TOPICS IN TOPOLOGY — 09/05/18

12:00

Def. X Hausdorff.
X compactly gen. $:\Longleftrightarrow$ [$A \subseteq X$ closed iff $A \cap C$ closed $\forall C \subseteq X$ compact]

Note: Topology on ΩX is done s.t. it is comp. gen.

Lemma 1

Suppose 1) X_q, $q \in \mathbb{N}$ Hausdorff and compactly gen.
2) $\forall q$, $f_q : X_q \to X_{q+1}$ closed inclusion

$\Rightarrow$ $X = \operatorname{colim}_{q \to \infty} X_q$ is comp. gen and
every compact subset is contained in X_q for some q.

Proof: Use the fact that the topology on
is made such that

$$g_p : X_p \hookrightarrow X$$

are continuous.
This tells you that $A \subseteq X$ closed iff
$A \cap X_q \subseteq X_q$ closed for all q.

The rest follows from unraveling the
definitions and using the assumption
$f_q : X_q \hookrightarrow X_{q+1}$ closed $\forall q$.

$\square$

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Def. Consider the sequence

$$S^0 \longrightarrow \Omega S^1 \longrightarrow \Omega^2 S^2 \longrightarrow \cdots$$

Note 1) $\Sigma S^q = S^1 \wedge S^q \cong S^{q+1}$, so
$\Omega^{q+1} S^{q+1} = \Omega^{q+1} \Sigma S^q = \Omega^q \Omega \Sigma S^q$.

2) If we have a cont. inclusion $X \hookrightarrow Y$, we get a cont. inclusion $\Omega X \hookrightarrow \Omega Y$

3) We always have a cont. inclusion
$X \hookrightarrow \Omega \Sigma X$.

So we first get a cont. inclusion $S^q \hookrightarrow \Omega \Sigma S^q = \Omega S^{q+1}$ and then inductively $\Omega^q S^q \hookrightarrow \Omega^q \Omega S^{q+1} = \Omega^{q+1} S^{q+1}$.

$\rightsquigarrow$ Can apply colimit of top. spaces and set

$$QS^0 := \operatorname*{colim}_{q \to \infty} \Omega^q S^q$$

called the 0-space of the sphere spectrum.

Prop. With the definitions given before,

$$\boxed{\pi_n^S = \pi_n(QS^0)\,.}$$

Proof: First note that (by definition) $\Omega^q S^q$ is compactly gen. and S^q is compactly gen. (since it is a smooth manifold.)

$\Rightarrow$ suffices to show that for comp. gen. Hausdorff spaces X_q,

$$\pi_n(\operatorname{colim}_{q\to\infty} X_q) \cong \operatorname{colim}_{q\to\infty} \pi_n X_q \quad (*)$$

Write $X = \operatorname{colim}_{q\to\infty} X_q$.

~~The natural map~~
Consider the natural map

$$\varphi: \operatorname{colim}_{q\to\infty} \pi_n X_q \longrightarrow \pi_n(\operatorname{colim}_{q\to\infty} X_q)$$

Have cont. inclusions $g_q : X_q \hookrightarrow X = \operatorname{colim} X_q$.

φ is defined as follows: Take some $[f] \in$ LHS.

Then $f: S^n \to X_q$ for some q.

$\rightsquigarrow$ get a map $g_q \circ f: S^n \to X$.

$\Rightarrow \varphi([f]) := [g_q \circ f] \in \pi_n X$.

This is well-defined and a group hom.

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<u>surjective</u>: Suppose $[f] \in \pi_n(X)$, $f: S^n \to X$ cont.

Lem. $\Rightarrow$ $\operatorname{im} f \subseteq X$ comp. $\Rightarrow$ $\operatorname{im} f \subseteq X_q$ for some q.

~~Bdk~~ $\Rightarrow$ Have $\tilde{f}: S^n \to X_q$ st. $g_q \circ \tilde{f} = f$, thus

$[f] = [g_q \circ \tilde{f}] = \varphi([\tilde{f}])$.

<u>injective</u>: Suppose we have some $f: S^n \to X_q$ with

$\varphi([f]) = [g_q \circ f] = [\text{const.}]$.

So we have a homotopy between

$g_q \circ f: S^n \to X$ and const. $: S^n \to X$

So a cont. map $H: S^n \times I \to X$.

$H(\cdot, 0) = g_p \circ f$, $H(\cdot, 1) = \text{const.}$

Again, by lemma 1 $\operatorname{im} H \subseteq X_r$ for some r

(since $S^n \times I$ <u>compact</u>) $\Rightarrow$ Can write this as

$$S^n \times I \xrightarrow{\tilde{H}} X_r \hookrightarrow X$$

Can assume q to be minimal, ie st. $\nexists\, s < q$,

$\hat{f}: S^n \to X_s$, $f = f_{q-1} \circ \dots \circ f_s \circ \hat{f}$. $\boxed{\Rightarrow r \geq q.}$

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In the colimit, we have

$$t[f] = [f_{r-1} \circ \dots \circ f_q \circ f]$$

And we have just seen that

$$\tilde{H} : S^n \times I \longrightarrow X_r$$

is a homotopy between $f_{r-1} \circ \dots \circ f_q \circ f$ and const.

$\Rightarrow [f_{r-1} \circ \dots \circ f_q \circ f] = [\mathrm{const}] = 0$ in $\pi_n X_r$

$\Rightarrow [f] = [f_{r-1} \circ \dots \circ f_q \circ f] = 0$ in $\operatorname{colim}_{q \to \infty} \pi_n X_q$.

$\Rightarrow \varphi$ is injective.

Stabilization of framed submanifolds

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Recall:

<u>Thm</u> (Pontrjagin-Thom)

M a smooth compact MF, $m = \dim M$.

$$\Rightarrow \quad [M, S^q] \xrightarrow{\sim} \Omega^{fr}_{m-q, M} \quad \cancel{\text{...}}.$$

Use this for $M = S^m$, so

$$\phi : [S^m, S^q] \xrightarrow{\sim} \Omega^{fr}_{m-q, S^m} .$$

Thus the sequence

$$[S^m, S^q] \xrightarrow{\Sigma} [S^{m+1}, S^{q+1}] \xrightarrow{\Sigma} [S^{m+2}, S^{q+2}] \xrightarrow{\Sigma} \dots$$

becomes $\quad (\sigma = \phi \circ \Sigma \circ \phi^{-1})$:

$$\Omega^{fr}_{m-q, S^m} \xrightarrow{\sigma} \Omega^{fr}_{m-q, S^{m+1}} \xrightarrow{\sigma} \Omega^{fr}_{m-q, S^{m+2}} \xrightarrow{\sigma} \dots$$

Recall that elements in Ω^{fr}_{m-q, S^m} are framed bordism classes of submanifolds of $S^m = \mathbb{A}^m \cup \{\infty\}$, each represented by a framed submanifold $Y \subseteq S^m$. WLOG $\infty \notin Y$, so $Y \subseteq \mathbb{A}^m$.

This identification tells us the following:

1) Ω^{fr}_{m-q, S^n} is an <u>abelian</u> group.
Addition in this group amounts to a disjoint union of submanifolds of $\mathbb{A}^m$.

2) Explicitly, σ is the map

$$\begin{array}{ccc} Y & \longmapsto & 0 \times Y \\ \cap & & \cap \\ \mathbb{A}^m & & \mathbb{A}^1 \times \mathbb{A}^m \end{array}$$

For a given framing of $Y \subseteq \mathbb{A}^m$, i.e. a ~~[illegible]~~ global basis of sections of the normal bundle, we get a framing of $0 \times Y \subseteq \mathbb{A}^{m+1}$ by adding the constant vector field $\frac{\partial}{\partial x_1}$ to this basis.

<u>Question</u>: Does this sequence stabilize?

<u>A</u>: It does, but to see this, we need a little more machinery.

~~Thm~~ Embeddings of compact MFs and the stabilization theorem

Def. Y a compact MF.

An *isotopy* of embeddings $Y \hookrightarrow \mathbb{A}^N$ is a smooth map

$$I : \Delta^1 \times Y \hookrightarrow \mathbb{A}^N$$

st. $I|_{\{t\} \times Y}$ is an embedding $\forall t \in \Delta^1$.

Note: For compact MF, every embedding is an injective immersion.

Thm Y a smooth comp. n-MF.

(i) (Whitney) $\exists$ embedding $i : Y \hookrightarrow \mathbb{A}^{2n+1}$.
If $Y \hookrightarrow \mathbb{A}^N$ an embedding with $N > 2n+1$,
$\exists$ isotopy of i to an embedding into an affine subspace $\mathbb{A}^{2n+1} \subseteq \mathbb{A}^N$.

(ii) $i_0, i_1 : Y \hookrightarrow \mathbb{A}^{2n+1}$ embeddings

$\Rightarrow$ Their stabilizations

$$\tilde{i}_k : Y \to \mathbb{A}^{2n+1} \times \mathbb{A}^{2n+1} \qquad k = 0, 1$$
$$y \mapsto (0, i_k(y))$$

are isotopic.

(iii) X a comp. $(n+1)$-MF with boundary.
$\Rightarrow \exists$ embedding $X \hookrightarrow \mathbb{A}^{2n+3}$
as a <u>neat</u> submanifold with boundary.

<u>Proof</u>: (i), (iii) as ~~[illegible]~~ blade boxes

(ii) use $\Delta^1 \cong [0,3]$. Define

$$I(t,y) := \begin{cases} (t\, i_0(y),\, i_0(y)) & t \in [0,1] \\ (i_0(y),\, (2-t)i_0(y) + (t-1)i_1(y)) & t \in [1,2] \\ ((3-t)i_0(y),\, i_1(y)) & t \in [2,3]. \end{cases}$$

so $(0, i_0) \rightsquigarrow (i_0, i_0) \rightsquigarrow (i_0, i_1) \rightsquigarrow (0, i_1)$.
This is an isotopy between $\tilde{i}_0$ and $\tilde{i}_1$.

We use this to prove the following thm:

~~Thm The map [illegible]~~

$$\boxed{\text{Thm The map } \sigma: \Omega^{fr}_{n,S^m} \to \Omega^{fr}_{n,S^{m+1}} \text{ is an isomorphism for } \underline{m \geq 2n+2}.}$$ (*)

Proof: (sketch)

Surjectivity:

Suppose $[Y] \in \Omega^{fr}_{n,S^{m+1}}$. Y n-MF.

Suppose we have an embedding $i_0 : Y \hookrightarrow \mathbb{A}^{m+1}$

(i) tells us that ($m+1 > m \geq 2n+2$) $\exists$ isotopy

$\Delta^1 \times Y \to \mathbb{A}^{m+1}$ to an embedding $i_1 : Y \hookrightarrow \mathbb{A}^m$.

This gives the preimage under σ.

Injectivity: Suppose $j_0 : Y \hookrightarrow \mathbb{A}^m$ is an embedding, $k_0 : X \hookrightarrow \mathbb{A}^{m+1}$ a null bordism of the composition $Y \xrightarrow{j_0} \mathbb{A}^m \subseteq \mathbb{A}^{m+1}$.

Then (iii) tells us there is an isotopy $k_t : \Delta^1 \times X \to \mathbb{A}^{m+1}$ s.t. $k_1(X) \subseteq \mathbb{A}^m$ is a null bordism of j_0.

Statement on the level of framings is taken as a black box. (nontrivial!) $\square$

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Get the Freudenthal suspension thm as a corollary:
Recall that we identified the sequences

$$[S^m, S^q] \xrightarrow{\overline{\Sigma}} [S^{m+1}, S^{q+1}] \xrightarrow{\overline{\Sigma}} [S^{m+2}, S^{q+2}] \xrightarrow{\overline{\Sigma}} \cdots$$

and

$$\Omega^{fr}_{m-q, S^m} \xrightarrow{\sigma} \Omega^{fr}_{m-q, S^{m+1}} \xrightarrow{\sigma} \Omega^{fr}_{m-q, S^{m+2}} \xrightarrow{\sigma} \cdots$$

via Pontryagin - Thom.

So rewriting (*) gives

<u>Corollary</u> (Freudenthal suspension thm)
The map $\overline{\Sigma} : [S^m, S^q] \to [S^{m+1}, S^{q+1}]$
is an isomorphism for $m \leq 2q-2$.

<u>Proof</u>:
$n = m-q$, so $m \geq 2n+2 \iff m \geq 2(m-q)+2$
$\iff m \leq 2q-2$.

□

<u>Note</u>: This might equivalently be stated as a statement about

$$\Sigma : \pi_m S^q \longrightarrow \pi_{m+1} S^{m+1}.$$