PROBLEM SET 8
Due: 14.06.2017

Organizational note

For the next two weeks after this, the usual weekly problem sets will be replaced by a take-home midterm, which will be distributed on June 14 and due June 28.

Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Wednesday lecture.

1. This problem involves the following four notions for continuous maps $f : X \to Y$:

   - f is a *covering map* if every point $y \in Y$ has a neighborhood $\mathcal{U} \subset Y$ that is *evenly covered*: the latter means that $f^{-1}(\mathcal{U}) = \bigcup_{\alpha \in I} \mathcal{V}_\alpha$ for some collection $\{\mathcal{V}_\alpha \subset X\}_{\alpha \in I}$ of disjoint sets such that $f|_{\mathcal{V}_\alpha} : \mathcal{V}_\alpha \to \mathcal{U}$ is a homeomorphism for each $\alpha \in I$.[1]
   - f is a *local homeomorphism* if for every $x \in X$, there exist neighborhoods $\mathcal{V} \subset X$ of x and $\mathcal{U} \subset Y$ of $f(x)$ such that $f|_{\mathcal{V}} : \mathcal{V} \to \mathcal{U}$ is a homoemorphism.
   - f is *open* if for every open subset $\mathcal{U} \subset X$, $f(\mathcal{U}) \subset Y$ is also open.
   - f is *proper*[2] if for every compact subset $K \subset Y$, $f^{-1}(K) \subset X$ is also compact.

   Prove each statement:

   (a) If $p : \widetilde{X} \to X$ is a covering map and $\mathcal{U} \subset X$ is evenly covered, then every subset of $\mathcal{U}$ is also evenly covered.

   (b) For any covering map $p : \widetilde{X} \to X$ and subspace $A \subset X$, the restriction $p|_{p^{-1}(A)} : p^{-1}(A) \to A$ is also a covering map.

   (c) For any covering map $p : \widetilde{X} \to X$, $f^{-1}(x)$ is a discrete subset of $\widetilde{X}$ for every $x \in X$.[3]

   (d) Every covering map is a local homeomorphism.

   (e) Every local homeomorphism is an open map.

   (f) $(*)$ The map $p : (0, 3\pi) \to S^1 : \theta \mapsto e^{i\theta}$ is a local homeomorphism but not a covering map. (Show this using the definitions directly, not using any theorems proved in lecture.)

   (g) For any covering map $p : \widetilde{X} \to X$, if $\widetilde{X}$ is compact, then X is also compact and $p^{-1}(x)$ is finite for all $x \in X$.

   (h) $(*)$ The converse of part (g).[4]
   Hint: Given an open cover $\widetilde{X} = \bigcup_\alpha \mathcal{U}_\alpha$, it suffices to find a finite cover of $\widetilde{X}$ by open sets such that each is contained in some $\mathcal{U}_\alpha$. (Why?) Start by showing that X can be covered by a finite collection of open neighborhoods which are evenly covered and small enough so that their (finitely many!) lifts to $\widetilde{X}$ are each contained in some $\mathcal{U}_\alpha$

   (i) If $f : X \to Y$ is a local homoemorphism and is proper, then $f^{-1}(y)$ is finite for every $y \in Y$.

   (j) A covering map $p : \widetilde{X} \to X$ is proper if and only if $p^{-1}(x)$ is finite for every $x \in X$.

[1] The notion of being *evenly covered* makes sense for arbitrary subsets $\mathcal{U} \subset X$, i.e. $\mathcal{U}$ need not be open. When we say that $f|_{\mathcal{V}_\alpha} : \mathcal{V}_\alpha \to \mathcal{U}$ is a homeomorphism, we mean with respect to the subspace topologies on $\mathcal{V}_\alpha \subset \widetilde{X}$ and $\mathcal{U} \subset X$.

[2] The German for "proper map" is "eigentliche Abbildung".

[3] We say that a subset A in a space X is *discrete* if the subspace topology induced by X on A is the same as the discrete topology.

[4] This is a corrected version of the problem sheet; the original version had a typo in #1(h) referring to "the converse of part (f)" instead of part (g).

2. (∗) Prove that $f : \mathbb{C} \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ defined by $f(z) = e^z$ is a covering map.
Advice: This map is easiest to understand using Cartesian coordinates on the domain and polar coordinates on the target, i.e. $f(x+iy) = e^x(\cos y + i \sin y)$. *Since* f *is continuously differentiable, you could for instance use the inverse function theorem from analysis to show that it is a local homeomorphism.*

3. Assume $p : \widetilde{X} \to X$ is a covering map and X is path-connected.

   (a) (∗) Use the lifting theorem to show that for any two points $x, y \in X$, lifting paths from x to y associates to each such path γ a bijection $\rho_\gamma : p^{-1}(x) \to p^{-1}(y)$, which depends only on the homotopy class of γ (with fixed end points).

   (b) Writing $I := p^{-1}(x)$ and applying part (a) in the case $x = y$ gives a map

   $$\rho : \pi_1(X, x) \to S(I) : [\gamma] \mapsto \rho_\gamma$$

   where $S(I)$ is the group of all bijections $I \to I$. Show that this map is a group homomorphism.

   (c) (∗) Write down the homomorphism $\rho : \pi_1(X, x) \to S(I)$ explicitly for the cover of $\mathbb{C}^*$ in Problem 2, with base point $1 \in \mathbb{C}^*$.

4. Convince yourself that the maps depicted in the figure below are covers, and determine their deck transformation groups. Which ones are regular?