Topology I
C. Wendl

Humboldt-Universität zu Berlin
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PROBLEM SET 11
Due: Never

Instructions

This problem set will not be graded and should not be handed in, but it involves material from the last week of lectures which may appear on the exam. Written solutions are appended, but I advise you to try the problems yourself before looking at them.

Problems

1. For any space X, abelian group G and integer $k \geqslant 1$, there is an isomorphism

$$S_* : H_k(X; G) \to H_{k+1}(SX; G),$$

where $SX := C_+X \cup_X C_-X$ denotes the suspension of X, defined by gluing together two homeomorphic copies of its cone CX. Letting $p_- \in C_-X \subset SX$ denote the tip of the bottom cone, one can construct S_* out of the following diagram:

$$\begin{tikzcd}
H_k(X;G) \arrow[rr, "S_*"] & & H_{k+1}(SX;G) \arrow[d, "k_*"] \\
H_{k+1}(C_+X, X; G) \arrow[u, "\partial_*"] \arrow[r, "i_*"] & H_{k+1}(SX\backslash\{p_-\}, C_-X\backslash\{p_-\}; G) \arrow[r, "j_*"] & H_{k+1}(SX, C_-X; G)
\end{tikzcd}$$

Here ∂_* is the connecting homomorphism from the long exact sequence of the pair (C_+X, X) and is an isomorphism due to the fact that the terms $H_k(C_+X; G)$ and $H_{k+1}(C_+X; G)$ in that sequence vanish, since C_+X is contractible. The other maps are all induced by the obvious inclusions of pairs and they are all also isomorphisms: i_* because i is a homotopy equivalence of pairs (see Example 23.3 in the lecture notes), j_* by the excision theorem, and k_* due to the fact that $H_k(C_-X; G) = H_{k+1}(C_-X; G) = 0$ in the long exact sequence of (SX, C_-X). One can use the diagram to write down a formula for S_*, but it isn't an immediately useful formula since it involves k_*^{-1} and ∂_*^{-1}, which we cannot so easily write in terms of cycles. But we can still characterize S_* in terms of cycles as follows:

(a) Show that for any k-cycle $b \in C_k(X; G) \subset C_k(SX; G)$, there exist two $(k+1)$-chains $c_\pm \in C_{k+1}(C_\pm X; G) \subset C_{k+1}(SX; G)$ such that

$$\partial c_+ = -\partial c_- = b \tag{1}$$

and S_* satisfies

$$S_*[b] = [c_+ + c_-]. \tag{2}$$

Moreover, show that the formula (2) holds for any pair of $(k+1)$-chains $c_\pm \in C_{k+1}(C_\pm X; G)$ satisfying (1). (Note that the assumption $\partial c_+ = -\partial c_-$ implies $c_+ + c_- \in C_{k+1}(SX; G)$ is a cycle.) *Hint: Start with a relative cycle representing $[c_+] \in H_{k+1}(C_+X, X; G)$, then follow the arrows wherever they lead you. When you get to the map k_*, deduce whatever you can from the fact that it is an isomorphism.*

(b) In the special case $X = S^k$, we have $SS^k \cong S^{k+1}$ and $H_1(S^1; \mathbb{Z}) \cong \pi_1(S^1) \cong \mathbb{Z}$, so induction on k gives $H_k(S^k; \mathbb{Z}) \cong \mathbb{Z}$ for all $k \in \mathbb{N}$. In this case we can define a distinguished generator $[S^k] \in H_k(S^k; \mathbb{Z})$, the *fundamental class* of S^k, inductively as follows:

- $[S^1]$ is the homotopy class of $\mathrm{Id} : S^1 \to S^1$ under the isomorphism $H_1(S^1; \mathbb{Z}) \cong \pi_1(S^1)$.
- $[S^{k+1}] := S_*[S^k]$ for $k \geqslant 1$.

Prove by induction that for each $k \in \mathbb{N}$, $[S^k]$ can be represented by a cycle of the form $\sum_i \epsilon_i \sigma_i$, where $\epsilon_i = \pm 1$ and $\sigma_i : \Delta^k \to S^k$ are parametrizations of the k-simplices in an oriented triangulation of S^k.
Hint: There may be multiple valid ways to do this, but in my solution, I end up with 2^k simplices of dimension k in the triangulation of S^k.

2. Let $i : A \hookrightarrow X$ denote the inclusion map for a pair (X, A). Show that for any given coefficient group G, the induced map $i_* : H_n(A; G) \to H_n(X; G)$ is an isomorphism for all n if and only if the relative homology groups $H_n(X, A; G)$ vanish for all n.

3. Work through the rest of the proof of Theorem 23.6 in the lecture notes, the existence of the long exact sequence resulting from any short exact sequence of chain complexes. Stop when you either finish, get tired, or simply decide that you believe the theorem.

See next page for solutions.

SOLUTIONS

1. (a) Suppose $c_+ \in C_{k+1}(C_+X;G)$ is a relative cycle in (C_+X,X), so $\partial c_+ \in C_k(X;G)$ and c_+ represents an arbitrary relative homology class $[c_+] \in H_{k+1}(C_+X,X;G)$. By the formula proved at the end of Lecture 23, we have

$$\partial_*[c_+] = [\partial c_+] \in H_k(X;G),$$

and this can be any element of $H_k(X;G)$ since ∂_* is an isomorphism. It follows that given any $[b] \in H_k(X;G)$ represented by a k-cycle $b \in C_k(X;G)$, b is homologous to ∂c_+ for some c_+ as above, which means $b - \partial c_+ = \partial x$ for some $x \in C_{k+1}(X;G)$, or equivalently, $b = \partial(c_+ + x)$. Regarding $C_{k+1}(X;G)$ as a subgroup of $C_{k+1}(C_+X;G)$, we are then free to replace the original c_+ with $c_+ + x$ and thus assume without loss of generality that

$$\partial c_+ = b.$$

Next, apply $j_* \circ i_*$ to c_+: since the maps j and i are just inclusions, this changes nothing except to view c_+ as a chain in SX (which contains C_+X) and also as a relative cycle in (SX, C_-X) (since ∂c_+ is a chain in X and C_-X contains the latter). Now since k_* is an isomorphism, we have $k_*[y] = [c_+] \in H_{k+1}(SX, C_-X;G)$ for some absolute cycle $y \in C_{k+1}(SX;G)$, which is unique up to homology, and by the definition of S_*,

$$S_*[b] = [y].$$

Since $k : (SX, \varnothing) \hookrightarrow (SX, C_-X)$ is just the inclusion of pairs, this means that the relative cycle $y - c_+$ in (SX, C_-X) is nullhomologous, so

$$y - c_+ = c_- + \partial z$$

for some $c_- \in C_{k+1}(C_-X;G)$ and $z \in C_{k+2}(SX;G)$, implying $y - \partial z = c_+ + c_-$. Since both terms on the left hand side of this relation are absolute cycles in SX, this proves that $c_+ + c_-$ is also an absolute cycle, which moreover represents the same homology class in $H_{k+1}(SX;G)$ as y, so we have proved that the formulas (1) and (2) hold for some $c_\pm$.

Now suppose $c_\pm \in C_{k+1}(C_\pm X;G)$ are any chains such that (1) holds for a given cycle $b \in C_k(X;G)$. Then c_+ is a relative cycle in (C_+X,X) satisfying $\partial_*[c_+] = [b]$, and if we follow the inclusions to regard it also as a relative cycle in (SX, C_-X) representing a class $[c_+] \in H_{k+1}(SX, C_-X;G)$, we have by definition

$$k_*S_*[b] = [c_+].$$

But by assumption $c_+ + c_-$ can be regarded as another absolute cycle in SX, and it clearly satisfies $k_*[c_+ + c_-] = [c_+] \in H_{k+1}(SX, C_-X;G)$ since c_- is contained in C_-X, so the injectivity of k_* now implies $S_*[b] = [c_+ + c_-]$.

(b) We fix a homeomorphism identifying Δ^1 with the unit interval I, giving for any space X an identification of singular 1-simplices with paths $\gamma : I \to X$ such that (under the obvious identification of singular 0-simplies in X with points in X)

$$\partial\gamma = \gamma(1) - \gamma(0).$$

Now define two paths $\gamma_\pm : I \to S^1$ by

$$\gamma_+(t) := e^{\pi i t}, \qquad \gamma_-(t) := e^{\pi i (t+1)},$$

so that the concatenation $\gamma_+ \cdot \gamma_-$ is the loop $t \mapsto e^{2\pi i t}$, also known as the identity loop under the usual identification of $I/\partial I$ with S^1. This means that if we regard $\gamma_\pm$ as singular 1-simplices in S^1, their sum is a cycle and

$$[S^1] = [\gamma_+ + \gamma_-].$$

Notice that both of the paths $\gamma_\pm$ are embeddings, and their images overlap only at their end points, so they are 1-simplices that overlap only at their common boundaries, thus forming a triangulation of S^1. The triangulation has an obvious orientation, defined by orienting both 1-simplices so that their vertices are ordered from $\gamma_\pm(0)$ to $\gamma_\pm(1)$.

Now by induction, assume we have an oriented triangulation of S^k giving rise to a k-cycle $\sum_i \epsilon_i \sigma_i$ as stated which represents the fundamental class $[S^k]$. Identifying S^{k+1} with $SS^k = C_+ S^k \cup_{S^k} C_- S^k$, let $p_\pm$ denote the points at the tips of the two cones $C_\pm S^k$. We claim that every k-simplex $\sigma : \Delta^k \to S^k$ in the triangulation naturally gives rise to an embedded $(k+1)$-simplex $\sigma^\pm : \Delta^{k+1} \to C_\pm S^k$ whose intersection with $S^k \subset C_\pm S^k$ is one of its boundary faces and matches σ, and moreover, that this collection of $(k+1)$-simplices triangulates $C_\pm S^k$.

Let us prove the claim for $C_+ S^k = ([0,1] \times S^k)/(\{1\} \times S^k)$, in which p_+ is literally the equivalence class of any point in $\{1\} \times S^k$. Given $\sigma : \Delta^k \to S^k$, we can then define $\sigma^+ : \Delta^{k+1} \to C_+ S^k$ by

$$\sigma^+(t_0, \dots, t_k, t_{k+1}) := \begin{cases} \left[\left(t_{k+1}, \sigma\left(\frac{t_0}{1-t_{k+1}}, \dots, \frac{t_k}{1-t_{k+1}}\right)\right)\right], & \text{if } t_{k+1} < 1, \\ p_+ & \text{if } t_{k+1} = 1. \end{cases}$$

Identifying Δ^k with the boundary face $\partial_{(k+1)} \Delta^{k+1}$ then gives $\sigma^+|_{\partial_{(k+1)}\Delta^{k+1}} = \sigma$, while σ^+ maps the opposite vertex $(0, \dots, 0, 1)$ to the point p_+. It is easy to check (but I will not explain it here) that this collection of $(k+1)$-simplices for all σ in the triangulation of S^k gives a triangulation of $C_+ S^k$. Similar $(k+1)$-simplices $\sigma^- : \Delta^{k+1} \to C_- S^k$ can be defined in an analogous way to define a triangulation of $C_- S^k$, and both sets of simplices together give a triangulation of $SS^k \cong S^{k+1}$.

We now orient this triangulation as follows: by assumption, each simplex $\sigma(\Delta^k)$ in the triangulation of S^k comes with an orientation determined by the sign $\epsilon_k = \pm 1$, so there is a preferred class of orderings of its vertices. Since each of the two $(k+1)$-simplices $\sigma^\pm(\Delta^{k+1})$ has the same vertices as $\sigma(\Delta^k)$ plus one extra vertex at $p_\pm$, we can assign to $\sigma^+(\Delta^{k+1})$ the orientation given by first writing the vertices of $\sigma(\Delta^k)$ in the preferred order, and then adding p_+ at the end of the list. Define the orientation of $\sigma^-(\Delta^{k+1})$ by this same prescription, but then reverse its orientation. The result is that the induced orientation of $\sigma(\Delta^k)$ as a boundary face of $\sigma^+(\Delta^{k+1})$ matches its given orientation, but is opposite its induced orientation as a boundary face of $\sigma^-(\Delta^{k+1})$. This makes our triangulation of $S^{k+1} \cong SS^k$ into an oriented triangulation, and writing

$$c_\pm := \sum_i \epsilon_i \sigma_i^\pm \in C_{k+1}(C_\pm S^k; \mathbb{Z})$$

then gives $\partial c_+ = -\partial c_- = \sum_i \epsilon_i \sigma_i \in C_k(S^k; \mathbb{Z})$. Using the induction hypothesis and the result of part (a), we conclude

$$\left[\sum_i \epsilon_i (\sigma_i^+ - \sigma_i^-)\right] = S_*[S^k] = [S^{k+1}] \in H_{k+1}(S^{k+1}; \mathbb{Z}).$$

2. The exact sequence of the pair (X, A) takes the form

$$\dots \to H_{n+1}(X, A; G) \to H_n(A; G) \xrightarrow{i_*} H_n(X; G) \to H_n(X, A; G) \to \dots,$$

so if $H_n(X, A; G) = 0$ for every n, this reduces to

$$0 \to H_n(A; G) \xrightarrow{i_*} H_n(X; G) \to 0$$

for every n, in which exactness immediately implies that i_* is an isomorphism.

Conversely, suppose we do not know $H_*(X, A; G)$ but it is given that $i_* : H_n(A; G) \to H_n(X; G)$ is an isomorphism for every n. Consider the following portion of the long exact sequence:

$$\dots \to H_n(A; G) \xrightarrow{i_*^n} H_n(X; G) \xrightarrow{j_*} H_n(X, A; G) \xrightarrow{\partial_*} H_{n-1}(A; G) \xrightarrow{i_*^{n-1}} H_{n-1}(X; G) \to \dots,$$

where superscripts have been added to i_* to distinguish the two versions of this map that appear. Since i_*^n is surjective, exactness at the term $H_n(X; G)$ implies $\ker j_* = \operatorname{im} i_*^n = H_n(X; G)$, which means j_*

is the trivial map, so its image is the trivial subgroup of $H_n(X, A; G)$. Exactness at $H_n(X, A; G)$ then implies $\ker \partial_* = \operatorname{im} j_* = 0$, so ∂_* is injective. The injectivity of i_*^{n-1} implies in turn via exactness at $H_{n-1}(A; G)$ that $\operatorname{im} \partial_* = \ker i_*^{n-1} = 0$, so we conclude that ∂_* is an injective homomorphism with trivial image. This is only possible if $H_n(X, A; G) = 0$.

3. I will not do this problem for you, but if you really want to read a proof instead of working it out yourself, you'll find one on pages 116–117 in Hatcher (Theorem 2.16 in particular).