Topology I
C. Wendl

Humboldt-Universität zu Berlin
Summer Semester 2018

PROBLEM SET 1
Due: 24.04.2018

Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture.

Problems

1. Suppose (X, d_X) is a metric space and $\sim$ is an equivalence relation on X, with the resulting set of equivalence classes denoted by $X/\sim$. For equivalence classes $[x], [y] \in X/\sim$, define

$$d([x],[y]) := \inf \left\{ d_X(x,y) \;\middle|\; x \in [x],\ y \in [y] \right\}. \tag{1}$$

(a) $(*)$ Show that d is a metric on $X/\sim$ if the following assumption is added: for every triple $[x],[y],[z] \in X/\sim$, there exist representatives $x \in [x]$, $y \in [y]$ and $z \in [z]$ such that

$$d_X(x,y) = d([x],[y]) \quad \text{and} \quad d_X(y,z) = d([y],[z]).$$

Comment: The hard part is proving the triangle inequality.

(b) Consider the *real projective plane*

$$\mathbb{RP}^2 := S^2/\sim,$$

where $S^2 := \{\mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| = 1\}$ and the equivalence relation identifies antipodal points, i.e. $\mathbf{x} \sim -\mathbf{x}$. If d_X is the metric on S^2 induced by the standard Euclidean metric on $\mathbb{R}^3$, show that the extra assumption in part (a) is satisfied, so that (1) defines a metric on $\mathbb{RP}^2$.

(c) For the metric defined on $\mathbb{RP}^2$ in part (b), show that the natural quotient projection $\pi : S^2 \to \mathbb{RP}^2$ sending each $\mathbf{x} \in S^2$ to its equivalence class $[\mathbf{x}] \in \mathbb{RP}^2$ is continuous, and a subset $\mathcal{U} \subset \mathbb{RP}^2$ is open if and only if $\pi^{-1}(\mathcal{U}) \subset S^2$ is open (with respect to the metric d_X).

(d) $(*)$ Here is a very different example of a quotient space. Define

$$X = (-1,1)^2 \setminus \{(0,0)\} \subset \mathbb{R}^2$$

with the metric d_X induced by the Euclidean metric on $\mathbb{R}^2$. Now fix the function $f : X \to \mathbb{R} : (x,y) \mapsto xy$ and define the relation $p_0 \sim p_1$ for $p_0, p_1 \in X$ to mean that there exists a continuous curve $\gamma : [0,1] \to X$ with $\gamma(0) = p_0$ and $\gamma(1) = p_1$ such that $f \circ \gamma$ is constant. Show that for this equivalence relation, the extra assumption of part (a) is not satisfied, and the distance function defined in (1) does not satisfy the triangle inequality.

(e) $(*)$ Despite our failure to define $X/\sim$ as a metric space in part (d), it is natural to consider the following notion: define a subset $\mathcal{U} \subset X/\sim$ to be *open* if and only if $\pi^{-1}(\mathcal{U})$ is an open subset of (X, d_X), where $\pi : X \to X/\sim$ denotes the natural quotient projection. We can then define a sequence $[x_n] \in X/\sim$ to be *convergent* to an element $[x] \in X/\sim$ if for every open subset $\mathcal{U} \subset X/\sim$ containing $[x]$, $[x_n] \in \mathcal{U}$ for all n sufficiently large. Find a sequence $[x_n] \in X/\sim$ and two elements $[x],[y] \in X/\sim$ such that

$$[x_n] \to [x] \quad \text{and} \quad [x_n] \to [y], \quad \text{but} \quad [x] \neq [y].$$

This could not happen if we'd defined convergence on $X/\sim$ in terms of a metric. (Why not?)

2. Suppose d_1 and d_2 are two metrics on the same set X. Show that the identity map defines a homeomorphism $(X, d_1) \to (X, d_2)$ if and only if the following condition is satisfied: for every sequence $x_n \in X$ and $x \in X$,

$$x_n \to x \text{ in } (X, d_1) \iff x_n \to x \text{ in } (X, d_2).$$

One says in this case that the metrics d_1 and d_2 are *equivalent*.

3. (a) Show that for any metric space (X, d),

$$d'(x, y) := \min\{1, d(x, y)\}$$

defines another metric on X which is equivalent to d (see Problem 2). In particular, this means that every metric is equivalent to one that is bounded.

(b) Suppose (X, d_X) and (Y, d_Y) are metric spaces satisfying

$$d_X(x, x') \le 1 \text{ for all } x, x' \in X, \qquad d_Y(y, y') \le 1 \text{ for all } y, y' \in Y.$$

Now let $Z = X \cup Y$, and for $z, z' \in Z$ define

$$d_Z(z, z') = \begin{cases} d_X(z, z') & \text{if } z, z' \in X, \\ d_Y(z, z') & \text{if } z, z' \in Y, \\ 2 & \text{if } (z, z') \text{ is in } X \times Y \text{ or } Y \times X. \end{cases}$$

Show that d_Z is a metric on Z with the following property: a subset $\mathcal{U} \subset Z$ is open in (Z, d_Z) if and only if it is the union of two (possibly empty) open subsets of (X, d_X) and (Y, d_Y). In particular, X and Y are each both open and closed subsets of Z. (Recall that subsets of metric spaces are closed if and only if their complements are open.)

(c) (∗) Suppose (Z, d) is a metric space containing two disjoint subsets $X, Y \subset Z$ that are each both open and closed. Show that there exists no continuous map $\gamma : [0, 1] \to Z$ with $\gamma(0) \in X$ and $\gamma(1) \in Y$.

(d) Suppose X is any set with the so-called *discrete metric*, defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \ne y. \end{cases}$$

Show that for every point $x \in X$, the subset $\{x\} \subset X$ is both open and closed, and moreover, every continuous map $\gamma : [0, 1] \to X$ is constant.

4. (∗) Assume (X, d_X) and (Y, d_Y) are metric spaces with $A \subset X$ a compact subset and $f : A \to Y$ a continuous map. Define the set

$$Z := X \cup_f Y := (X \cup Y) \big/ \sim,$$

where the equivalence relation is defined by $a \sim f(a)$ for each $a \in A$. Assume additionally that f is an *isometry* onto its image, meaning it satisfies

$$d_X(a, b) = d_Y(f(a), f(b)) \quad \text{for all } a, b \in A;$$

notice that f must then be injective, so we can regard both X and Y naturally as subsets of Z which intersect along A. We can then define a metric d_Z on Z such that $d_Z(x, y) = d_X(x, y)$ for $x, y \in X$, $d_Z(x, y) = d_Y(x, y)$ for $x, y \in Y$, and for $(x, y) \in X \times Y$,

$$d_Z(x, y) := \min \left\{ d_X(x, a) + d_Y(f(a), y) \;\middle|\; a \in A \right\}.$$

Verify the following case of the triangle inequality for d_Z:

$$d_Z(x, z) \le d_Z(x, y) + d_Z(y, z) \quad \text{whenever} \quad x \in X,\ y \in Y \text{ and } z \in X.$$

Hint: Notice that in the definition of d_Z, it says "min" instead of "inf". The minimum always exists because A is compact!
Advice: If you want something to visualize for intuition, a concrete example of this construction is mentioned below in Problem 5.

5. In the first lecture, we discussed the fact that $\mathbb{RP}^2$ is homeomorphic to an object constructed by gluing a disk $\mathbb{D}^2 = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| \le 1\}$ to a Möbius strip $\mathbb{M} = \{(\theta, t\cos(\pi\theta), t\sin(\pi\theta)) \in S^1 \times \mathbb{R}^2 \mid \theta \in S^1,\ t \in [-1, 1]\}$, where $S^1 := \mathbb{R}/\mathbb{Z}$. One can now make this precise using metrics of the types defined in Problems 1(b) and 4 respectively on $\mathbb{RP}^2$ and the glued object $\mathbb{D}^2 \cup_f \mathbb{M}$ (for a suitable homeomorphism f between the boundaries of $\mathbb{D}^2$ and $\mathbb{M}$). Work out the details until you get bored.