Topics covered in the lectures

1. We. 19.04.2017: general overview on problems and results in topology: classification of manifolds up to homeomorphism/diffeomorphism, Brouwer fixed point theorem and sketch of the proof
2. Th. 20.04.2017: metric spaces, continuity and sequential continuity, examples, product metric, equivalence of metrics, example of two inequivalent metrics on $C^0([-1, 1])$
3. We. 26.04.2017: interiors and closures, axioms of a topological space, continuity, neighborhoods, convergence of sequences, bases and subbases, basic examples (metric spaces, discrete/indiscrete/cofinite topologies), stronger/finer/larger vs. weaker/coarser/smaller topologies, induced topologies on subspaces, disjoint unions
4. Th. 27.04.2017: product topology in finite and infinite cases, topology of pointwise convergence for functions, relation between continuity and sequential continuity, first and second countability axioms, convergence of nets
5. We. 3.05.2017: compact spaces and subsets, examples, closed subsets in compact sets are compact, images of compact sets are compact, Hausdorff axiom, compact + Hausdorff $\Rightarrow$ closed, compactness of products and Tychonoff's theorem (statement without proof), cluster points and convergent subnets, proof that compact implies all nets have convergent subnets
6. Th. 4.05.2017: first countable compact spaces are sequentially compact, proof that all nets have cluster points iff compact, second countable and sequentially compact implies compact, proof (via nets) that finite products of compact spaces are compact, local compactness, the quotient topology, connected and path-connected spaces, local (path-)connectedness, path-connected implies connected (and a counterexample to the converse), connected + locally path-connected $\Rightarrow$ path-connected
7. We. 10.05.2017: corrected definitions of connectedness and local (path-)connectedness, separation axioms T_0, T_1, T_2 (Hausdorff), T_3 (regular) and T_4 (normal). FUNDAMENTAL GROUP: motivational questions from knot theory, multiplication of paths, homotopy, definition of $\pi_1(X, p)$ and proof that it is a group, simple connectedness
8. Th. 11.05.2017: pointed spaces and pointed (i.e. base point preserving) maps, $\pi_1(X, p)$ is independent of the base point p if X is path-connected, definition of X/A for a subset $A \subset X$, maps $X \to Y$ constant on A descend continuously to X/A, interpretation of $\pi_1(X, p)$ as pointed homotopy classes of maps $(S^1, t_0) \to (X, P)$, $[\gamma] = \mathrm{Id} \in \pi_1(X, p)$ if and only if $\gamma : S^1 \to X$ extends over $\mathbb{D}^2$, X is simply connected if and only if there is a unique homotopy class of paths between any two points, homomorphisms on π_1 induced by pointed maps, homeomorphic path-connected spaces have isomorphic fundamental groups, statement of $\pi_1(S^1) \cong \mathbb{Z}$
9. We. 17.05.2017: the notation $[X, Y]$ and $[X, Y]_+$, retractions, wedge sum $X \vee Y$, inclusions of retracts induce injections on π_1, proof of the Brouwer fixed point theorem in dimension 2, deformation retractions and examples, homotopy equivalence, $f : X \to Y$ homotopy equivalence implies $f_* : \pi_1(X, p) \to \pi_1(Y, f(p))$ isomorphism, contractible spaces are simply connected
10. Th. 18.05.2017: π_1 of Cartesian products (e.g the tori $\mathbb{T}^n$), weak version of van Kampen's theorem (conditions for $\pi_1(A)$ and $\pi_1(B)$ to generate $\pi_1(A \cup B)$), $\pi_1(S^n) = 0$ for $n \geq 2$, $\mathbb{R}^2$ and $\mathbb{R}^n$ are not homeomorphic for $n \geq 2$, gluing of spaces, cone and suspension, X path-connected implies ΣX simply connected
11. We. 24.05.2017: free products of groups, examples, normal subgroups generated by subsets, statement of the Seifert-van Kampen theorem, $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$, free groups generated by sets, $\pi_1(\mathbb{R}^3 \setminus K) \cong \pi_1(S^3 \setminus K)$ for any compact subset K, presentations of groups via generators and relations

12. We. 31.05.2017: restatement of the Seifert-van Kampen theorem with finite group presentations, $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$, the torus knots $K_{p,q}$, $\pi_1(\mathbb{R}^3 \setminus K_{p,q}) \cong \{a, b \mid a^p = b^q\}$, surfaces represented as polygons with edges identified (examples $\mathbb{T}^2$ and $\mathbb{RP}^2$)

13. Th. 1.06.2017: π_1 for polygons with edges identified, the surface Σ_g of genus g (defined via a polygon), $\pi_1(\Sigma_g) \cong \{x_1, y_1, \dots, x_g, y_g \mid x_1 y_1 x_1^{-1} y_1^{-1} \dots x_g y_g x_g^{-1} y_g^{-1} = e\}$, $\Sigma_{g+h} \cong \Sigma_g \# \Sigma_h$, generalized Seifert-van Kampen theorem (with more than two open sets) and its proof

14. We. 7.06.2017: proof that $\pi_1(S^1) \cong \mathbb{Z}$, covering spaces/maps, degree of a cover, lifting theorem and corollaries (lifts from simply connected spaces always exist, $p_* : \pi_1(\widetilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective, every cover of a simply connected space is simply connected), application to defining the logarithm on domains in $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$

15. Th. 8.06.2017: proof of the lifting theorem via path lifting and homotopy lifting properties, maps and isomorphisms of covers, deck transformations, two covers of X are equivalent iff they inject to the same subgroup of $\pi_1(X)$ (with attention to base points), uniqueness of deck transformations, $|\operatorname{Aut}(p)| \le \deg(p)$, regular/normal covers, $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is regular iff $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$

16. We. 14.06.2017: classification of covers up to equivalence ("Galois correspondence"), uniqueness of the universal cover, covers $p : (Y, y_0) \to (X, x_0)$ give bijections between $p^{-1}(x_0)$ and the right cosets of $H := p_*(\pi_1(Y, y_0))$ in $G := \pi_1(X, x_0)$, in particular $\deg(p) = [G : H]$, p regular $\Rightarrow \operatorname{Aut}(p) \cong G/H$, automorphism group of the universal cover of X is $\pi_1(X)$, application to $\pi_1(\mathbb{T}^n)$ and $\pi_1(\mathbb{RP}^n)$, proof of existence of the universal cover (sketch)

17. Th. 15.06.2017: the automorphism group of any cover $p : (Y, y_0) \to (X, x_0)$ is $N(H)/H$ for $H = p_*(\pi_1(Y, y_0)) \subset \pi_1(X, x_0) =: G$ and $N(H)$ the normalizer of H in G (stated without proof), topological groups, continuous group actions on topological spaces, action of $\pi_1(X, x_0)$ on the universal cover of X, $\mathbb{D}^2$ is not a nontrivial covering space of anything (follows via Brouwer fixed point theorem), G acts freely and properly discontinuously on X implies the projection $X \to X/G$ is a covering map, proof (via actions on the universal cover) that the Galois correspondence is surjective, $\pi_1(G)$ is abelian for any topological group G, the universal cover of a topological group is also a topological group (example of $\mathrm{SU}(2) \to \mathrm{SO}(3)$ sketched very briefly)

18. We. 21.06.2017: DIFFERENTIAL TOPOLOGY: topological manifolds, charts and transition maps, smooth manifolds, manifolds are connected if and only if path-connected, smooth maps between smooth manifolds, diffeomorphisms, tangent vectors and the tangent space T_pM at a point $p \in M$, preview of the mapping degree for continuous maps between compact oriented manifolds

19. Th. 22.06.2017: manifolds with boundary, brief survey of results in differential topology (existence and uniqueness of smooth structures up to dimension 3, exotic 7-spheres and non-smoothable 4-manifolds, classification of 1- and 2-manifolds, geometrization and Poincaré conjecture, higher-dimensional and smooth versions), the tangent map $Tf : TM \to TN$ of a smooth map $f : M \to N$, regular and critical points/values, statement of the implicit function theorem in manifolds, statement of Sard's theorem, application to continuous maps $f : M \to S^n$ for $n > \dim M$ (they are all homotopic to a constant)

20. We. 28.06.2017: approximation of continuous maps by smooth maps (proof for case $S^m \to S^n$ using Stone-Weierstrass), Sard's theorem and proof in the setting of second-countable smooth manifolds (using case $\mathbb{R}^n \supset \mathcal{U} \to \mathbb{R}^m$ as a black box), implicit function theorem in smooth manifolds without/with boundary, the mod 2 mapping degree and its homotopy invariance

21. Th. 29.06.2017: the space of bases of a real vector space has two connected components, orientations of a real vector space, orientations of smooth manifolds, $\mathbb{RP}^2$ is not orientable, boundary orientation, product orientation, implicit function theorem with orientations, orientation-preserving/-reversing smooth maps and counting $f^{-1}(q)$ with signs, the $\mathbb{Z}$-valued mapping degree

22. We. 5.07.2017: HOMOLOGY: motivation (how π_1 detects holes or not, the higher homotopy groups $\pi_k(X)$, oriented bordism groups $\Omega_n^{\mathrm{SO}}(X)$), introduction to simplicial homology (the standard n-simplex, oriented simplices, example of a simplicial decomposition of $\mathbb{T}^2$)

23. Th. 6.07.2017: basic homological algebra ($\mathbb{Z}$-graded chain complexes, homology of a chain complex, tensor product of abelian groups, homology with coefficients), simplicial complexes and simplicial homology, sketch of computation of $H_1^{\Delta}(\mathbb{T}^2)$, CW-complexes and cellular homology, computation of $H_*^{\mathrm{CW}}(\mathbb{RP}^2)$ and $H_*^{\mathrm{CW}}(\mathbb{RP}^2; \mathbb{Z}_2)$

24. We. 12.07.2017: review of cellular homology, computation for $\mathbb{T}^2$, S^2, S^n (with two different cell decompositions), $H_*^{\mathrm{CW}}(X)$ is independent of cell decompositions (proof next semester), Euler characteristic $\chi(X)$ as alternating sum of Betti numbers and alternating sum of numbers of cells

25. Th. 13.07.2017: definition of singular homology, calculation of $H_0(X; G)$, interpretation of singular 1-cycles, $H_1(X)$ is the abelianization of $\pi_1(X)$ (proof is homework), example of singular n-cycles, homomorphisms $f_* : H_*(X; G) \to H_*(Y; G)$ induced by continuous maps $f : X \to Y$, chain maps, $f \sim_h g \Rightarrow f_* = g_*$ and invariance of $H_*(X)$ under homotopy equivalence, chain homotopies, $\chi(X)$ depends only on the homotopy type of X

26. We. 19.07.2017: retractions $X \to A$ induce surjections $H_*(X) \to H_*(A)$, Brouwer fixed point theorem as corollary of $H_n(S^n) \cong \mathbb{Z}$, relative singular homology for pairs, homotopy of pairs, example $S^n = \mathbb{D}^n_+ \cup_{S^{n-1}} \mathbb{D}^n_-$ and deformation retraction of $(S^n \setminus \{p_-\}, \mathbb{D}^n_- \setminus \{p_-\})$ to $(\mathbb{D}^n_+, S^{n-1})$, statement of excision theorem, exact sequences, short exact sequences of chain complexes induce long exact sequences on homology (diagram chasing), application to pairs, proof that $H_{k-1}(S^{n-1}) \cong H_k(\mathbb{D}^n_+, S^{n-1})$ and $H_k(S^n, \mathbb{D}^n_-) \cong H_k(S^n)$.

27. Th. 20.07.2017: Computation of $H_k(S^n)$, barycentric subdivision and proof of the excision theorem, interpretation of the connecting homomorphism $H_n(X) \to H_{n-1}(A)$ in the long exact sequence of (X, A), explicit picture of the isomorphism $H_k(S^n) \to H_{k-1}(S^{n-1})$ for $k \geq 2$ and $n \geq 1$, the fundamental class $[S^n] \in H_n(S^n)$, characterization of $[S^n]$ in terms of simplicial decomposition