PROBLEM SET 4
Due: 22.05.2018

Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture.

Problems

1. Prove that $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic for any $n \geqslant 2$.
Hint: If $\mathbb{R}$ and $\mathbb{R}^n$ are homeomorphic, then so are $\mathbb{R}\backslash\{t\}$ and $\mathbb{R}^n\backslash\{x\}$ for some $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Show that one of those spaces is connected and the other is not.

2. $(*)$ In lecture we proved that in any space X that is locally compact and Hausdorff, every neighborhood of every point $x \in X$ contains a compact neighborhood of x. Extend this result to prove the following: if X is locally compact and Hausdorff, then for any nested pair of subsets $K \subset \mathcal{U} \subset X$ with K compact and $\mathcal{U}$ open, there exists an open set $\mathcal{V} \subset X$ with compact closure $\bar{\mathcal{V}}$ such that $K \subset \mathcal{V} \subset \bar{\mathcal{V}} \subset \mathcal{U}$.

3. Given a space X, the collection of all connected components of X can be viewed as a collection of topological spaces $\{X_\alpha\}_{\alpha\in I}$, where each X_α is endowed with the subspace topology as a subset of X. Each therefore comes with a continuous inclusion map $i_\alpha : X_\alpha \hookrightarrow X$, and these can be assembled into a map
$$i : \coprod_{\alpha\in I} X_\alpha \to X,$$
defined by the property that $i|_{X_\alpha} = i_\alpha$ for each $\alpha \in I$. This map is obviously a bijection, and the definition of the disjoint union topology implies that it is continuous. Show that i is a homeomorphism if and only if every X_α is an open subset of X.[1]

4. (a) Prove that if X and Y are both connected, then so is $X \times Y$.[2]
Hint: Start by showing that for any $x \in X$ and $y \in Y$, the subsets $\{x\} \times Y$ and $X \times \{y\}$ in $X \times Y$ are connected. Then think about continuous maps $X \times Y \to \{0, 1\}$.
   (b) Show that for any collection of path-connected spaces $\{X_\alpha\}_{\alpha\in I}$, the space $\prod_{\alpha\in I} X_\alpha$ is path-connected in the usual product topology.
Hint: You might find Problem Set 2 #3(d) helpful.
   (c) Consider $\mathbb{R}^{\mathbb{N}}$ with the "box topology" which we discussed in Problem Set 2 #5. Show that the set of all elements $f \in \mathbb{R}^{\mathbb{N}}$ represented as functions $f : \mathbb{N} \to \mathbb{R}$ that satisfy $\lim_{n\to\infty} f(n) = 0$ is both open and closed, hence $\mathbb{R}^{\mathbb{N}}$ in the box topology is not connected (and therefore also not path-connected).

5. For each of the following spaces, determine whether it is (i) Hausdorff, (ii) locally compact, (iii) connected, (iv) locally path-connected.[3]
   (a) $(*)$ The irrational numbers $\mathbb{R}\backslash\mathbb{Q}$
   (b) $\{0\} \cup \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$
   (c) The quotient group $\mathbb{R}/\mathbb{Q}$

[1] Recall that this condition holds whenever X is locally connected or has finitely many connected components, but e.g. it does not hold for $X = \mathbb{Q}$.
[2] The analogous statement about infinite products is also true, but it takes more work to prove it.
[3] You should always assume unless otherwise specified that $\mathbb{R}$ is endowed with its standard topology, and all spaces derived from it as subsets/products/quotients etc. carry the natural subspace/product/quotient topology.

6. There is a cheap trick to view any topological space as a compact space with a single point removed. For a space X with topology $\mathcal{T}$, let $\{\infty\}$ denote a set consisting of one element that is not in X, and define the *one point compactification* of X as the set $X^* = X \cup \{\infty\}$ with topology $\mathcal{T}^*$ consisting of all subsets in $\mathcal{T}$ plus all subsets of the form $(X \backslash K) \cup \{\infty\} \subset X^*$ where $K \subset X$ is closed and compact.

   (a) Verify that $\mathcal{T}^*$ is a topology and that X^* is always compact.
   (b) $(*)$ Show that if X is first countable and Hausdorff, a sequence in $X \subset X^*$ converges to $\infty \in X^*$ if and only if it has no convergent subsequence with a limit in X. Conclude that if X is first countable and Hausdorff, X^* is sequentially compact.[4]
   (c) Show that for $X = \mathbb{R}$, X^* is homeomorphic to S^1. *(More generally, one can use stereographic projection to show that the one point compactification of $\mathbb{R}^n$ is homeomorphic to S^n.)*
   (d) Show that if X is already compact, then X^* is homeomorphic to the disjoint union $X \amalg \{\infty\}$.
   (e) Show that X^* is Hausdorff if and only if X is both Hausdorff and locally compact.

   Notice that $\mathbb{Q}$ is not locally compact, since every neighborhood of a point $x \in \mathbb{Q}$ contains sequences without convergent subsequences, e.g. any sequence of rational numbers that converges to an irrational number sufficiently close to x. The one point compactification $\mathbb{Q}^*$ is a compact space, and by part (b) it is also sequentially compact, but those are practically the only nice things we can say about it.

   (f) Show that for any $x \in \mathbb{Q}$, every neighborhood of x in $\mathbb{Q}^*$ intersects every neighborhood of ∞, so in particular, $\mathbb{Q}^*$ is not Hausdorff.
   Advice: Do not try to argue in terms of sequences with non-unique limits (cf. part (g) below), and do not try to describe precisely what arbitrary compact subsets of $\mathbb{Q}$ can look like (the answer is not nice). One useful thing you can say about arbitrary compact subsets of $\mathbb{Q}$ is that they can never contain the intersection of $\mathbb{Q}$ with any open interval. (Why not?)
   (g) Show that every convergent sequence in $\mathbb{Q}^*$ has a unique limit. *(Since $\mathbb{Q}^*$ is not Hausdorff, this implies via a result we proved in lecture that $\mathbb{Q}^*$ is not first countable—in particular, ∞ does not have a countable neighborhood base.)*
   (h) Find a point in $\mathbb{Q}^*$ with a neighborhood that does not contain any compact neighborhood.

7. Given spaces X and Y, let $C(X, Y)$ denote the set of all continuous maps from X to Y, and consider the natural *evaluation map*

$$\mathrm{ev} : C(X, Y) \times X \to Y : (f, x) \mapsto f(x).$$

   It is easy to show that ev is a continuous map if we assign the discrete topology to $C(X, Y)$, but usually one can also find more interesting topologies on $C(X, Y)$ for which ev is continuous. The *compact-open topology* is defined via a subbase consisting of all subsets of the form

$$\mathcal{U}_{K,V} := \{ f \in C(X, Y) \mid f(K) \subset V \},$$

   where K ranges over all compact subsets of X, and V ranges over all open subsets of Y.

   (a) Show that if Y is a metric space, then convergence of a sequence $f_n \in C(X, Y)$ in the compact-open topology means that f_n converges uniformly on all compact subsets of X.
   (b) Show that if $C(X, Y)$ carries the topology of pointwise convergence (i.e. the subspace topology defined via the obvious inclusion $C(X, Y) \subset Y^X$), then ev is not sequentially continuous in general.
   (c) Show that if $C(X, Y)$ carries the compact-open topology, then ev is always sequentially continuous.
   (d) $(*)$ Show that if $C(X, Y)$ carries the compact-open topology and X is locally compact and Hausdorff, then ev is continuous.
   (e) $(*)$ Show that every topology on $C(X, Y)$ for which ev is continuous contains the compact-open topology. (This proves that if X is locally compact and Hausdorff, the compact-open topology is the weakest topology for which the evaluation map is continuous.)
   Hint: If $(f_0, x_0) \in \mathrm{ev}^{-1}(V)$ where $V \subset Y$ is open, then $(f_0, x_0) \in \mathcal{O} \times \mathcal{U} \subset \mathrm{ev}^{-1}(V)$ for some open $\mathcal{O} \subset C(X, Y)$ and $\mathcal{U} \subset X$. Is $\mathcal{U}_{K,V}$ a union of sets $\mathcal{O}$ that arise in this way?

[4] This is a revised version of the problem sheet. The Hausdorff condition has been added, as the statement is not true without it. See the written solution posted on the website.

(f) Show that for the compact-open topology on $C(\mathbb{Q}, \mathbb{R})$, $\mathrm{ev} : C(\mathbb{Q}, \mathbb{R}) \times \mathbb{Q} \to \mathbb{R}$ is not continuous.