Lecture 5: The stable framed bordism ring

The ring structure of $\pi_\bullet^s$

Recall: Last week, we looked at the following sequence of group homomorphisms of (abelian) homotopy groups:

$$(*) \quad \pi_m S^q \xrightarrow{\Sigma} \pi_{m+1} S^{q+1} \xrightarrow{\Sigma} \pi_{m+2} S^{q+2} \xrightarrow{\Sigma} \cdots \quad m \geq q > 0,$$

where Σ denotes the suspension applied to both, spaces and maps. ($\Sigma X = S^1 \wedge X$)

We then defined the nth stable homotopy group of the sphere, or the nth stable stem π_n^s, as the colimit of the sequence $(*)$, i.e.

$$\pi_n^s = \operatorname*{colim}_{q \to \infty} \pi_{n+q} S^q.$$

This is an abelian group for any $n \in \mathbb{N}$, and according to Freudenthal, we have isomorphisms

$$\pi_{n+q} S^q \longrightarrow \pi_n^s \quad \text{for } q \gg 1,$$

because the sequence $(*)$ stabilizes.

Prop. $\pi_\bullet^s := \bigoplus_{n \geq 1} \pi_n^s$ is a $\mathbb{Z}$-graded commutative ring.

Proof: • As the direct sum of abelian groups, $\pi_\bullet^s$ certainly is an abelian group.

• Consider two homogeneous elements in $\pi_\bullet^s$, say $a_1 \in \pi_{n_1}^s$, $a_2 \in \pi_{n_2}^s$.

From above, it follows that there exist $q_1, q_2 \gg 1$, s.t.

$$\pi_{n_1}^s \cong \pi_{n_1+q_1} S^{q_1}, \quad \pi_{n_2}^s \cong \pi_{n_2+q_2} S^{q_2} \quad \text{and}$$

$$\pi_{n_1+n_2}^s \cong \pi_{n_1+n_2+q_1+q_2} S^{q_1+q_2}.$$

Thus, a_1 and a_2 can be identified with homotopy classes with representatives

$$f_1 : S^{q_1+n_1} \longrightarrow S^{q_1} \quad \text{and} \quad f_2 : S^{q_2+n_2} \longrightarrow S^{q_2}.$$

Then,
$$\begin{array}{ccc} f_1 \wedge f_2 : S^{q_1+n_1} \wedge S^{q_2+n_2} & \longrightarrow & S^{q_1} \wedge S^{q_2} \\ \wr\!\!\!\text{| homeo} & & \wr\!\!\!\text{| homeo} \\ S^{q_1+q_2+n_1+n_2} & & S^{q_1+q_2} \end{array}$$

$\Rightarrow [f_1 \wedge f_2] \in \pi_{n_1+n_2+q_1+q_2} S^{q_1+q_2} \cong \pi^s_{n_1+n_2}$

Therefore, we identify the product $a_1 \cdot a_2 \in \pi^s_\bullet$ with the homotopy class of the smash product in order to obtain a well-defined multiplication on $\pi^s_\bullet$.

(extend to inhomogeneous elements)

As seen in the last talk, it is commutative.

Furthermore it satisfies $\pi^s_{n_1} \cdot \pi^s_{n_2} \subset \pi^s_{n_1+n_2}$, and thus, $\pi^s_\bullet = \bigoplus_{n \geq 1} \pi^s_n$ is a $\mathbb{Z}$-graded commutative ring. $\square$

Stable framings

Def: Consider a short exact sequence of vector bundles over a (smooth) manifold Y,

$$0 \longrightarrow E' \xrightarrow{i} E \underset{s}{\overset{j}{\rightleftarrows}} E'' \longrightarrow 0.$$

A splitting of this sequence is a linear map $s: E'' \longrightarrow E$ s.t. $j \circ s = id_{E''}$.

Note: A splitting s determines an isomorphism of vector bundles

$$s \oplus i : E'' \oplus E' \xrightarrow{\sim} E.$$

Rem: A splitting as defined above (of a s.e.s. of vector bundles over a smooth manifold) always exists.

(Pf: partition of unity, Riemannian metric)

Notation: We denote by $\underline{U}$ the constant vector bundle with fiber U.

Recall: M a closed manifold, $Y \subset M$ submanifold. S.e.s. of vector bundles:

$$0 \longrightarrow TY \longrightarrow TM|_Y \longrightarrow \underset{\substack{\uparrow \\ \text{normal bundle}}}{\nu} \longrightarrow 0$$

A framing of Y is a trivialization of the normal bundle ν, i.e. an isomorphism of vector bundles $\underline{\mathbb{R}^q} \xrightarrow{\sim} \nu$, where q is the codimension of Y in M.

Def: Let Y be a (smooth) manifold and $E \longrightarrow Y$ a vector bundle of rank q, i.e. all the fibers are of dimension q.

(a) A stable framing or stable trivialization of $E \longrightarrow Y$ is an isomorphism of vector bundles $\phi : \underline{\mathbb{R}^{k+q}} \xrightarrow{\sim} \underline{\mathbb{R}^k} \oplus E$ for some $k \geq 0$.

(b) A homotopy of stable framings is a homotopy of the isomorphism ϕ.

(c) Identifying ϕ with $id_{\underline{\mathbb{R}^\ell}} \oplus \phi : \underline{\mathbb{R}^{\ell+k+q}} \xrightarrow{\sim} \underline{\mathbb{R}^{\ell+k}} \oplus E$, we define a set of homotopy classes of stable framings.

Note: Every framing of a submanifold $Y \subset M$ is a stable framing of its normal bundle, i.e. a stable normal framing of Y.

Ex: (The stable framing of the tangent bundle of the sphere) Consider the standard unit sphere S^m as a submanifold of $\mathbb{R}^{m+1}$. Then the vector field

$$\sum_i x_i \frac{\partial}{\partial x_i},$$

restricted to S^m, gives a trivialization of the normal bundle ν to $S^m \subset \mathbb{R}^{m+1}$, i.e. $\underline{\mathbb{R}} \xrightarrow{\sim} \nu$.
We have the following s.e.s.

$$0 \longrightarrow TS^m \longrightarrow T\mathbb{R}^{m+1}|_{S^m} \longrightarrow \nu \longrightarrow 0,$$

which becomes

$$0 \longrightarrow TS^m \longrightarrow \underline{\mathbb{R}^{m+1}} \longrightarrow \underline{\mathbb{R}} \longrightarrow 0,$$

as the tangent bundle to $\mathbb{R}^{m+1}$ is the constant bundle $\underline{\mathbb{R}^{m+1}}$.
A splitting of this sequence determines an isomorphism

$$\underline{\mathbb{R}} \oplus TS^m \cong \underline{\mathbb{R}^{m+1}},$$

which is a stable framing of the tangent bundle to the sphere, i.e. a stable tangential framing.

Prop: Let $Y \subset S^m$ be a submanifold. Then there exists a bijection between homotopy classes of stable normal framings of Y and stable tangential framings of Y.

Proof: $Y \subset S^m$ a submanifold of dimension n with a stable normal framing

$$\underline{\mathbb{R}}^{k+q} \xrightarrow{\sim} \underline{\mathbb{R}}^k \oplus \mu \quad , \; k \geq 0,$$

where μ is the normal bundle of Y of rank $q = m-n$ defined by the s.e.s.

$$0 \longrightarrow TY \longrightarrow TS^m|_Y \longrightarrow \mu \longrightarrow 0. \qquad (*)$$

This induces

$$0 \longrightarrow TY \longrightarrow \underline{\mathbb{R}}^k \oplus TS^m|_Y \longrightarrow \underline{\mathbb{R}}^k \oplus \mu \longrightarrow 0.$$

Choosing a splitting, we obtain

$$(\underline{\mathbb{R}}^k \oplus \mu) \oplus TY \cong \underline{\mathbb{R}}^k \oplus TS^m|_Y .$$

Using the stable normal framing of Y and the isomorphism we got for the tangent bundle of the sphere above, we get

$$\begin{tikzcd}
\underline{\mathbb{R}}^{k+q} \oplus TY \arrow[rr, bend right=15, "\sim"'] \arrow[r, phantom, "\cong (\underline{\mathbb{R}}^k \oplus \mu) \oplus TY \cong"] & \underline{\mathbb{R}}^k \oplus TS^m|_Y \arrow[d, phantom, "\cong"] & \left(\cong \underline{\mathbb{R}}^{k-1} \oplus \underline{\mathbb{R}}^{1} \oplus TS^m|_Y \right. \arrow[d, phantom, "\cong"] \\
& \underline{\mathbb{R}}^{k+m} & \left. \cong \underline{\mathbb{R}}^{k-1} \oplus \underline{\mathbb{R}}^{m+1} \right)
\end{tikzcd}$$

This defines a stable tangential framing of Y. $(k+m = k+q+n)$

Similarly, we get from a stable tangential framing $\underline{\mathbb{R}}^k \oplus TY \xrightarrow{\sim} \underline{\mathbb{R}}^{k+n}$ of Y for $k \geq 0$ a stable normal framing:

$(*)$ induces also

$$0 \longrightarrow \underline{\mathbb{R}}^k \oplus TY \longrightarrow \underline{\mathbb{R}}^k \oplus TS^m|_Y \longrightarrow \mu \longrightarrow 0$$

and by choosing a splitting, we obtain

$$(\underline{\mathbb{R}}^k \oplus TY) \oplus \mu \cong \underline{\mathbb{R}}^k \oplus TS^m|_Y .$$

Using the stable tangential framing of Y as above, we get

$$\underline{\mathbb{R}}^{k+n} \oplus \mu \cong \underline{\mathbb{R}}^{k+m} .$$

This is a stable normal framing of Y. $(k+m = k+n+q)$

Both maps preserve homotopy and are thus well-defined. They are also clearly mutually inverse. $\square$

Application to framed bordism

Recall the stabilization sequence of normally framed submanifolds $Y \subset S^m$ as seen last lecture:

$$\Omega^{fr}_{n;S^m} \xrightarrow{\sigma} \Omega^{fr}_{n;S^{m+1}} \xrightarrow{\sigma} \Omega^{fr}_{n;S^{m+2}} \xrightarrow{\sigma} \cdots, \qquad (*)$$

where $\Omega^{fr}_{n;S^m}$ is the abelian group of framed bordism classes of normally framed n-dimensional submanifolds of S^m.
Identifying S^m with $\mathbb{A}^m \cup \{\infty\}$ and arranging Y such, that $\infty \notin Y$,

$$\sigma : \mathbb{A}^m \longrightarrow \mathbb{A}^1 \times \mathbb{A}^m$$
$$y \longmapsto 0 \times y \quad ,$$

where furthermore the constant vector field $\frac{\partial}{\partial x_1}$ is added to the basis of sections of the normal bundle to extend the framing.

<u>Prop:</u> The colimit of $(*)$ is the bordism group Ω^{fr}_n of n-manifolds with a stable tangential framing.

<u>Proof:</u> Let $Y \subset S^m$ be a normally framed submanifold of dimension n.
Then, the normal framing of Y, which is always a stable normal framing, induces a stable tangential framing by the proposition before.
Also, the homotopy class of the stable tangential framing of Y is unchanged under the stabilization map σ.
Conversely, let Y be an n-dimensional manifold with a stable tangential framing $\underline{\mathbb{R}}^{k+n} \xrightarrow{\sim} \underline{\mathbb{R}}^k \oplus TY$.
By the Whitney embedding theorem, we can realize $Y \subset S^m$ as a submanifold for some m.
By the proposition before, this induces a stable normal framing of Y $\underline{\mathbb{R}}^{k+q} \xrightarrow{\sim} \underline{\mathbb{R}}^k \oplus \mu$.
This is a framing of the normal bundle to $Y \subset S^{m+k}$, which defines an element of $\Omega^{fr}_{n;S^{m+k}}$.
These maps are mutually inverse. □

As a corollary to the Pontrjagin-Thom theorem and above, we have:

Corollary: (Stable Pontrjagin-Thom)
There is an isomorphism $\phi : \pi_n^s \longrightarrow \Omega_n^{fr}$ for all $n \in \mathbb{N}$.

Prop: $\Omega_\bullet^{fr} := \bigoplus_{n \geq 1} \Omega_n^{fr}$ is a commutative $\mathbb{Z}$-graded ring.

"Proof:" Letting ~~an~~ n vary, we get an isomorphism of $\mathbb{Z}$-graded abelian groups, $\phi : \pi_\bullet^s \longrightarrow \Omega_\bullet^{fr}$.
We know that $\pi_\bullet^s$ is a $\mathbb{Z}$-graded ring and thus, there is a corresponding ring structure on $\Omega_\bullet^{fr}$.
It is given by the cartesian product.
This can be seen by assuming that the representatives f_1 and f_2 of two classes a_1, a_2 in the stable stem π_n^s are pointed and send ∞ to ∞.
As ϕ was defined in the proof of the Pontrjagin-Thom theorem, the classes a_1 and a_2 are sent to submanifolds Y_1 and Y_2, defined as the inverse images of $p_i \in S^{q_1} \setminus \{\infty\}$.
(regular values)
Recall that we defined the ring structure on π_n^s as taking the homotopy class $[f_1 \wedge f_2]$ of the representatives and thus, we have to consider the inverse image of $(p_1, p_2) \in S^{q_1} \wedge S^{q_2}$ under $f_1 \wedge f_2$, which is $Y_1 \times Y_2$.
Check rest.

□