Topology I
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Humboldt-Universität zu Berlin
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TAKE-HOME MIDTERM

Instructions

To receive credit for this assignment, you must hand it in by **Wednesday, June 28** before the Übung. The solutions will be discussed in the Übung on that day.

You are free to use any resources at your disposal, but you should not discuss the problems with your comrades and **must write up your solutions alone**. Solutions may be written up in German or English, this is up to you.

There are 100 points in total; a score of 75 points or better will boost your final exam grade according to the formula that was indicated in the course syllabus. Note that the number of points assigned to each part of each problem is usually proportional to its conceptual importance/difficulty.

If a problem asks you to prove something, then unless it says otherwise, a **complete argument** is typically expected, not just a sketch of the idea. Partial credit may sometimes be given for incomplete arguments if you can demonstrate that you have the right idea, but for this it is important to write as clearly as possible. Less complete arguments can sometimes be sufficient, e.g. in cases where you want to show that two spaces are homotopy equivalent and can justify it with a very convincing picture (use your own judgement). You are free to make use of all results we've proved in lectures or problem sets, without reproving them. (When using a result from a problem set, say explicitly which one.)

One more piece of general advice: if you get stuck on one part of a problem, it may often still be possible to move on and do the next part.

You are free to ask for clarification or hints via e-mail or in office hours; of course we reserve the right not to answer such questions.

Problems

1. [35 pts total] Recall that if $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on the same set X, we say that $\mathcal{T}_1$ is **stronger** than $\mathcal{T}_2$ and $\mathcal{T}_2$ is **weaker** than $\mathcal{T}_1$ whenever $\mathcal{T}_2 \subset \mathcal{T}_1$. By this terminology, the strongest topology on X is always the discrete topology, and the weakest is the trivial (i.e. "indiscrete") topology.

   Given two topological spaces X and Y, let $C(X, Y)$ denote the set of all continuous maps $X \to Y$. There is a natural "evaluation map"

   $$\mathrm{ev} : C(X,Y) \times X \to Y : (f, x) \mapsto f(x).$$

   There are in general many possible topologies that one might imagine assigning to $C(X, Y)$, but one good criterion for finding the most "natural" one is that the map ev should be continuous. We saw one candidate in Problem Set 3 #7: the **compact-open topology** on $C(X, Y)$ is defined via the subbase consisting of all sets of the form

   $$\mathcal{U}_{K,V} = \left\{ f \in C(X,Y) \;\middle|\; f(K) \subset V \right\}$$

   for arbitrary compact subsets $K \subset X$ and open subsets $V \subset Y$.

   (a) [4 pts] Show that if $C(X, Y)$ is assigned the discrete topology, then ev is continuous.
   Remark: This is not meant to be an interesting observation, but just a demonstration that the goal of making ev *continuous is not hopeless. The goal will then be to identify more interesting topologies on $C(X, Y)$ that also have the desired property.*

(b) [10 pts] Show that every topology on $C(X,Y)$ for which ev is continuous contains the compact-open topology.
This is a bit tricky, but here are some pointers to get you started. Given a topology $\mathcal{T}$ on $C(X,Y)$ such that ev *is continuous, it suffices to show that $\mathcal{U}_{K,V} \in \mathcal{T}$ for every compact $K \subset X$ and open $V \subset Y$. (Why?) Remember that in order to prove a set (such as $\mathcal{U}_{K,V}$) is open in some topology (such as $\mathcal{T}$), it is enough to show that every $f_0 \in \mathcal{U}_{K,V}$ has an open neighborhood (open with respect to $\mathcal{T}$) contained in $\mathcal{U}_{K,V}$, as $\mathcal{U}_{K,V}$ will then be the union of all these open neighborhoods and therefore open. Now, the main piece of information given to you is that since $V \subset Y$ is open and* ev *is continuous, $\mathrm{ev}^{-1}(V) \subset C(X,Y) \times X$ is open with respect to the product topology determined by $\mathcal{T}$ and the topology of X. Deduce from this that if $f_0 \in \mathcal{U}_{K,V}$, and $x \in K$, then there exist neighborhoods $\mathcal{F}_x \subset C(X,Y)$ of f_0 and $\mathcal{O}_x \subset X$ of x such that $f(\mathcal{O}_x) \subset V$ for every $f \in \mathcal{F}_x$. (Remember how the product topology is defined?) Then use the fact that K is compact.*

(c) [8 pts] Show that if X is locally compact and Hausdorff and $C(X,Y)$ is assigned the compact-open topology, then ev is continuous.
Hint: Everything you need to know about locally compact Hausdorff spaces for this part is provided by the lemma in Problem Set 3 #6.

We have just proved: if X is a locally compact Hausdorff space, then the compact-open topology on $C(X,Y)$ is the *weakest* topology for which the evaluation map is continuous.

Now let's focus on maps from X to itself. A group G with a topology is called a **topological group** if the maps

$$G \times G \to G : (g,h) \mapsto gh \quad \text{and} \quad G \times G : g \mapsto g^{-1}$$

are both continuous. Given a topological space X, consider the group

$$\mathrm{Homeo}(X) = \left\{ f \in C(X,X) \;\middle|\; f \text{ is bijective and } f^{-1} \in C(X,X) \right\},$$

with the group operation defined via composition of maps. We would like to know what topologies can be assigned to $C(X,X)$ so that $\mathrm{Homeo}(X) \subset C(X,X)$, with the subspace topology, becomes a topological group. Notice again that the discrete topology clearly works; this is immediate because all maps between spaces with the discrete topology are automatically continuous, so there is nothing to check. But the discrete topology is not very interesting. Let $\mathcal{T}_H$ denote the topology on $C(X,X)$ with subbase consisting of all sets of the form $\mathcal{U}_{K,V}$ and $\mathcal{U}_{X \setminus V, X \setminus K}$, where again $K \subset X$ can be any compact subset and $V \subset X$ any open subset.

(d) [8 pts] Show that if X is locally compact and Hausdorff, then $\mathrm{Homeo}(X)$ with the topology $\mathcal{T}_H$ is a topological group.
Hint: Show first that $f \mapsto f^{-1}$ is continuous, using the fact that $f(K) \subset V$ if and only if $f^{-1}(X \setminus V) \subset X \setminus K$. Most of the rest of what you need was done already in Problem Set 3 #7(d).

(e) [5 pts] Conclude that if X is Hausdorff and compact, then $\mathrm{Homeo}(X)$ with the compact-open topology is a topological group.

Remark: The result proved in part (e) is true somewhat more generally, e.g. it suffices to know that X is Hausdorff, locally compact and locally connected. If you're interested, a (quite clever) proof of this can be found in: R. Arens, Topologies for homeomorphism groups, Amer. J. Math. ***68*** *(1946) 593–610.*

Just for fun, here's an example to show that just being locally compact and Hausdorff is not enough: let $X = \{0\} \cup \{e^n \mid n \in \mathbb{Z}\} \subset \mathbb{R}$ with the subspace topology, and notice that X is neither compact (since it is unbounded) nor locally connected (since every neighborhood of 0 is disconnected). Consider the sequence $f_k \in \mathrm{Homeo}(X)$ defined for $k \in \mathbb{N}$ by $f_k(0) = 0$, $f_k(e^n) = e^{n-1}$ for $n \le -k$ or $n > k$, $f_k(e^n) = e^n$ for $-k < n < k$, and $f_k(e^k) = e^{-k}$. It is not hard to show that in the compact-open topology on $C(X,X)$, $f_k \to \mathrm{Id}$ but $f_k^{-1} \not\to \mathrm{Id}$ as $k \to \infty$, hence the map $\mathrm{Homeo}(X) \to \mathrm{Homeo}(X) : f \mapsto f^{-1}$ is not continuous.

2. [30 pts total] Given a space X and a continuous map $\gamma : S^1 \to X$, consider the space

$$X' = X \cup_\gamma \mathbb{D}^2 := \left(X \sqcup \mathbb{D}^2\right)/\sim,$$

where the equivalence relation is defined by $z \sim \gamma(z)$ for every $z \in \partial\mathbb{D}^2 = S^1$. We say that X' is constructed by "attaching a 2-cell to X along γ". Note that γ need not be injective, so the equivalence relation may identify distinct points in $\partial\mathbb{D}^2$ to the same point in X', but both X and the interior of $\mathbb{D}^2$ are naturally subspaces of X'.

(a) [5 pts] Show that if X is compact and Hausdorff, then so is X'.

(b) [10 pts] Fixing the base point $p = \gamma(1) \in X$, suppose $\pi_1(X, p)$ is a finitely presented group, so we have an isomorphism

$$\pi_1(X, p) \cong \{g_1, \dots, g_n \mid r_1, \dots, r_m\}$$

where $\{g_1, \dots, g_n\}$ is a finite set of generators and $\{r_1, \dots, r_m\}$ a finite set of relations. Given a word $w = g_{i_1}^{p_1} \dots g_{i_k}^{p_k}$ that represents $[\gamma] \in \pi_1(X, p)$ under this isomorphism, show that $\pi_1(X', p)$ is then obtained from $\pi_1(X, p)$ by adding the one extra relation "$w = e$", that is,

$$\pi_1(X', p) \cong \{g_1, \dots, g_n \mid r_1, \dots, r_m,\ w = e\}.$$

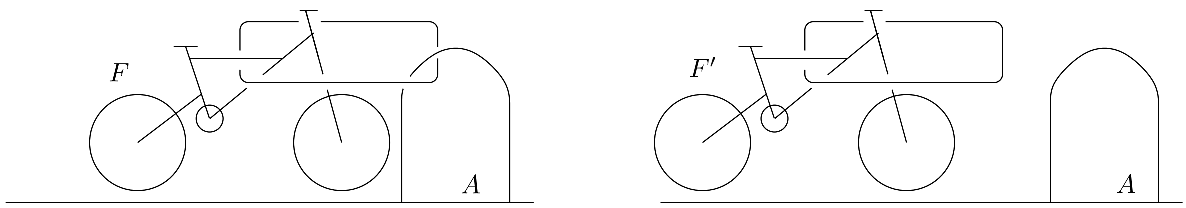
(c) [10 pts] Prove that every finitely presented group is isomorphic to the fundamental group of some compact Hausdorff space.
Hint: Start with a wedge of circles, then start attaching 2*-cells.*

(d) [5 pts] Use the procedure behind part (c) to construct and draw a picture of a compact Hausdorff space X with

$$\pi_1(X) \cong \left\{a, b, c, d \mid abc^{-1} = e,\ ca^{-1}d^{-1} = e,\ db^{-1} = e\right\}.$$

Include loops in the picture representing each of the four generators in the above presentation of $\pi_1(X)$.
Hint: This presentation of $\pi_1(X)$ is more complicated than it needs to be. If you first simplify it, you might recognize the group and be able to guess what X looks like.

3. [10 pts] The picture on the left below shows a pair of subsets $F, A \subset \mathbb{R}^3$ that are disjoint. In the picture at the right, A is the same subset, but F has been replaced by another set $F' \subset \mathbb{R}^3$, which is also disjoint from A and is homeomorphic to F but is translated some distance to the left.



Prove that the inclusion map $F \hookrightarrow \mathbb{R}^3 \setminus A$ is not homotopic to any map whose image is F'.
Advice: There is a lot of extraneous information in the picture. Ignore as much of it as you can.

4. [25 pts total]

(a) [5 pts] Show that every covering map of degree 2 is regular.
Hint: Just write down the required deck transformation—nothing fancy is needed here.

(b) [10 pts] Prove that every covering map of the torus $\mathbb{T}^2 = S^1 \times S^1$ is regular.

(c) [5 pts] Write down an example of a space $\widetilde{X}$ with a covering map $p : \widetilde{X} \to \mathbb{T}^2$ of degree 3. (No need to prove that it's a cover, just write it down.) Then write down its three deck transformations.
Hint: What can you say in advance about $\pi_1(\widetilde{X})$? This might help you guess what $\widetilde{X}$ should be.

(d) [5 pts] Prove that there exists a non-regular covering map of degree 3. (Not of the torus, obviously.)
Advice: There are two approaches one could take to this—one could try to find an explicit example, or one could apply some general theorems that we've proved in the class (or elsewhere on this take-home). Either approach is reasonable.