# PROBLEM SET 4
## Solutions for #6–7

These problems were harder than usual, and the way I stated #6(b) in particular was slightly wrong. (The grader has been instructed to give everyone full credit on that one.) The notions of one point compactification and the compact-open topology are also conceptually important, thus it's worth devoting somewhat more attention to them than we had time for in the Übung this week. So here are some written solutions.

6. (a) The collection $\mathcal{T}^*$ consists of all open sets $\mathcal{U} \subset X$ plus all sets of the form $(X \backslash K) \cup \{\infty\} \subset X^*$ for closed compact subsets $K \subset X$. Note first that this collection includes $\varnothing$ and $X^*$; the latter in particular because $\varnothing \subset X$ is closed and compact. Next, consider unions: any union of open sets in $X$ is another open set in $X$, and a union of sets of the form $(X \backslash K_\alpha) \cup \{\infty\} \in \mathcal{T}^*$ for $\{K_\alpha \subset X\}_{\alpha \in I}$ a collection of closed compact subsets has the form

$$\bigcup_{\alpha \in I} \left( (X \backslash K_\alpha) \cup \{\infty\} \right) = \left( X \backslash \bigcap_{\alpha \in I} K_\alpha \right) \cup \{\infty\} = (X \backslash K) \cup \{\infty\}$$

where we define

$$K := \bigcap_{\alpha \in I} K_\alpha. \tag{1}$$

Since every $K_\alpha$ is closed, the intersection $K$ is also closed, and since it is contained in each of the compact sets $K_\alpha$, $K$ is therefore also compact, implying $(X \backslash K) \cup \{\infty\} \in \mathcal{T}^*$. It remains only to observe that for any $\mathcal{U} \subset X$ open and $K \subset X$ closed and compact,

$$\mathcal{U} \cup \left( (X \backslash K) \cup \{\infty\} \right) = \Big( X \backslash \big( (X \backslash \mathcal{U}) \cap K \big) \Big) \cup \{\infty\} = (X \backslash K') \cup \{\infty\}$$

where

$$K' := (X \backslash \mathcal{U}) \cap K. \tag{2}$$

The latter is the intersection of two closed sets and is thus closed, and since it is contained in the compact set $K$, it is also compact, so $(X \backslash K') \cup \{\infty\} \in \mathcal{T}^*$. This proves that arbitrary unions of sets in $\mathcal{T}^*$ are also in $\mathcal{T}^*$.

For intersections of two sets $\mathcal{U}, \mathcal{V} \in \mathcal{T}^*$, we have three cases to consider: first, if $\mathcal{U}$ and $\mathcal{V}$ are both open subsets of $X$ then obviously $\mathcal{U} \cap \mathcal{V}$ is as well. If $\mathcal{U} \subset X$ is open and $\mathcal{V} = (X \backslash K) \cup \{\infty\}$ for $K \subset X$ closed and compact, then

$$\mathcal{U} \cap \mathcal{V} = \mathcal{U} \cap (X \backslash K) \tag{3}$$

is the intersection of two open subsets and is thus open. Finally, if both contain $\infty$ and the complements of closed compact subsets $K, K' \subset X$, then

$$\mathcal{U} \cap \mathcal{V} = \big( X \backslash (K \cup K') \big) \cup \{\infty\},$$

where $K \cup K'$ is closed and compact since it is a finite union of closed compact sets. This proves that finite intersections of sets in $\mathcal{T}^*$ are also in $\mathcal{T}^*$, hence $\mathcal{T}^*$ is a topology on $X^*$.

To see that $X^*$ is compact, note that any given open cover of $X^*$ must contain at least one open set that contains $\infty$, so let us pick such a set and call it $\mathcal{U}_\infty$, denoting the entire open cover then by

$$X^* = \mathcal{U}_\infty \cup \left( \bigcup_{\alpha \in I} \mathcal{U}_\alpha \right).$$

Since $\mathcal{U}_\infty \backslash \{\infty\} = X \backslash K$ for some closed compact set $K \subset X$, the open sets $\mathcal{U}'_\alpha := X \cap \mathcal{U}_\alpha \subset X$ must form an open cover of $K$, which therefore has a subcover

$$K \subset \bigcup_{i=1}^{N} (X \cap \mathcal{U}_{\alpha_i})$$

for some finite subset $\{\alpha_1, \ldots, \alpha_N\} \subset I$. We therefore have a finite subcover of $X^*$ consisting of $\mathcal{U}_\infty$ and the sets $\mathcal{U}_{\alpha_i}$ for $i = 1, \ldots, N$. This concludes the solution.

*I would like to add a further remark in connection with the detail that got me into trouble in part (b) below. One might wonder if one could instead define $\mathcal{T}^*$ so that open sets containing $\infty$ are of the form $(X \backslash K) \cup \infty$ for* arbitrary *compact sets $K \subset X$, rather than requiring $K$ to be both compact and closed.*[1] *The answer is that if this were the definition, then $\mathcal{T}^*$ would not generally be a topology, i.e. it would not be closed under arbitrary unions and finite intersections. Indeed, at three points in the above argument (namely at Equations (1), (2) and (3)), we explicitly used the assumption that our compact subsets $K \subset X$ were also closed: this way we can be sure that their intersections with each other or with other closed subsets are also closed and therefore (as closed subsets of compact sets) compact. It is not true in general that an intersection of compact subsets must always be compact. This probably goes against your intuition since it can only happen in non-Hausdorff spaces, but here is an easy example: let $X = (\mathbb{R} \times \{0,1\})/\sim$ with the quotient topology, where $(x,0) \sim (x,1)$ for every $x \neq 0$, i.e. this defines the "line with two zeroes". Since the quotient projection $\pi : \mathbb{R} \times \{0,1\} \to X$ is continuous, the sets $K_i := \pi([-1,1] \times \{i\}) \subset X$ for $i = 0, 1$ are continuous images of compact sets and are therefore compact. But since each of them contains a different copy of 0 and not the other one, $K_0 \cap K_1$ is homeomorphic to the union of intervals $[-1,0) \cup (0,1]$, and in particular it has sequences that converge to both copies of 0 but have no convergent subsequence with limit in $K_0 \cap K_1$. This can happen only because while $K_0$ and $K_1$ are both compact, they are not closed, e.g. the closure of $K_0$ also contains $[(0,1)]$, which is not in $K_0$!*

*You should be aware that in much of the literature, authors require their space $X$ to be both Hausdorff and locally compact before even attempting to define its one point compactification, and in this case they do not always include the word "closed" together with "compact" since it is automatic. This restriction is not necessary, but it does make life easier.*

(b) The statement was false as written, but a correct version would be as follows:

> *If $X$ is first countable and Hausdorff, then a sequence $x_n \in X$ converges to $\infty \in X^*$ as a sequence in $X^*$ if and only if it has no convergent subsequence as a sequence in $X$. In particular, $X^*$ is sequentially compact if $X$ is first countable and Hausdorff.*

Without Hausdorff the statement is not true; see below for a counterexample. We will use the fact (proved in lecture) that every compact subset of a Hausdorff space is also closed. Note that if $x_n \in X$ is a sequence converging to the point $x \in X$, then

$$\{x, x_1, x_2, x_3, \ldots\} \subset X$$

is a compact set. Indeed, any open cover includes a set that is a neighborhood of $x$ and thus (by the definition of convergence) also contains $x_n$ for all $n \geqslant N$ if $N \in \mathbb{N}$ is sufficiently large. Then a finite subcover can be found that includes this neighborhood of $x$ plus a finite collection of sets containing the finite set of points $x_1, \ldots, x_{N-1}$. If $X$ is Hausdorff, it follows that $\{x, x_1, x_2, x_3, \ldots\}$ is also closed. (This need not be true if $X$ is not Hausdorff, e.g. we have seen spaces in which one point subsets $\{x\} \subset X$ are not always closed, in which case the union of points in the constant sequence $x_n := x$ is a compact but non-closed subset.)

To prove the statement, assume $X$ is Hausdorff and first countable and $x_n \in X \subset X^*$ is a sequence. We then have $x_n \to \infty$ if and only if for every closed compact set $K \subset X$, $x_n \notin K$ for all $n$ sufficiently large. If this is true but $x_n$ has a subsequence $x_{k_n}$ converging to some $x \in X$, then $K := \{x, x_{k_1}, x_{k_2}, x_{k_3}, \ldots\} \subset X$ is an example of a closed compact set, implying $x_n \notin K$ for all $n$ sufficiently large, but this is clearly false. Conversely, if $x_n$ does not converge to $\infty$, then $x_n$ has a subsequence $x_{k_n}$ that stays away from some neighborhood of $\infty$, meaning there is a compact set $K \subset X$ such that $x_{k_n} \notin X \backslash K$ for all $n$, or equivalently, $x_{k_n} \in K$. Since $X$ (and therefore also $K \subset X$ with the subspace topology) is first countable, $K$ is sequentially compact, so $x_{k_n}$ has a further subsequence that converges to a limit in $K$.

[1] This detail makes no difference of course if $X$ is Hausdorff.

*Here is the counterexample I promised if the Hausdorff condition is dropped. Define a topology $\mathcal{T}$ on $\mathbb{R}$ consisting of $\varnothing$ plus every set of the form $\mathcal{U} \cup \{0\}$ for $\mathcal{U} \subset \mathbb{R}$ open in the standard topology. In the topological space $X := (\mathbb{R}, \mathcal{T})$, every nonempty open subset contains 0, thus the only closed subset $A \subset X$ containing 0 is $A = X$. It follows that 0 is not contained in any closed compact subset, and it is therefore in every neighborhood of $\infty \in X^*$, so that the constant sequence $x_n := 0 \in X$ converges both to 0 and to $\infty$.*

(c) As usual it is convenient to view $S^1$ as the unit circle in $\mathbb{C}$ so that we can use complex exponentials. Choose a homeomorphism between $\mathbb{R}$ and $S^1 \backslash \{1\}$: one can do this by first fixing any homeomorphism $\varphi : \mathbb{R} \to (0, 1)$ and then defining

$$\Phi : \mathbb{R} \to S^1 \backslash \{1\} : t \mapsto e^{2\pi i \varphi(t)}. \tag{4}$$

This extends in an obvious way to a bijection $\Phi : \mathbb{R}^* \to S^1$ by setting $\Phi(\infty) := 1$. Is it a homeomorphism?

One can reframe this question as follows. Consider two topologies on $S^1$: its standard topology $\mathcal{T}$ (defined as the subspace topology with respect to the standard topology of $\mathbb{C} = \mathbb{R}^2$), and another topology $\mathcal{T}'$ whose open sets are $\Phi(\mathcal{U})$ for all open sets $\mathcal{U} \subset \mathbb{R}^*$. In this way $\Phi$ becomes tautologically a homeomorphism between $\mathbb{R}^*$ and $(S^1, \mathcal{T}')$, so now the question is: are the two topologies $\mathcal{T}$ and $\mathcal{T}'$ the same?

The way we constructed $\mathcal{T}'$ allows us to think of $(S^1, \mathcal{T}')$ as the one point compactification of $S^1 \backslash \{1\}$, but with the role of $\infty$ played by the point $1 \in S^1$. This means two things in particular. First, the open sets in $(S^1, \mathcal{T}')$ that do not contain 1 are simply the open sets in $S^1 \backslash \{1\}$ with its standard topology; this results from the fact that the map (4) is a homeomorphism. Second, the open sets in $(S^1, \mathcal{T}')$ that contain 1 are all the sets of the form

$$\left((S^1 \backslash \{1\}) \backslash K\right) \cup \{1\} = S^1 \backslash K$$

for subsets $K \subset S^1 \backslash \{1\}$ that are closed and compact in the standard topology. Since $S^1$ is compact and Hausdorff, its compact subsets are precisely its closed subsets, so this just means the sets in $\mathcal{T}'$ containing 1 are precisely the complements of closed subsets that do not contain 1, i.e. the open sets (in the standard topology) containing 1. This proves that the two topologies are the same.

*By the way, one can use the same strategy to prove that the one point compactification of $\mathbb{R}^n$ is homeomorphic to $S^n$ for every $n \in \mathbb{N}$.*

(d) If $X$ is compact, then since it is also closed (because $\varnothing \subset X$ is open), $(X \backslash X) \cup \{\infty\} = \{\infty\}$ is an open subset of $X^*$. Notice that $X$ is also an open subset of $X^*$ since it is an open subset of $X$, hence $\{\infty\} \subset X^*$ and $X \subset X^*$ are each both open and closed. One can now show as in Problem 3 that the natural map $X \amalg \{\infty\} \hookrightarrow X^*$ defined via the inclusions $X \hookrightarrow X^*$ and $\{\infty\} \hookrightarrow X^*$ is a homeomorphism.

(e) Suppose $X$ is locally compact and Hausdorff. Since open sets in $X$ are also open in $X^*$, this proves immediately that any two distinct points in $X$ have disjoint neighborhoods in $X^*$. We still need to show that every $x \in X$ has a neighborhood in $X^*$ that is disjoint from some neighborhood of $\infty$. For this, choose $K \subset X$ to be a compact neighborhood of $x$, and note that $K$ is also closed since $X$ is Hausdorff. Then $(X \backslash K) \cup \{\infty\}$ is a neighborhood of $\infty$ disjoint from $K$, so we are done.

Conversely, suppose $X^*$ is Hausdorff. Notice that for any open set $\mathcal{U} \subset X^*$, $\mathcal{U} \cap X$ is an open subset of $X$: indeed, this is obvious if $\infty \notin \mathcal{U}$, and otherwise we have $\mathcal{U} = (X \backslash K) \cup \{\infty\}$ for a closed set $K \subset X$, so that

$$\mathcal{U} \cap X = X \backslash K$$

is also open. Now given distinct points $x, y \in X$ with disjoint neighborhoods $\mathcal{U}_x \subset X^*$ of $x$ and $\mathcal{U}_y \subset X^*$, we obtain disjoint neighborhoods of these two points in $X$ by setting

$$\mathcal{U}'_x := \mathcal{U}_x \cap X \quad \text{and} \quad \mathcal{U}'_y := \mathcal{U}_y \cap X,$$

and this proves that $X$ is Hausdorff. Furthermore, for every $x \in X$, there is an open neighborhood $\mathcal{U} \subset X$ of $x$ which is disjoint from some open neighborhood $(X \backslash K) \cup \{\infty\}$ of $\infty$, where $K \subset X$ is closed and compact. Then $\mathcal{U} \cap (X \backslash K) = \varnothing$ implies $\mathcal{U} \subset K$, hence $K$ is a compact neighborhood of $x$, proving that $X$ is locally compact.

(f) We start with the following claim: if $K \subset \mathbb{Q}$ is a compact subset of $\mathbb{Q}$, then $K$ does not contain any nonempty open subset of $\mathbb{Q}$. This means in particular that $K$ does not contain any set of the form $\mathbb{Q} \cap (a, b)$ for real numbers $a < b$; note that since $\mathbb{Q}$ inherits its topology from $\mathbb{R}$ as a subspace, every open set in $\mathbb{Q}$ is a union of sets of this form. To prove the claim, suppose $a < b$ and $\mathbb{Q} \cap (a, b) \subset K \subset \mathbb{Q}$. Then there exists an irrational number $x \in (a, b)$ and a sequence of rational numbers $x_n \in \mathbb{Q} \cap (a, b)$ with $x_n \to x$, so $x_n$ is a sequence in $K$ that has no convergent subsequence with a limit in $K$. This proves that $K$ cannot be sequentially compact, and since it is a subset of the metric space $\mathbb{R}$, it follows that $K$ also cannot be compact.

Now suppose $x \in \mathbb{Q}$ has a neighborhood $\mathcal{U} \subset \mathbb{Q}^*$ that is disjoint from some neighborhood $(\mathbb{Q} \backslash K) \cup \{\infty\} \subset \mathbb{Q}^*$ of $\infty$, where $K \subset \mathbb{Q}$ is compact. Then $\infty \notin \mathcal{U}$, so $\mathcal{U}$ is an open subset of $\mathbb{Q}$, but $\mathcal{U} \cap (\mathbb{Q} \backslash K) = \varnothing$ implies $\mathcal{U} \subset K$ and thus contradicts the claim above.

(g) Suppose $x_n \in \mathbb{Q}^*$ is a sequence convergent to $x \in \mathbb{Q}$. Since $x$ has neighborhoods that do not contain $\infty$, we then have $x_n \neq \infty$ for sufficiently large $n$, so without loss of generality $x_n$ is a sequence in $\mathbb{Q}$. Since $\mathbb{Q}$ is Hausdorff, $x_n$ cannot also converge to some other point $y \neq x$ in $\mathbb{Q}$. Now observe that $K := \{x, x_1, x_2, x_3, \ldots\} \subset \mathbb{Q}$ is a compact subset of $\mathbb{Q}$ (we proved this in part (b) above), and therefore also closed since $\mathbb{Q}$ is Hausdorff. Thus $(\mathbb{Q} \backslash K) \cup \{\infty\}$ is a neighborhood of $\infty$ in $\mathbb{Q}^*$ that does not contain $x_n$ for any $n$, implying $x_n \not\to \infty$.

*Remark: $\mathbb{Q}$ is a metric space and therefore first countable, so each $x \in \mathbb{Q}$ has a countable neighborhood base, which can also serve as a countable neighborhood base of $x$ in $\mathbb{Q}^*$. But the result of this problem together with part (f) shows that $\mathbb{Q}^*$ cannot be first countable, as we proved in lecture that every non-Hausdorff first countable space has a sequence with two limits. It follows that $\infty$ is the unique point in $\mathbb{Q}^*$ that fails to have a countable neighborhood base.*

(h) We already observed that $\mathbb{Q}$ is not locally compact: in fact no nonempty open subset of $\mathbb{Q}$ is contained in a compact set, hence no neighborhood in $\mathbb{Q}$ of any point is compact. Since $\mathbb{Q}$ is also an open subset of $\mathbb{Q}^*$, $\mathbb{Q}$ is a neighborhood in $\mathbb{Q}^*$ of each $x \in \mathbb{Q}$, but the previous sentence implies that $\mathbb{Q}$ does not contain any compact neighborhood of $x$.

7. (a) Denote the metric on $Y$ by $d$ and suppose $f_n \in C(X, Y)$ is a sequence that converges uniformly on compact sets to $f \in C(X, Y)$. Concretely, this means that for every compact subset $K \subset X$ and every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(f_n(x), f(x)) < \epsilon$ for all $n \geqslant N$ and $x \in K$. To show that $f_n$ also converges to $f$ in the compact-open topology, we need to show that for every open set $V \subset Y$ such that $f \in \mathcal{U}_{K,V}$, we also have $f_n \in \mathcal{U}_{K,V}$ for all large $n$; equivalently, if $f(K) \subset V$, then $f_n(K) \subset V$ for all $n$ sufficiently large. The proof is based on the fact that since $K$ is compact, the distance from $f(K)$ to the complement of $V$ is positive. More generally, suppose $A, B \subset Y$ are disjoint closed subsets such that $A$ is compact, and write

$$d(A, B) := \inf_{x \in A} d(x, B) \quad \text{where} \quad d(x, B) := \inf_{y \in B} d(x, y).$$

Since $A \subset X \backslash B$ and the latter is open, $d(x, B) > 0$ for every $x \in A$. Now if $d(A, B) = 0$, then we can find a sequence $x_n \in A$ such that $d(x_n, B) \to 0$, and since $A$ is compact, we can replace $x_n$ with a subsequence that converges to some point $x \in A$. (We are using the fact that $A$ is a metric space so that compactness implies sequential compactness.) But $d(x, B) > 0$, so as soon as $n$ is large enough to have $x_n \in B_{\frac{1}{2}d(x,B)}(x)$, we find that for all $y \in B$,

$$d(x, B) \leqslant d(x, y) \leqslant d(x, x_n) + d(x_n, y)$$

and thus

$$d(x_n, y) \geqslant d(x, B) - d(x, x_n) > \frac{1}{2} d(x, B),$$

which is a contradiction since $d(x_n, B) \to 0$. This proves $d(A, B) > 0$. Applying this in the case at hand, we have

$$\epsilon := d(f(K), X \backslash V) > 0 \quad \text{and thus} \quad d(f(x), X \backslash V) \geqslant \epsilon > 0 \text{ for all } x \in K.$$

Now if $N \in \mathbb{N}$ is chosen so that $d(f_n(x), f(x)) < \epsilon$ for all $n \geqslant N$ and $x \in K$, it follows that $f_n(x) \in V$ for all $x \in K$, hence $f_n \in \mathcal{U}_{K,V}$ as claimed.

I will give two arguments to prove the converse: one that is relatively straightforward but works only if $X$ is first countable, and another that works in general but requires a bit more creativity.

For the first version, we prove that if $f_n$ does not converge to $f$ uniformly on all compact subsets, then it also does not converge in the compact-open topology. The assumption means there exists a compact subset $K \subset X$, a number $\epsilon > 0$, a sequence $x_n \in K$ and a subsequence $f_{k_n}$ of $f_n$ such that

$$d(f_{k_n}(x_n), f(x_n)) \geqslant \epsilon \quad \text{for all } n.$$

If $X$ is first countable, then $K$ (with the subspace topology) is also first countable and therefore sequentially compact, so we are free to replace $x_n$ with a subsequence that converges to some point $x \in K$. Let

$$V := B_{\epsilon/2}(f(x)) \subset Y.$$

Since $f : X \to Y$ is continuous, $f^{-1}(V) \subset X$ is an open neighborhood of $x$, so the convergence $x_n \to x$ implies $x_n \in f^{-1}(V)$ for all $n \geqslant N$ if $N \in \mathbb{N}$ is sufficiently large. Now $K' := \{x, x_N, x_{N+1}, x_{N+2}, \ldots\} \subset X$ is a compact subset satisfying $f(K') \subset V$, hence $f \in \mathcal{U}_{K',V}$. But for each $n \geqslant N$, the triangle inequality gives

$$d(f_{k_n}(x_n), f(x)) \geqslant d(f_{k_n}(x_n), f(x_n)) - d(f(x_n), f(x)) > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2},$$

implying $f_{k_n}(K') \not\subset V$ and thus $f_{k_n} \notin \mathcal{U}_{K',V}$ for all $n \geqslant N$. This proves that $f_n$ does not converge to $f$ in the compact-open topology.

Without assuming $X$ is first countable, we can still argue as follows. Assuming $f_n \to f$ in the compact-open topology, we need to show that for every compact $K \subset X$ and every $\epsilon > 0$, $\sup_{x \in K} d(f_n(x), f(x)) < \epsilon$ for all $n$ sufficiently large. To start with, note that for each $x \in K$, the continuity of $f$ allows us to choose an open neighborhood $\mathcal{V}_x \subset X$ of $x$ such that

$$f(\mathcal{V}_x) \subset B_{\epsilon/3}(f(x)).$$

It follows that[2]

$$f(\bar{\mathcal{V}}_x) \subset \overline{B_{\epsilon/3}(f(x))} \subset B_{\epsilon/2}(f(x)).$$

Since $K$ is compact and the sets $\mathcal{V}_x$ for $x \in K$ cover $K$, we can also find a finite set $x_1, \ldots, x_N \in K$ such that $K \subset \mathcal{V}_{x_1} \cup \ldots \cup \mathcal{V}_{x_N}$. Notice that in the subspace topology on $K$, the subsets $\bar{\mathcal{V}}_x \cap K \subset K$ are closed and are therefore also compact. Defining $K_i := \bar{\mathcal{V}}_{x_i} \cap K \subset X$ and $V_i := B_{\epsilon/2}(f(x_i)) \subset Y$ for each $i = 1, \ldots, N$, we then have an open set

$$\mathcal{U} := \bigcap_{i=1}^{N} \mathcal{U}_{K_i, V_i} \subset C(X, Y)$$

such that $f \in \mathcal{U}$, since $f(\bar{\mathcal{V}}_{x_i}) \subset B_{\epsilon/2}(f(x_i))$ for every $i = 1, \ldots, N$. Convergence in the compact-open topology then implies $f_n \in \mathcal{U}$ for all $n$ sufficiently large. Assuming this, we can find for every $x \in K$ some $i \in \{1, \ldots, N\}$ such that $x \in \mathcal{V}_{x_i}$, and it follows that $f_n(x) \in B_{\epsilon/2}(f(x_i))$, thus

$$d(f_n(x), f(x)) \leqslant d(f_n(x), f(x_i)) + d(f(x_i), f(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

[2] Here is a straightforward but necessary exercise: show that for any continuous map $f : X \to Y$ between topological spaces and any subset $\mathcal{U} \subset X$, $f(\bar{\mathcal{U}}) \subset \overline{f(\mathcal{U})}$.

(b) We need to find an example of a pointwise convergent sequence $f_n \to f \in C(X, Y)$ together with a convergent sequence $x_n \to x \in X$ such that $f_n(x_n)$ does not converge to $f(x)$. I suspect you have seen such an example before in analysis: e.g. let $X = Y = [0, 1]$, and take $f_n : [0, 1] \to [0, 1]$ to be any sequence of continuous functions such that $f_n|_{[1/n,1]} \equiv 0$ but $f_n(1/2n) = 1$. (Draw a picture: you need the graph of $f_n$ to have a sharp "bump" in the interval $[0, 1/n]$, with steeper slopes for larger values of $n$.) Then $f_n$ converges pointwise to $f := 0 \in C(X, Y)$ and $x_n := 1/2n \in X$ converges to $x := 0 \in X$, but $f_n(x_n) = 1$ does not converge to $f(x) = 0$.

(c) We need to show that if $f_n \to f \in C(X, Y)$ converges in the compact-open topology and $x_n \to x$ in $X$, then $\mathrm{ev}(f_n, x_n) = f_n(x_n) \to f(x) = \mathrm{ev}(f, x)$ in $Y$. Given an open neighborhood $V \subset Y$ of $f(x)$, we know $f^{-1}(V) \subset X$ is an open neighborhood of $x$ since $f$ is continuous, so the convergence of $x_n$ implies $x_n \in f^{-1}(V)$ for all $n \geqslant N$ if $N \in \mathbb{N}$ is sufficiently large. Consider the compact set $K := \{x, x_N, x_{N+1}, x_{N+2}, \ldots\} \subset X$, which by the previous remarks satisfies $f(K) \subset V$ and thus $f \in \mathcal{U}_{K,V}$. The convergence of $f_n$ then implies $f_n \in \mathcal{U}_{K,V}$ for all $n \geqslant N'$ for some $N' \in \mathbb{N}$ sufficiently large, in which case taking $n \geqslant \max\{N, N'\}$ gives

$$\mathrm{ev}(f_n, x_n) = f_n(x_n) \subset f_n(K) \subset V,$$

proving $\mathrm{ev}(f_n, x_n) \to \mathrm{ev}(f, x)$.

(d) Assuming $X$ is locally compact and Hausdorff, we need to show that for any open set $V \subset Y$, $\mathrm{ev}^{-1}(V) \subset C(X, Y) \times X$ is open, where $C(X, Y)$ carries the compact-open topology and the topology on $C(X, Y) \times X$ is the resulting product topology. Open sets in $C(X, Y) \times X$ are unions of "boxes" of the form $\mathcal{O} \times \mathcal{W}$ for $\mathcal{O} \subset C(X, Y)$ and $\mathcal{W} \subset X$ both open. Thus another way to frame the problem is that for any given open set $V \subset Y$ and $(f_0, x_0) \in \mathrm{ev}^{-1}(V)$, there exist open neighborhoods $\mathcal{O} \subset C(X, Y)$ for $f_0$ and $\mathcal{W} \subset X$ of $x_0$ such that $\mathrm{ev}(\mathcal{O} \times \mathcal{W}) \subset V$. To show this, choose $\mathcal{W} \subset X$ to be an open neighborhood of $x_0$ with compact closure $K := \overline{\mathcal{W}} \subset f_0^{-1}(V)$, which exists because $X$ is Hausdorff and locally compact. Then $\mathcal{U}_{K,V}$ is an open neighborhood of $f_0$ in $C(X, Y)$. For any $f \in \mathcal{U}_{K,V}$ and $x \in \mathcal{W}$, we then have

$$\mathrm{ev}(f, x) = f(x) \subset f(K) \subset V,$$

proving $\mathcal{U}_{K,V} \times \mathcal{W} \subset \mathrm{ev}^{-1}(V)$.

(e) In this problem we assume $C(X, Y)$ carries some topology $\mathcal{T}$ for which $\mathrm{ev} : C(X, Y) \times X \to Y$ is continuous, and the goal is to show that for every compact $K \subset X$ and open $V \subset Y$, the sets $\mathcal{U}_{K,V} \subset C(X, Y)$ which define the subbase of the compact-open topology also belong to $\mathcal{T}$. One way to do this is by showing that for any $f_0 \in \mathcal{U}_{K,V}$, there exists a set $\mathcal{O} \in \mathcal{T}$ that contains $f_0$ and is also a subset of $\mathcal{U}_{K,V}$; indeed, if this is true, then since the element $f_0 \in \mathcal{U}_{K,V}$ is arbitrary, $\mathcal{U}_{K,V}$ is a union of sets in $\mathcal{T}$ and must therefore also belong to $\mathcal{T}$.

To proceed, we know that since $V \subset Y$ is open and ev is continuous, $\mathrm{ev}^{-1}(V)$ is also open in the product topology on $C(X, Y) \times X$. By assumption $\mathrm{ev}^{-1}(V)$ contains $(f_0, x)$ for every $x \in K$. Being open in the product topology then means that for each $x \in K$, there exists a "box neighborhood" of $(f_0, x)$ in $\mathrm{ev}^{-1}(V)$, meaning a pair of open sets $\mathcal{O}_x \in \mathcal{T}$ and $\mathcal{U}_x \subset X$ such that

$$(f_0, x) \in \mathcal{O}_x \times \mathcal{U}_x \subset \mathrm{ev}^{-1}(V).$$

The sets $\{\mathcal{U}_x\}_{x \in K}$ then form an open cover of $K$, which has a finite subcover since $K$ is compact: choose $x_1, \ldots, x_N \in K$ such that

$$K \subset \bigcup_{i=1}^{N} \mathcal{U}_{x_i}.$$

The finite intersection

$$\mathcal{O} := \bigcap_{i=1}^{N} \mathcal{O}_{x_i} \subset C(X, Y)$$

then also belongs to the topology $\mathcal{T}$, and it contains $f_0$. Finally, we see that for any $f \in \mathcal{O}$ and $x \in K$, we have $x \in \mathcal{U}_{x_i}$ for some $i \in \{1, \ldots, N\}$, and $f \in \mathcal{O}_{x_i}$ then implies $f(x) \in V$ since $\mathcal{O}_{x_i} \times \mathcal{U}_{x_i} \subset \mathrm{ev}^{-1}(V)$. This establishes $f(K) \subset V$, thus $f \in \mathcal{U}_{K,V}$ and therefore $\mathcal{O} \subset \mathcal{U}_{K,V}$, completing the proof.

(f) When asked to prove that something is not continuous, the temptation is always to look for a sequence that violates sequential continuity, but in this case part (c) tells you that that will not work. Instead we use the definition of continuity directly. Arguing by contradiction, suppose $\text{ev} : C(\mathbb{Q}, \mathbb{R}) \times \mathbb{Q} \to \mathbb{R}$ is continuous, so for every open set $\mathcal{W} \subset \mathbb{R}$, $\text{ev}^{-1}(\mathcal{W}) \subset C(\mathbb{Q}, \mathbb{R}) \times \mathbb{Q}$ is open. Let us assume $\mathcal{W} \neq \mathbb{R}$. Here $C(\mathbb{Q}, \mathbb{R}) \times \mathbb{Q}$ carries the product topology with respect to the compact-open topology on $C(\mathbb{Q}, \mathbb{R})$, so for any $(f_0, x_0) \in \text{ev}^{-1}(\mathcal{W})$, openness means the existence of a box neighborhood

$$(f_0, x_0) \in \mathcal{O} \times \mathcal{U} \subset \text{ev}^{-1}(\mathcal{W}),$$

where $\mathcal{O} \subset C(\mathbb{Q}, \mathbb{R})$ and $\mathcal{U} \subset \mathbb{Q}$ are each open. By the definition of the compact-open topology, $\mathcal{O}$ is a union of finite intersections of sets of the form $\mathcal{U}_{K,V}$, so in particular there exists a finite collection of compact sets $K_i \subset \mathbb{Q}$ and open sets $V_i \subset \mathbb{R}$ indexed by $i = 1, \ldots, N$ such that

$$f_0 \in \bigcap_{i=1}^{N} \mathcal{U}_{K_i, V_i} \subset \mathcal{O}.$$

Let $K := K_1 \cup \ldots \cup K_N$ and observe that this is also a compact subset of $\mathbb{Q}$. As we observed in the solution to #6(f), compactness implies that $K$ cannot contain any nonempty open subset of $\mathbb{Q}$, so in particular there exist points in $\mathcal{U}$ that are not in $K$, and in fact the set of such points is open in $\mathcal{U}$ (and therefore open in $\mathbb{Q}$) since $K$ is closed in $\mathbb{Q}$. It follows that there exists a nonempty open interval $(a, b) \subset \mathbb{R}$ such that

$$(a, b) \cap \mathbb{Q} \subset \mathcal{U} \quad \text{and} \quad (a, b) \cap K = \emptyset.$$

Now we use the fact that no nonempty open set in $\mathbb{Q}$ is connected: for instance, $(a, b) \cap \mathbb{Q}$ can be split into three disjoint open subsets

$$(a, b) \cap \mathbb{Q} = I_- \cup I_0 \cup I_+$$

which are the intersections of $\mathbb{Q}$ with disjoint open intervals $(a, r_-)$, $(r_-, r_+)$ and $(r_+, b)$ respectively for some *irrational* numbers $r_\pm$. The point is that since $r_\pm \notin \mathbb{Q}$, $I_0$ is an open and closed subset of $\mathbb{Q}$. Finally, define a function $f \in C(\mathbb{Q}, \mathbb{R})$ such that $f(x)$ is a point in $\mathbb{R} \backslash \mathcal{W}$ for every $x \in I_0$ but $f(x) = f_0(x)$ everywhere else. Here we do not need to worry about any discontinuities at $x = r_\pm$ since these two points do not belong to the domain of $f$. Since $I_0 \cap K = \emptyset$, $f$ satisfies the same constraints $f(K_i) \subset V_i$ that $f_0$ does and thus belongs to $\bigcap_{i=1}^{N} \mathcal{U}_{K_i, V_i} \subset \mathcal{O}$, and since $I_0 \subset (a, b) \cap \mathbb{Q} \subset \mathcal{U}$, we have

$$(f, x) \in \mathcal{O} \times \mathcal{U}$$

for every $x \in I_0$. But by construction, $(f, x) \notin \text{ev}^{-1}(\mathcal{W})$, so this is a contradiction.

*Remark: We've just shown that* $\text{ev} : C(\mathbb{Q}, \mathbb{R}) \times \mathbb{Q} \to \mathbb{R}$ *is sequentially continuous but not continuous, thus* $C(\mathbb{Q}, \mathbb{R}) \times \mathbb{Q}$ *is not first countable. Since* $\mathbb{Q}$ *clearly is first countable, it follows easily that* $C(\mathbb{Q}, \mathbb{R})$ *is not.*