Talk 3: Framed Bordism & Pontrjagin-Thom Construction

Let M be a closed m-mfd, Y_0 & Y_1 be closed n-submfds of M. We want to define a bordism "within M".

(All mfds are assumed to be smooth)

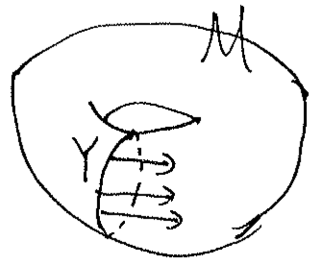
$Y \subset M$ submfd, there is a short exact seq of vector bundles:

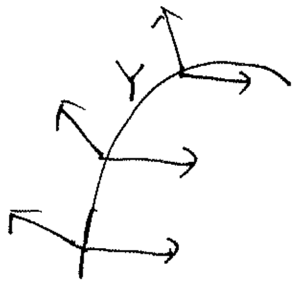
$$0 \to TY \to TM|_Y \to \nu \to 0.$$

ν is defined as the quotient bundle and is called the normal bundle.

<u>Def</u>: A framing of submfd $Y \subset M$ is a trivilization of ν.

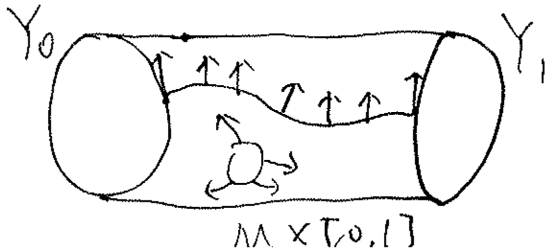
Rmk: Equivalently, a framing of $Y \subset M$ is a smooth function b which assigns to each $x \in Y$ a basis $b(x) = (v^1(x), \cdots, v^{m-n}(x))$ of $TY_x^{\perp} \subset TM_x$.





The pair (Y, b) is called a framed submfd of M.

<u>Def</u>: (Y_0, b_0) is framed bordant to (Y_1, b_1) in M if there is a framed submfd with boundary $X \subset [0,1] \times M$ s.t. $X \cap (\{i\} \times M) = Y_i$, $i = 0, 1$, and the framing match at the boundary.



Let $f: M \to N$ smooth, $p \in N$ regular value,

then $Y := f^{-1}(p)$ is a smooth submfd of M with an induced framin

given a basis $b = (v_1, \cdots, v_n)$ of $T_p N$, b pulls back to a basis of the normal bundle at each $y \in Y$.

denote $w = f^*b$, $(f^{-1}(p), f^*b)$ is called the Pontrjagin mfd associated with f.

Now assume $N = S^q$, we have the following Theorem.

Thm (Pontrjagin - Thom)

There is an isomorphism $[M, S^q] \to \Omega^{fr}_{m-q;M}$

where $[M, S^q]$ is the homotopy classes of maps from M to S^q, $\Omega^{fr}_{m-q,M}$ is the framed bordism classes of submfds with codim q within M.

Several things to show:

① take $[f] \in [M, S^q]$, $p \in S^q$ regular under f,

then $[f^{-1}(p)] \in \Omega^{fr}_{m-q;M}$:

Why is $[f] \mapsto [f^{-1}(p)]$ well-defined?

② How to construct an inverse $\Omega^{fr}_{m-q;M} \to [M, S^q]$.

i.e.: given a framed submfd Y, how can we construct a map $f: M \to S^q$ s.t. $Y = f^{-1}(p)$ for some regular value?

Following Milnor's book Chapter 7,

we have the following theorems:

Thm A: If p and p' are both regular values of f, b and b' are positively oriented basis for $T_p(S^q)$, $T_{p'}(S^q)$, respectively. then $(f^{-1}(p), f^*b)$ is framed bordant to $(f^{-1}(p'), f^*b')$.

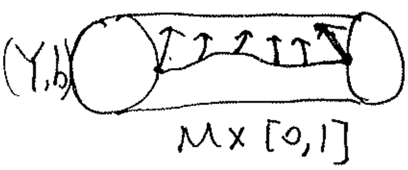
Thm B: Two mappings from M to S^q are smoothly homotopic if and only if the associated Pontrjagin mfds are framed bordant

Thm C: Any compact framed submfd (Y, w) of codim q in M occurs as Pontrjagin mfd for some smooth $f: M \to S^q$.

Lemma 1: b & b' two different positively oriented bases at p, then $(f^{-1}(p), f^*b)$ is framed bordant to $(f^{-1}(p), f^*b')$.

Pf: choose a path from b to b' in the space of positively oriented bases of $T_p S^q$, which can be identified as $GL^+(q, R)$

The path gives rise to the required framing.



$X = Y \times [0,1]$

$\square$

Lemma 2: p regular of f, $\tilde{p}$ close to p, then $f^{-1}(\tilde{p})$ is framed bordant to $f^{-1}(p)$.

Pf: $f(C) = \{\text{critical values}\}$ is compact.

$\exists\, \varepsilon > 0$, ε-nbhd of p contains only regular values.

choose a smooth one-parameter family of rotations,

$r_t: S^q \to S^q$, $r_1(p) = \tilde{p}$,

1) $r_t = Id$ for $0 \leq t < \varepsilon'$

2) $r_t = r_1$ for $t \in [1-\varepsilon', 1]$

3) each $r_t^{-1}(\tilde{p})$ is a regular value of f

then $F: M \times [0,1] \longrightarrow S^q$, $F(x,t) = r_t f(x)$

$\Rightarrow \check{p}$ is a regular value for F.

$F^{-1}(\check{p}) \subset M \times [0,1]$ is a framed mfd and gives a framed bordism between $f^{-1}(p)$ and $f^{-1}(\check{p})$. ▨

Lemma 3: f & g are smoothly homotopic and p regular for both, then $f^{-1}(p)$ is framed bordant to $g^{-1}(p)$.

Pf: choose a homotopy F, with $F(x,t) = f(x) \quad 0 \leqslant t < \varepsilon$

$$F(x,t) = g(x) \quad 1-\varepsilon < t \leqslant 1$$

choose a regular value $\tilde{p}$ for F which is close enough to p

s.t. $f^{-1}(\tilde{p})$ is framed bordant to $f^{-1}(p)$, $g^{-1}(\tilde{p})$ is framed bordant to $g^{-1}(p)$

then $F^{-1}(\check{p})$ is a framed mfd, provides a framed bordism between $f^{-1}(p)$ & $g^{-1}(p)$.

▨

Pf of Thm A: Give p & p' two regulare values for f, there is a family of rotations $r_t: S^q \to S^q$

$r_0 = Id$, $r_1(p) = p'$, which gives a homotopy from f to $r_1 \circ f$,

$\overset{\text{Lemma 3}}{\Longrightarrow}$ $f^{-1}(p')$ is framed bordant to $(r_1 \circ f)^{-1}(p') = f^{-1}(p)$. ▨

Pf of Thm C:

Let $Y \subset M$ be a compact submfd (without boundary)

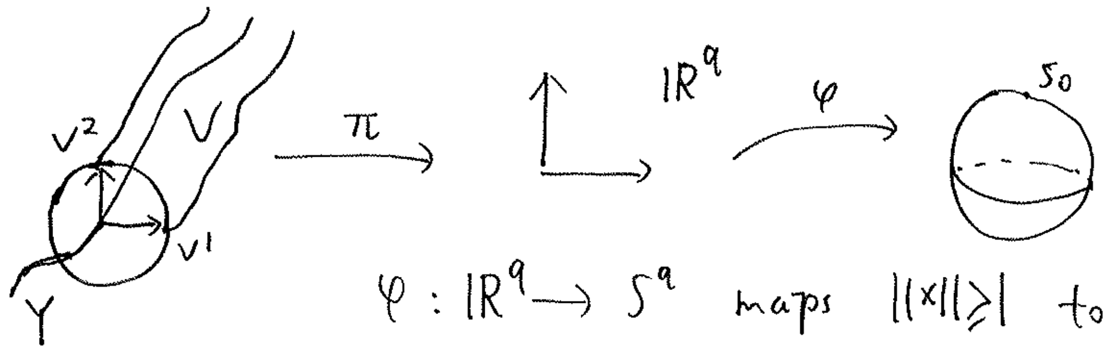
by tubular heighborhoods Thm,

$\exists$ nbhd V of Y, V is isomorphic to the normal bundle of Y

choose a base point $s_0 \in S^q$, define

$$f: \begin{array}{l} M \longrightarrow S^q \\ f(x) = s_0 \quad \forall x \notin V \\ f(x) = \varphi(\pi(x)) \text{ for } x \in V \end{array}$$

~~where $\varphi: \mathbb{R}^q \to S^q \setminus \{s_0\}$ is a diffeomorphism, $\pi: V \cong Y \times \mathbb{R}^q \xrightarrow{\text{proj}} \mathbb{R}^q$.~~



$\varphi : \mathbb{R}^q \to S^q$ maps $\|x\| \geq 1$ to s_0;
and $\varphi|_{B_1(0)}$ is diffeomorphic to $S^q \setminus \{s_0\}$.

$\pi : V \cong Y \times \mathbb{R}^q \longrightarrow \mathbb{R}^q$ is the projection.

$\varphi(0)$ is a regular value of f,
since $f^{-1}(\varphi(0)) = \pi^{-1}(0) = Y$, this proves Thm C. ☒

To prove Thm B, we show that the Pontrjagin mfd of a map determines its homotopy class.

Let $f, g : M \to S^q$ smooth, with a common regular value p.

<u>Lemma 4</u>: If $(f^{-1}(p), f^* b)$ is equal to $(g^{-1}(p), g^* b)$ then f is smoothly homotopic to g.

<u>Pf</u>: Write $Y = f^{-1}(p)$. $f^* b = g^* b$ means $df_x = dg_x, \forall x \in Y$.

If f coincides with g in a nbhd V of Y;

let $h : S^q \setminus \{p\} \to \mathbb{R}^q$ stereographic proj.

$$F(x,t) = f(x) \qquad x \in V$$

$$F(x,t) = h^{-1}[t\, h(f(x)) + (1-t)\, h(g(x))] \qquad x \in M \setminus Y.$$

defines a smooth homotopy between f & g.

Thus it suffices to deform f so that it coincides g in a small nbhd of Y, without mapping any new points into p during deformation.

choose V small nbhd of Y s.t. $V \overset{\sim}{=} Y \times \mathbb{R}^q$, and

$f(V), g(V)$ do not contain $-P$. , identify $S^q - \{-p\}$ with $\mathbb{R}^q$

we get $F, G: Y \times \mathbb{R}^q \longrightarrow \mathbb{R}^q$

$$F^{-1}(0) = G^{-1}(0) = Y \times \{0\}.$$

$$dF_{(x,0)} = dG_{(x,0)} = (\text{projection to } \mathbb{R}^q) \quad \forall x \in Y.$$

$\|F(x,u) - u\| \leqslant c_1 \|u\|^2$ for $\|u\| \leqslant 1$, some $c_1 > 0$ (Taylor expansion)

$\Rightarrow \langle F(x,u), u \rangle \geqslant \|u\|^2 - c_1\|u\|^3 > 0$ for $0 < \|u\| < \min(c_1^{-1}, 1)$

Similarly, $\langle G(x,u), u \rangle > 0$ for $0 < \|u\|$ small enough.

$\Rightarrow$ the homotopy $(1-t)F(x,u) + tG(x,u)$ does not map

new points to 0 for $\|u\| \leqslant c$.

choose a cut-off function $\lambda : \mathbb{R}^p \rightarrow \mathbb{R}$: $\quad \lambda(u) = 1 \quad \|u\| \leqslant \frac{c}{2}$

$\lambda(u) = 0 \quad \|u\| \geqslant c$.

then $F_t(x,u) = [1 - \lambda(u)t] F(x,u) + \lambda(u) t \, G(x,u)$.

deforms $F_0 = F$ to F_1

F_1 satisfies (1) $F_1 = G$ for $\|u\| < \frac{c}{2}$

(2) $F_1 = F$ for $\|u\| \geqslant c$

(3) F_1 has no new zeros $\quad$ ////

Pf of Thm B

If f & g are smoothly homotopic,

Lemma 3 $\Longrightarrow$ $f^{-1}(p)$ & $g^{-1}(p)$ framed bordant.

Conversely, given a framed bordism (X, ω) between $f^{-1}(p)$ & $g^{-1}(p)$, we can construct a homotopy

$$F: M \times [0,1] \longrightarrow S^q$$

with $(F^{-1}(y), F^* b) = (X, \omega)$ (analogous to Thm C)

Let $F_t(x) = F(x,t)$, then F_0 and f have the same Pontrjagin mfd

$\overset{\text{Lemma 4}}{\Longrightarrow}$ $F_0 \sim f$, similarly $F_1 \sim g$. $\Rightarrow f \sim g$. ////

<u>Rmk</u>: The Thms can be generalized to mfds with bdrys (See Freed's lecture Notes)

Application of Pontrjagin - Thom construction

<u>The Hopf Thm</u> Let M be a closed connected mfd of dim m.

(1) If M is orientable, then $[M, S^m] \cong \mathbb{Z}$ given by the integer degree

(2) If M is not orientable, then $[M, S^m] \cong \mathbb{Z}_2$ given by the mod 2 degree.

Notice that framed submfds of codim m in M are finite sets of points with preferred basis at each

when M is orientable, each basis gives a sign, $sgn(x) = \pm 1$,

$\sum sgn(x) = \deg: M \to S^m$.

when M non-orientable, the frame bundle is connected, thus $\{x\} \amalg \{y\}$ is null bordant,

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