

PROBLEM SET 11
Due: 19.07.2017

Instructions

Problems marked with (∗) will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without (∗), you do not need to write up your solutions, but it is highly recommended that you think through them before the next Wednesday lecture.

1. For each of the following, you may use without proof the theorem (to be proved next semester) that the singular homology $H_*(X)$ of any finite cell complex X matches its cellular homology $H_*^{\mathrm{CW}}(X)$.

   (a) (∗) Compute $H_n(\mathbb{RP}^3)$, $H_n(\mathbb{RP}^3;\mathbb{Q})$ and $H_n(\mathbb{RP}^3;\mathbb{Z}_2)$ for each $n = 0, 1, 2, 3$, and prove $\chi(\mathbb{RP}^3) = 0$.
   Hint: To find a nice cell decomposition of $\mathbb{RP}^3$, start with a $\mathbb{Z}_2$-invariant cell decomposition of S^3.
   Remark: You should find that $H_2(\mathbb{RP}^3) = H_2(\mathbb{RP}^3;\mathbb{Q}) = 0$ but $H_2(\mathbb{RP}^3;\mathbb{Z}_2) \neq 0$. This has to do with the fact that $\mathbb{RP}^3$ contains a submanifold homeomorphic to $\mathbb{RP}^2$, which is not orientable.

   (b) (∗) Let Σ_g denote the closed orientable surface of genus $g \geqslant 0$ and, for $k \geqslant 0$, let $\Sigma_{g,k} := \Sigma_g \backslash \{k \text{ points}\}$. Show that $\Sigma_{g,k}$ has Euler characteristic $\chi(\Sigma_{g,k}) = 2 - 2g - k$.
   Hint: You only need a cell decomposition of something homotopy equivalent to $\Sigma_{g,k}$. (Why?)

2. (∗) Show that for the 1-point space $\{\mathrm{pt}\}$ and any coefficient group G, singular homology satisfies[1]

$$H_n(\{\mathrm{pt}\};G) \cong \begin{cases} G & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases}$$

   Hint: For each integer $n \geqslant 0$, there is exactly one singular n-simplex $\Delta^n \to \{\mathrm{pt}\}$, so the chain groups $C_n(\{\mathrm{pt}\}) \otimes G$ are all naturally isomorphic to G. What is $\partial : C_n(\{\mathrm{pt}\}) \otimes G \to C_{n-1}(\{\mathrm{pt}\}) \otimes G$?

3. In this problem, we prove that $H_1(X)$ for a path-connected space X is isomorphic to the abelianization of its fundamental group. Fix a base point $x_0 \in X$ and abbreviate $\pi_1(X) := \pi_1(X, x_0)$, so elements of $\pi_1(X)$ are represented by paths $\gamma : I \to X$ with $\gamma(0) = \gamma(1) = x_0$. Identifying the standard 1-simplex

$$\Delta^1 := \left\{ (t_0, t_1) \in \mathbb{R}^2 \;\middle|\; t_0 + t_1 = 1,\ t_0, t_1 \geqslant 0 \right\}$$

   with $I := [0,1]$ via the homeomorphism $\Delta^1 \to I : (t_0, t_1) \mapsto t_0$, every path $\gamma : I \to X$ corresponds to a singular 1-simplex $\Delta^1 \to X$, which we shall denote by $\tilde{h}(\gamma)$ and regard as an element of the singular 1-chain group $C_1(X)$. Show that $\tilde{h}$ has each of the following properties:

   (a) If $\gamma : I \to X$ satisfies $\gamma(0) = \gamma(1)$, then $\partial \tilde{h}(\gamma) = 0$.

   (b) For any constant path $e : I \to X$, $\tilde{h}(e) = \partial\langle\sigma\rangle$ for some singular 2-simplex $\sigma : \Delta^2 \to X$.

   (c) (∗) For any paths $\alpha, \beta : I \to X$ with $\alpha(1) = \beta(0)$, the concatenated path $\alpha \cdot \beta : I \to X$ satisfies $\tilde{h}(\alpha) + \tilde{h}(\beta) - \tilde{h}(\alpha \cdot \beta) = \partial\langle\sigma\rangle$ for some singular 2-simplex $\sigma : \Delta^2 \to X$.
   Hint: Imagine a triangle whose three edges are mapped to X via the paths α, β and $\alpha \cdot \beta$. Can you extend this map continuously over the rest of the triangle?

   (d) If $\alpha, \beta : I \to X$ are two paths that are homotopic with fixed end points, then $\tilde{h}(\alpha) - \tilde{h}(\beta) = \partial f$ for some singular 2-chain $f \in C_2(X)$.
   Hint: If you draw a square representing a homotopy between α and β, you can decompose this square into two triangles.

   (e) Applying $\tilde{h}$ to paths that begin and end at the base point x_0, deduce that $\tilde{h}$ determines a group homomorphism $h : \pi_1(X) \to H_1(X) : [\gamma] \mapsto [\tilde{h}(\gamma)]$.

[1] This is one of the Eilenberg-Steenrod axioms for homology theories, which we will discuss next semester. It is called the *dimension axiom.*

We call $h : \pi_1(X) \to H_1(X)$ the **Hurewicz homomorphism**. Notice that since $H_1(X)$ is abelian, $\ker h$ automatically contains the commutator subgroup $[\pi_1(X), \pi_1(X)] \subset \pi(X)$ (see Problem Set 6 #2), thus h descends to a homomorphism on the abelianization of $\pi_1(X)$,

$$\Phi : \pi_1(X) \Big/ [\pi_1(X), \pi_1(X)] \to H_1(X).$$

We will now show that this is an isomorphism by writing down its inverse. For each point $p \in X$, choose arbitrarily a path $\omega_p : I \to X$ from x_0 to p, and choose ω_{x_0} in particular to be the constant path. Regarding singular 1-simplices $\sigma : \Delta^1 \to X$ as paths $\sigma : I \to X$ under the usual identification of I with Δ^1, we can then associate to every singular 1-simplex $\sigma \in C_1(X)$ a concatenated path

$$\widetilde{\Psi}(\sigma) := \omega_{\sigma(0)} \cdot \sigma \cdot \omega_{\sigma(1)}^{-1} : I \to X$$

which begins and ends at the base point x_0, hence $\widetilde{\Psi}(\sigma)$ represents an element of $\pi_1(X)$. Let $\Psi(\sigma)$ denote the equivalence class represented by $\widetilde{\Psi}(\sigma)$ in the abelianization $\pi_1(X)/[\pi_1(X), \pi_1(X)]$, and observe that by Problem Set 10 #1(f), this uniquely determines a homomorphism

$$\Psi : C_1(X) \to \pi_1(X)/[\pi_1(X), \pi_1(X)].$$

(f) $(*)$ Show that $\Psi(\partial\langle\sigma\rangle) = 0$ for every singular 2-simplex $\sigma : \Delta^2 \to X$, and deduce that Ψ descends to a homomorphism $\Psi : H_1(X) \to \pi_1(X)/[\pi_1(X), \pi_1(X)]$.

(g) Show that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are both the identity map.

(h) For a closed surface Σ_g of genus $g \geqslant 1$, find an example of a nontrivial element in the kernel of the Hurewicz homomorphism $\pi_1(\Sigma_g) \to H_1(\Sigma_g)$. *Hint: See Problem Set 7 #3.*

4. Suppose (C_*, ∂) is a chain complex such that C_n is a free abelian group for every $n \in \mathbb{Z}$, and $\mathbb{K}$ is a field with characteristic zero. The goal is to prove that the natural maps

$$H_n(C_*, \partial) \otimes \mathbb{K} \to H_n(C_* \otimes \mathbb{K}, \partial) : ([x] \otimes k) \mapsto [x \otimes k] \tag{1}$$

are isomorphisms for every n. It follows via Problem 1 from last week that for any space X whose singular homology $H_n(X)$ is finitely generated, $\operatorname{rank} H_n(X) = \dim_{\mathbb{K}} H_n(X; \mathbb{K})$, thus one can compute $\operatorname{rank} H_n(X)$ by looking at e.g. the rational vector space $H_n(X; \mathbb{Q})$ and using linear algebra.

(a) Show by example that $H_n(C_*, \partial) \otimes \mathbb{K}$ and $H_n(C_* \otimes \mathbb{K}, \partial)$ need not be isomorphic when $\mathbb{K} = \mathbb{Z}_2$. *Hint: See for instance Problem Set 10 #2(b).*

(b) Show that if $\mathbb{K}$ has characteristic zero and G is any free abelian group, then $G \to G \otimes \mathbb{K} : g \mapsto g \otimes 1$ defines an injective group homomorphism.

(c) Let us distinguish the boundary maps on (C_*, ∂) and $(C_* \otimes \mathbb{K}, \partial)$ by writing $\partial_n : C_n \to C_{n+1}$ for the former and $\partial_n^{\mathbb{K}} : C_n \otimes \mathbb{K} \to C_{n-1} \otimes \mathbb{K}$ for the latter. Using the injective homomorphism in part (b), we can regard C_n as a subgroup of $C_n \otimes \mathbb{K}$. Show that

$$\begin{aligned} \ker \partial_n^{\mathbb{Q}} &= \left\{ x \in C_n \otimes \mathbb{Q} \;\middle|\; mx \in \ker \partial_n \subset C_n \text{ for some } m \in \mathbb{N} \right\} \\ \operatorname{im} \partial_{n+1}^{\mathbb{Q}} &= \left\{ x \in C_n \otimes \mathbb{Q} \;\middle|\; mx \in \operatorname{im} \partial_{n+1} \subset C_n \text{ for some } m \in \mathbb{N} \right\}. \end{aligned}$$

(d) Deduce that there are natural isomorphisms $\ker \partial_n \otimes \mathbb{Q} \to \ker \partial_n^{\mathbb{Q}}$ and $\operatorname{im} \partial_{n+1} \otimes \mathbb{Q} \to \operatorname{im} \partial_{n+1}^{\mathbb{Q}}$. *Hint: The maps are trivial to define, but you need part (c) in order to write down their inverses.*

(e) Show that for any abelian groups $H \subset G$ and K, there is a natural isomorphism $(G/H) \otimes K \to (G \otimes K)/i(H \otimes K)$, where $i : H \otimes K \to G \otimes K$ is naturally induced by the inclusion $H \hookrightarrow G$.

(f) Deduce that (1) is an isomorphism in the case $\mathbb{K} = \mathbb{Q}$.

One can use linear algebra to extend this result to any field $\mathbb{K}$ that contains $\mathbb{Q}$, i.e. any field of characteristic zero. This starts with the observation that $\mathbb{Q} \otimes \mathbb{K}$ is naturally isomorphic to $\mathbb{K}$, so one can view the complex $(C_* \otimes \mathbb{K}, \partial)$ as the tensor product (in the sense of rational vector spaces) of $\mathbb{K}$ with $(C_* \otimes \mathbb{Q}, \partial)$, and then repeat the above steps in a vector space context. Alternatively, the general result can be viewed as a corollary of the universal coefficient theorem, *which we'll discuss next semester.*