Topology I
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Humboldt-Universität zu Berlin
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PROBLEM SET 9
Due: 5.07.2017

Instructions

Problems marked with (∗) will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without (∗), you do not need to write up your solutions, but it is highly recommended that you think through them before the next Wednesday lecture.

1. The goal of this problem is to prove the following statement.

   ***Proposition**: If M is a topological n-manifold, then it is not also a topological m-manifold for $m \neq n$.*

   We need to show in particular that if we are given two neighborhoods $\mathcal{U}, \mathcal{U}' \subset M$ of a point $p \in M$ and charts $\varphi : \mathcal{U} \to \Omega$ and $\psi : \mathcal{U}' \to \Omega'$ where $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^m$ are both open subsets, then $n = m$. By composing both charts with translations in Euclidean space, we can assume without loss of generality that both map p to the origin in $\mathbb{R}^n$ or $\mathbb{R}^m$ respectively. The transition map $\psi \circ \varphi^{-1}$ then takes some neighborhood of $0 \in \mathbb{R}^n$ homeomorphically to some neighborhood of $0 \in \mathbb{R}^m$.

   (a) Prove that if $\psi \circ \varphi^{-1}$ is continuously differentiable, then $m = n$. *(This proves the weaker statement that a* smooth *n-manifold cannot also be a* smooth *m-manifold for $m \neq n$.)*

   (b) Suppose $\delta > 0$ and $\epsilon > 0$ are small enough so that the balls $B^n_\delta(0) \subset \mathbb{R}^n$ and $B^m_\epsilon(0) \subset \mathbb{R}^m$ about the origin are contained in the domains of $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ respectively, and $\psi \circ \varphi^{-1}(B^n_\delta(0)) \subset B^m_\epsilon(0)$. Now show that the map

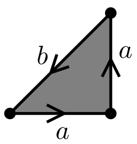
   $$f : S^{n-1} \to B^n_\delta(0) \setminus \{0\} : \mathbf{v} \mapsto \frac{\delta}{2}\mathbf{v}$$

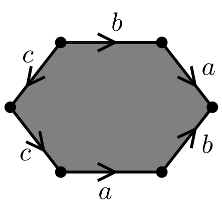
   is not homotopic to a constant among continuous maps $S^{n-1} \to B^n_\delta(0) \setminus \{0\}$.
   Hint: Reduce this to a statement about maps $S^{n-1} \to S^{n-1}$ and consider their degrees.

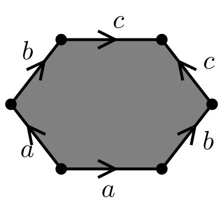
   (c) Show that if $m > n$, then the map $\psi \circ \varphi^{-1} \circ f : S^{n-1} \to B^m_\epsilon(0) \setminus \{0\}$ is homotopic (among maps $S^{n-1} \to B^m_\epsilon(0) \setminus \{0\}$) to a constant. Then compose it with $\varphi \circ \psi^{-1}$ to derive a contradiction to part (b), proving $m \leq n$.

   To conclude the proof, one can use the same argument replacing $\psi \circ \varphi^{-1}$ by $\varphi \circ \psi^{-1}$ in order to derive a similar contradiction if $n > m$.

2. Each of the following pictures defines a topological space by identifying all vertices of the polygon to a single point and identifying any pairs of edges with matching letters via a homeomorphism that matches the arrows. Determine whether each space is (i) a 2-manifold (without boundary), (ii) a 2-manifold with boundary, or (iii) neither.

   (a)
   

   (b)
   

   (c)
   

   (d) For any cases in parts (a)–(c) where the space is a manifold or a manifold with boundary, describe it in terms of familiar surfaces such as the disk, the Möbius band, the sphere, the torus, the projective plane, and connected sums of these.

   Remark: You can answer parts (a)–(c) without doing part (d), but if you can see how to do part (d), then it's a good way to verify whether your answers to parts (a)–(c) were correct. In some cases, you might also be able to compute the fundamental groups of the spaces in part (d) and compare the result with whatever the polygon pictures tell you.

3. Recall that an *atlas* on a manifold M is a collection of charts $\{\varphi_\alpha : \mathcal{U}_\alpha \to \Omega_\alpha\}_{\alpha \in I}$ such that the sets $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ form an open cover of M.[1]

   (a) What is the minimum number of charts required to form an atlas on S^n?
   Hint: The answer is not 1.

   (b) Suppose M is a connected 1-manifold (without boundary) admitting an atlas that consists of precisely two charts $\varphi : \mathcal{U} \to \Omega \subset \mathbb{R}$ and $\psi : \mathcal{V} \to \Omega' \subset \mathbb{R}$. Show that $\mathcal{U} \cap \mathcal{V}$ has either one or two connected components.

   (c) Under the same assumptions as in part (b), show that M is homeomorphic to $\mathbb{R}$ if $\mathcal{U} \cap \mathcal{V}$ is connected, and M is otherwise homeomorphic to S^1.

   (d) Prove that every connected 1-manifold is homeomorphic to either S^1 or $\mathbb{R}$.
   Hint: Though you don't know whether M admits an atlas with only one or two charts, you know it admits a finite atlas if M is compact, and more generally, the second countability axiom guarantees that it always admits a countable atlas. (Why?) Show that M must be S^1 in the compact case, and in the noncompact case, use the countable atlas to inductively construct a homeomorphism to $\mathbb{R}$.

   If you're wondering what additional connected 1*-manifolds would be possible if we did not assume the second countability axiom, see* `https://en.wikipedia.org/wiki/Long_line_(topology)`.

4. $(*)$ Show that for the unit circle $S^1 \subset \mathbb{C}$, the degree of a map $f : S^1 \to S^1$ is equivalent to the *winding number* $\operatorname{wind}(f; 0) \in \mathbb{Z}$ defined in Problem Set 5 #1.

5. $(*)$ Use the implicit function theorem to prove that the orthogonal group

$$\mathrm{O}(n) := \{\mathbf{A} \in \mathrm{GL}(n, \mathbb{R}) \mid \mathbf{A}^T \mathbf{A} = \mathbb{1}\}$$

   is a smooth manifold. What is its dimension?
   Hint: $\mathrm{O}(n) = f^{-1}(\mathbb{1})$ *for the smooth map* $f : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n} : \mathbf{A} \mapsto \mathbf{A}^T \mathbf{A}$ *on the vector space* $\mathbb{R}^{n \times n}$ *of real n-by-n matrices, but $\mathbb{1}$ is not a regular value of this map. In fact, f is also very far from being surjective, as its image lies in a linear subspace of $\mathbb{R}^{n \times n}$. Can you replace the target space of f with something smaller so that the identity matrix $\mathbb{1}$ becomes a regular value?*

6. Suppose N is a closed and connected smooth n-manifold, and X is a compact smooth $(n+1)$-manifold with boundary such that ∂X has two connected components M_+ and M_-. We call X in this case an *unoriented cobordism* between M_+ and M_-. Denote the mod 2 degree for maps $f : M_\pm \to N$ by $\deg_2(f) \in \mathbb{Z}_2$.

   (a) $(*)$ Show that if $f_\pm : M_\pm \to N$ are two smooth maps that are restrictions $f|_{M_\pm}$ of some smooth map $f : X \to N$, then $\deg_2(f_+) = \deg_2(f_-)$.

   (b) Suppose X is compact but has connected boundary $\partial X = M$ and $f : M \to N$ extends to a smooth map $F : X \to N$. What can you conclude about $\deg_2(f)$?

7. Compute the degrees $\deg(f) \in \mathbb{Z}$ of the following maps.

   (a) $(*)$ $f : S^2 \to S^2$ where $S^2 = \mathbb{C} \cup \{\infty\}$ is the one-point compactification of $\mathbb{C}$ and f is the unique continuous extension of a complex polynomial $\mathbb{C} \to \mathbb{C}$ with degree $n \geq 0$.

   (b) $(*)$ $f : S^n \times S^m \to S^m \times S^n : (x, y) \mapsto (-y, -x)$ for some $m, n \in \mathbb{N}$.
   Hint: f is a diffeomorphism either way, but does it preserve or reverse orientations? The answer may depend on m and n.

   (c) $f : \mathbb{T}^2 \to \mathbb{T}^2 : [\mathbf{v}] \mapsto [\mathbf{A}\mathbf{v}]$, where $\mathbb{T}^2$ is identified with $\mathbb{R}^2/\mathbb{Z}^2$ and $\mathbf{A}$ is a 2-by-2 matrix with integer entries. (Note that f is well defined because $\mathbf{A} : \mathbb{R}^2 \to \mathbb{R}^2$ preserves the lattice $\mathbb{Z}^2$.)
   Hint: You can at least guess the answer if you think about the geometric meaning of the determinant in terms of areas of regions in $\mathbb{R}^2$.

[1] Note that an atlas is not always required to be *maximal*, but e.g. it is common to specify a smooth structure on a manifold by forming an atlas out of finitely many smoothly compatible charts. The actual smooth structure is then the unique maximal collection of smoothly compatible charts containing that atlas.