FINAL EXAM / KLAUSUR
October 9, 2017

Instructions

You have three hours in total, though the exam is designed to be doable in significantly less time than that. For reference, you may use any notes or books that you bring with you, but nothing electronic, i.e. no calculators or smartphones.

Answers can be written in German or English, and all answers require justification (within reason) in order to receive full credit. You may use results that were proved in the lectures or on problem sets without reproving them, but state clearly which results you are using. If you would like to use a result that you've found in a book but it was not covered in the class, then you need to explain the proof.

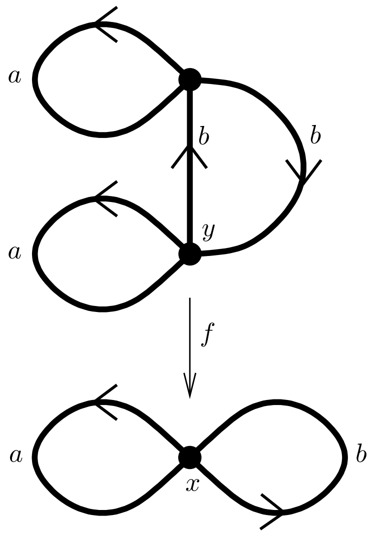
Keep in mind that if you get stuck on one part of a problem, it may sometimes be possible to skip it and do the next part.

Problems [100 pts total]

1. Regard each of the following subsets of $\mathbb{R}$ as topological spaces with the subspace topology (induced by the standard topology of $\mathbb{R}$). Give brief justifications for each answer.
   (a) [8 pts] Is the set $\mathbb{Q}$ of rational numbers locally compact?
   (b) [7 pts] Is $\{1/n \mid n \in \mathbb{N}\}$ locally path-connected?

2. [15 pts] Suppose $\mathbf{A}$ is a 3-by-3 matrix with integer entries, so the linear transformation $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ maps $\mathbb{Z}^3$ to $\mathbb{Z}^3$. Writing $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, it follows that $\mathbf{A}$ descends to a continuous map
$$f_{\mathbf{A}} : \mathbb{T}^3 \to \mathbb{T}^3.$$
   Compute the degree of $f_{\mathbf{A}}$ in the case where $\mathbf{A}$ is a *diagonal* matrix
$$\mathbf{A} := \begin{pmatrix} k & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & n \end{pmatrix}, \qquad k, m, n \in \mathbb{Z}.$$
   Tip: The situation where $kmn = 0$ should be handled as a special case.

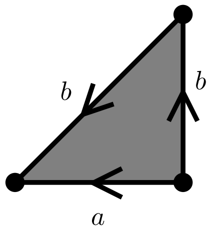
3. The picture at the right describes a base-point preserving covering map $f : (Y, y) \to (X, x)$, where $X \cong S^1 \vee S^1$, the two generators of $\pi_1(X, x) \cong \mathbb{Z} * \mathbb{Z}$ are labeled by loops a and b based at x, and the preimages of these loops in Y are given the same labels. Let us write elements of $\pi_1(X, x)$ accordingly as words in the letters a and b.



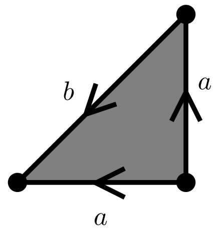
   (a) [10 pts] What is the subgroup $f_* (\pi_1(Y, y)) \subset \pi_1(X, x)$? Is it normal? (How do you know?)
   Describe it as $\langle S \rangle$ or $\langle S \rangle_N$, meaning the smallest subgroup or normal subgroup respectively containing some specific subset $S \subset \pi_1(X, x)$.
   (b) [15 pts] Draw a similar picture of a covering map $g : (Z, z) \to (X, x)$ such that $g_* (\pi_1(Z, z)) = \langle a^2, b^2, ab, ba \rangle_N$.
   Hint: First determine the degree of the cover. Then consider loops in X based at x, and determine whether their lifts to Z starting at the base point should close up or not.

4. [10 pts] Each of the two pictures below defines a topological space by identifying all vertices of the triangle to a single point and identifying any pairs of edges with matching letters via a homeomorphism that matches the arrows. One of these spaces is a 2-dimensional manifold (possibly with boundary), and the other is not. Determine which is which, and for the one that is not a manifold, explain why not.

(i)



(ii)



5. Consider the picture in Problem 4(i) as defining a topological space X with a cell decomposition consisting of one 0-cell (represented by every vertex, all of them identified), two 1-cells (the edges labeled a and b, with two copies of the latter identified), and one 2-cell (the interior of the triangle, attached to its edges).

   (a) [10 pts] Compute $\pi_1(X)$, and simplify the answer as much as you can.

   (b) [10 pts] Show that the cellular homology group $H_1^{\mathrm{CW}}(X)$ is isomorphic to $\mathbb{Z}$, and find a specific 1-cycle that generates it in the cellular chain complex.
   Note: One could in principle deduce the answer from part (a) via the theorem that $H_1^{\mathrm{CW}}(X)$ is isomorphic to the singular homology $H_1(X)$, which in turn is the abelianization of $\pi_1(X)$. But since we have not proved $H_1^{\mathrm{CW}}(X) \cong H_1(X)$, you may not use this argument.

6. [15 pts] Suppose X is a topological space with a subspace $A \subset X$ and the inclusion map $i : A \hookrightarrow X$. Show that the induced map on singular homology $i_* : H_n(A) \to H_n(X)$ is an isomorphism for all n if and only if the relative homology groups $H_n(X, A)$ vanish for every n.
Hint: Write down the long exact sequence for the pair (X, A).