Freudenthal suspension theorem

defn: (operations on pointed spaces) $(X_i, *_i)$ pointed spaces $(i \in \{1,2\})$

(i) The wedge product:

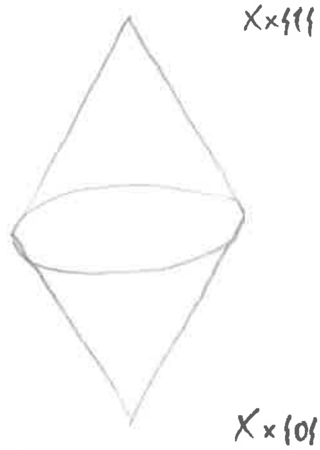
$$X_1 \vee X_2 := X_1 \amalg X_2 / *_1 \amalg *_2$$

(ii) The smash product:

$$X_1 \wedge X_2 := X_1 \times X_2 / X_1 \vee X_2$$

(iii) The suspension of X:

$$\Sigma X := S^1 \wedge X$$



remarks: (i) Freed's suspension $\neq$ Hatcher's suspension:

$$SX := (X \times I) / (x_1, 0) \sim (x_2, 0) \text{ and } (x_1, 1) \sim (x_2, 1)$$

i.e.:

$\neq$

$$\Sigma X := S^1 \wedge X \overset{\text{homeo}}{\cong} (X \times I) / (X \times \{0\}) \amalg (X \times \{1\}) \amalg (\{x\} \times I)$$

base point.

(ii) If X_1, X_2 compact + Hausdorff $\Rightarrow$ $X_1 \wedge X_2 \overset{\text{homeo}}{\cong} ((X_1 \setminus \{x_1\}) \times (X_2 \setminus \{x_2\})) \cup \{\infty\}$

one point compactification.

pf: the map:

$$\underbrace{X_1 \setminus \{x_1\} \times X_2 \setminus \{x_2\} \overset{i}{\hookrightarrow} X_1 \times X_2 \xrightarrow{\pi} X_1 \wedge X_2}_{\pi \circ i}$$

is a homeomorphism onto $\mathrm{im}(\pi \circ i)$ which is $X_1 \wedge X_2$ without the base point

$$\Rightarrow ((X_1 \setminus \{x_1\}) \times (X_2 \setminus \{x_2\})) \cup \{\infty\} \overset{\text{homeo}}{\cong} X_1 \wedge X_2$$

$\Rightarrow$ in particular: $S^n \wedge S^m \cong \mathbb{R}^{n+m} \cup \{\infty\} \cong S^{n+m}$

(iii) $\exists$ induced maps: If $f_i : X_i \to Y_i$ are pointed maps of pointed spaces we have $f_1 \vee f_2 : X_1 \vee X_2 \longrightarrow Y_1 \vee Y_2$ defined the natural way.

$\rightsquigarrow$ $f_1 \wedge f_2$ induced by the natural map on the cartesian product $(f_1 \times f_2)$ and $f_1 \vee f_2$.

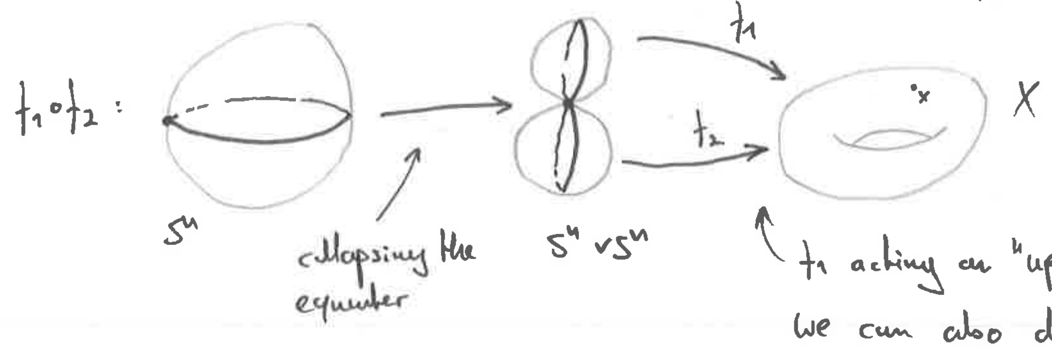
defn: (n^{th} homotopy group): Let $(X, *)$ be a pointed space and $n \in \mathbb{N}$

$$\pi_n(X, *) := [(S^n, *), (X, *)]$$

the n^{th} homotopy group.

remarks: (i) S^n has a "natural base point" (boundary of collapsed Disk)

(ii) group structure: for f_1, f_2 representatives of $[f_1], [f_2] \in \pi_n(X, *)$ can define:



f_1 acting on "upper" S^n, f_2 on "lower"

we can also define this as $f_1 \vee f_2 : S^n \vee S^n \to X \vee X$ but then you have to fold X on X to obtain the desired map.

$\Rightarrow$ (I) associative

(II) constant map is neutral element

(III) $\exists$ inverse maps

(IV) composition is commutative for $n \geq 2$

Freudenthal suspension theorem:

there is a sequence of group homomorphisms:

$$[S^m, S^q] \xrightarrow{\Sigma} [S^{m+1}, S^{q+1}] \xrightarrow{\Sigma} [S^{m+2}, S^{q+2}] \xrightarrow{\Sigma} \dots \qquad m \geq q$$

which stabilizes in the sense that all but fin. many maps are isomorphisms.

remark: (i) $[S^m, S^q]$ means we are working with base points but the group is independent of choice of basepoint (see lemma below)

(ii) the connecting homomorphism is defined the following way: (consider $[S^m, S^q]$ as the start)

(I) apply suspension to both spaces (and use $\Sigma X = S^1 \wedge S^m = S^{m+1}$, $X = S^m$, after using the induced map)

(II) for f rep. of $[f] \in [S^m, S^q]$ use:

$$id \wedge f : \Sigma S^m \to \Sigma S^q .$$

(III) proceed inductively.

lemma: We can introduce basepoints without changing the groups i.e.:

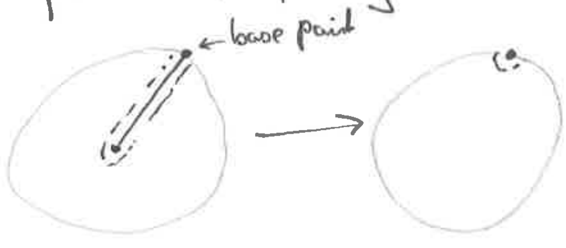
if $m,q \geq 1$ then $[S^m, S^q]_* = [S^m, S^q]$

Pf: every base point preserving map is in particular a map therefore we can use the ~~inj~~ inclusion $[S^m, S^q]_* \hookrightarrow [S^m, S^q]$

surj: Suppose $f: S^m \to S^q$. We can compose f with a continuous family of rotations from $R_0 = id$ to R_1, a rotation which maps $f(*) \in S^q$ to $* \in S^q$.

$\to$ Found homotopy between f and a pointed map $\hat{f}$

inj: If $F: D^{m+1} \to S^q$ is a null homotopy of a pointed map $f: S^m \to S^q$ precompose with the following h.e.



$\to$ found a pointed homotopy to the constant map.

$\to [S^m, S^q]_* = [S^m, S^q]$

$\to$ Can rewrite the Freudenthal sequence as:

$$\pi_m S^q \xrightarrow{\Sigma} \pi_{m+1} S^{q+1} \xrightarrow{\Sigma} \pi_{m+2} S^{q+2} \xrightarrow{\Sigma} \dots$$

Q: Is there some sort of limiting group?

defn: (direct limit) $(I, \leq)$ a directed set; $\{A_i \mid i \in I\}$ a family of objects indexed by I and $f_{ij}: A_i \to A_j$ morphisms $\forall i \leq j$ s.t.:

(i) $f_{ii} = id_{A_i}$

(ii) $f_{ik} = f_{jk} \circ f_{ij} \quad \forall \quad i \leq j \leq k$

then the direct limit is defined as $\varinjlim A_i := \begin{cases} \coprod_{i \in I} A_i/\sim & obj = \text{top. spaces} \\ \bigoplus_{i \in I} A_i/\sim & obj = \text{groups.} \end{cases}$

where if $x_i \in A_i$, $x_j \in A_j$ then:

$$x_i \sim x_j \iff \exists\, k \in I \text{ with } i \leq k \text{ and } j \leq k \text{ s.t. } f_{ik}(x_i) = f_{jk}(x_j)$$

defn: (n^{th} stable homotopy group of spheres)

$$\pi_n^S := \lim_{q \to \infty} \pi_{n+q} S^q$$

(groups: $\pi_{n+q} S^q \quad q \in \mathbb{N}$

mor: Σ

Q: Is there a top. space X s.t. $\pi_n^S = \pi_n X$

defn (based loop space) Let $(X,*)$ be a pointed space. The based loop space of $(X,*)$ is the set of continuous maps:

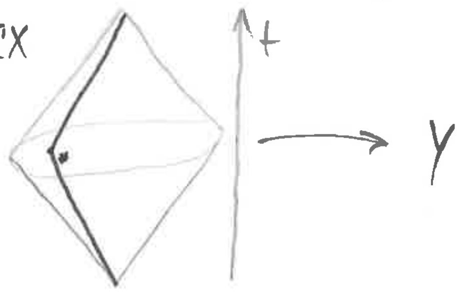
$$\Omega X = \{ \gamma : S^1 \to X \mid \gamma(*) = * \}$$

(we can introduce a topology on ΩX namely the compact-open topology)

lemma: Let X, Y be pointed spaces, then there is a isomorphism of sets:

$$\mathrm{Map}_*(\Sigma X, Y) \xrightarrow{\cong} \mathrm{Map}_*(X, \Omega Y)$$

pf: ΣX



$\forall x \in X$ we get a map: $\gamma : S^1 \to Y$ where S^1 is sitting inside ΣX indexed by t as shown in the picture

$\Rightarrow$ isomphism.

note: If we introduce a topology we can make this map to a homeo.

$\Rightarrow$ We can rewrite the Freudenthal sequence as:

$$\pi_n(S^0) \to \pi_n(\Omega S^1) \to \pi_n(\Omega^2 S^2) \to$$

$$\text{(start from } [S^n, S^0] \xrightarrow{\Sigma} [\Sigma S^n, \Sigma S^0] \overset{\text{lemma}}{=} [S^n, \underbrace{\Omega \Sigma S^0}_{\cong \Omega S^1}] \xrightarrow{\Sigma} \dots)$$