PROBLEM SET 3
Due: 10.05.2017

Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Wednesday lecture.

Problems

1. Recall that $[0,1]^{\mathbb{R}}$ denotes the set of all functions $f : \mathbb{R} \to [0,1]$, with the topology of pointwise convergence. Tychonoff's theorem implies that $[0,1]^{\mathbb{R}}$ is compact, but one can show that it is not first countable, so it need not be sequentially compact.

   (a) For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, let $x_{(n)} \in \{0, \ldots, 9\}$ denote the nth digit to the right of the decimal point in the decimal expansion of x. Now define a sequence $f_n \in [0,1]^{\mathbb{R}}$ by setting $f_n(x) = \frac{x_{(n)}}{10}$. Show that for any subsequence f_{k_n} of f_n, there exists $x \in \mathbb{R}$ such that $f_{k_n}(x)$ does not converge, hence f_n has no pointwise convergent subsequence.

   (b) $(*)$ The compactness of $[0,1]^{\mathbb{R}}$ does imply that every sequence has a convergent *subnet*, or equivalently, a cluster point. Use this to deduce that for every sequence $f_n \in [0,1]^{\mathbb{R}}$, there exists $f \in [0,1]^{\mathbb{R}}$ such that for any given finite subset $X \subset \mathbb{R}$, some subsequence of f_n converges to f at all points in X. *Attention: The choice of subsequence can depend on the choice of subset X!*

   Challenge: Find a direct proof of the statement in part (b), without passing through Tychonoff's theorem. (I do not know how to do this, and I suspect that it's approximately as difficult as actually proving Tychonoff's theorem—in any case, it very likely requires the axiom of choice.)

2. Consider the space $X = \left\{ f \in [0,1]^{\mathbb{R}} \;\middle|\; f(x) \neq 0 \text{ for at most countably many points } x \in \mathbb{R} \right\}$, with the subspace topology that it inherits from $[0,1]^{\mathbb{R}}$.

   (a) Show that X is sequentially compact.
   Hint: For any sequence $f_n \in X$, the set $\bigcup_{n\in\mathbb{N}} \{x \in \mathbb{R} \mid f_n(x) \neq 0\}$ is also countable.

   (b) For each $x \in \mathbb{R}$, define $\mathcal{U}_x = \{f \in X \mid -1 < f(x) < 1\}$.[1] Show that the collection $\{U_x \subset X \mid x \in \mathbb{R}\}$ forms an open cover of X that has no finite subcover, hence X is not compact.

3. There is a cheap trick to view any topological space as a compact space with a single point removed. For a space X with topology $\mathcal{T}$, let $\{\infty\}$ denote a set consisting of one element that is not in X, and define the *one point compactification* of X as the set $X^* = X \cup \{\infty\}$ with topology $\mathcal{T}^*$ consisting of all subsets in $\mathcal{T}$ plus all subsets of the form $(X \setminus K) \cup \{\infty\} \subset X^*$ where $K \subset X$ is closed and compact.

   (a) Verify that $\mathcal{T}^*$ is a topology and that X^* is always compact.

   (b) Show that if X is first countable,[2] then a sequence in $X \subset X^*$ converges to $\infty \in X^*$ if and only if it has no convergent subsequence in X.

   (c) Show that X^* is Hausdorff if and only if X is both Hausdorff and locally compact.

   (d) Show that for $X = \mathbb{R}$, X^* is homeomorphic to S^1. *(More generally, one can use stereographic projection to show that the one point compactification of $\mathbb{R}^n$ is homeomorphic to S^n.)*

   (e) Show that if X is already compact, then X^* is homeomorphic to the disjoint union $X \sqcup \{\infty\}$.

[1] This is a corrected version of the problem sheet. In the version handed out in class, the definition of $\mathcal{U}_x$ was misstated.
[2] The original version of this problem sheet did not mention the first countability condition in Problem 3(b), but it is rather important.

4. For each of the following spaces, determine whether it is (i) Hausdorff, (ii) locally compact, (iii) connected, (iv) locally path-connected.[3]

   (a) The irrational numbers $\mathbb{R} \setminus \mathbb{Q}$
   (b) $\{0\} \cup \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$
   (c) The quotient group $\mathbb{R}/\mathbb{Q}$, i.e. the set of equivalence classes of real numbers where $x \sim y$ if and only if $x - y \in \mathbb{Q}$
   (d) $(*)$ The one point compactification of $\mathbb{Q}$ (cf. Problem 3)

5. Suppose X and Y are topological spaces, $x \in X$, $K \subset Y$ is compact, and $\mathcal{U} \subset X \times Y$ is an open subset such that $\{x\} \times K \subset \mathcal{U}$. Prove that $\mathcal{V} \times K \subset \mathcal{U}$ for some neighborhood $\mathcal{V} \subset X$ of x.

6. The main goal of this problem is to prove the following important lemma.

   ***Lemma**: Suppose X is a locally compact Hausdorff space with subsets $K \subset \mathcal{U} \subset X$ such that K is compact and $\mathcal{U}$ is open. Then there exists an open subset $\mathcal{V} \subset X$ with compact closure $\bar{\mathcal{V}}$ such that $K \subset \mathcal{V} \subset \bar{\mathcal{V}} \subset \mathcal{U}$.*

   (a) Show that the lemma becomes false if X is not locally compact.
   Hint: Every finite subset is compact.
   (b) In any topological space X, the *boundary* of a subset $K \subset X$ is defined by

   $$\partial K = \{y \in X \mid \text{every neighborhood of } y \text{ intersects both } K \text{ and } X \setminus K\}.$$

   Show that if K is compact, then ∂K is a closed subset of K (and is therefore compact).
   (c) $(*)$ Show that if X is locally compact and Hausdorff, then for every point $x \in X$, every compact neighborhood $K \subset X$ of x contains an open neighborhood $\mathcal{V} \subset X$ of x which is disjoint from some neighborhood of ∂K.[4] *Hint: The argument you need here is similar to our proof in lecture that compact subsets of Hausdorff spaces are always closed.*
   (d) Prove the lemma in the case where K is a one point subset.
   (e) Use a finite covering argument to complete the proof of the lemma.

7. For any topological spaces X and Y, the set $C(X, Y)$ of continuous maps from X to Y can be assigned a topology with subbase consisting of all sets of the form

   $$\mathcal{U}_{K,V} = \{f \in C(X, Y) \mid f(K) \subset V\}$$

   for $K \subset X$ compact and $V \subset X$ open. The resulting topology on $C(X, Y)$ is called the *compact-open topology.*

   (a) Show that if Y is a metric space, then a sequence in $C(X, Y)$ converges in the compact-open topology if and only if it converges uniformly on all compact subsets.
   (b) $(*)$ In algebraic topology, two continuous maps $f, g : X \to Y$ are called *homotopic* if there exists a continuous map $H : [0, 1] \times X \to Y$ with $H(0, \cdot) = f$ and $H(1, \cdot) = g$. Show that if f and g are homotopic, then they belong to the same path component of $C(X, Y)$ in the compact-open topology. *Hint: You might find #5 helpful, and also #3(b) from Problem Set 2.*
   (c) $(*)$ Prove that if X is locally compact and Hausdorff, then the converse of the statement in part (b) holds: any two maps in the same path component of $C(X, Y)$ are homotopic.
   Hint: The lemma of #6 is needed here, or at least the case of it where K is a one point subset.
   (d) Show that for any three topological spaces X, Y and Z such that Y satisfies the conclusion of the lemma in #6 (so in particular if Y is locally compact and Hausdorff), the natural map

   $$C(X, Y) \times C(Y, Z) \to C(X, Z) : (f, g) \mapsto g \circ f$$

   is continuous.

[3] You should always assume unless otherwise specified that $\mathbb{R}$ is endowed with its standard topology, and all spaces derived from it as subsets/products/quotients etc. carry the natural subspace/product/quotient topology.

[4] In case I forgot to say this in lecture, for any subset $A \subset X$ of a space X, a set $B \subset X$ is called a *neighborhood of A* if it contains an open set that contains A.