# Characteristic classes II

Recall (Chern classes)

DEF. ~~Take~~ $y^k \in H^{2k}(\mathbb{P}(E))$, it satisfies

$$y^k + c_1(E)\, y^{k-1} + c_2(E)\, y^{k-2} + \dots + c_k(E) = 0$$

for some unique classes $c_i(E) \in H^{2i}(M, \mathbb{Z})$

The class $c_i(E)$ is the $i^{th}$ Chern class of $E \to M$ cpx. vector bundle

* recall properties

DEF. Let $V \to M$ be a real vector bundle of rank $k$. We can define its complexification as $V \otimes_{\mathbb{R}} \mathbb{C} \to M$, then its total Pontrjagin class is

$$p(V) = 1 + p_1(V) + \dots + p_k(V) = \dots$$

$$= 1 + c_1(V \otimes_{\mathbb{R}} \mathbb{C}) + \dots + c_k(V \otimes_{\mathbb{R}} \mathbb{C}) \in H^{*}(M, \mathbb{Z})$$

* complexification (more)

Remark It follows from Chern class properties that Pontjagin class is functorial and satisfies Whitney formula for $V, V'$ real vect. bundle

$$p(V \oplus V') = p(V)\, p(V')$$

Furthermore, since $V \otimes \mathbb{C}$ is isomorphic to its conjugate $\overline{V \otimes \mathbb{C}}$ and so

↳ transition functions of $V \otimes \mathbb{C}$ are the same of $V$

$$c_i(V \otimes \mathbb{C}) = c_i(\overline{V \otimes \mathbb{C}}) = (-1)^i\, c_i(V \otimes \mathbb{C})$$

It means that for odd $i$ $2c_i(V \otimes \mathbb{C}) = 0$, thus the odd Pontrjagin classes are torsion of order 2 in cohomology.

Hence we can consider only the terms

$$p_i(V) := (-1)^i\, c_{2i}(V \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(M, \mathbb{Z})$$

Remark For a complex manifold $M$ the Pontrjagin class is defined to be that of the underlying real manifold $M_{\mathbb{R}}$, which is defined as that of its tangent bundle. Let $T$ be the holomorphic tangent bundle to $M$ then

$$p(M) = p(T_{\mathbb{R}}) = c(T_{\mathbb{R}} \otimes \mathbb{C}) = c(T \oplus \overline{T}) = c(T)\, c(\overline{T})$$

And, since the total Chern class of $M$ is $c(M) = \prod_i (1 + x_i)$, the Pontryagin class is $p(M) = \prod_i (1 + x_i^2)$

<u>Example</u> ($p(S^n)$)

Since the sphere $S^n$ is orientable, its normal bundle $N$ in $\mathbb{R}^{n+1}$ is trivial, from the exact sequence

$$0 \to TS^n \to T\mathbb{R}^{n+1}|_{S^n} \to N \to 0$$

we see by the Whitney formula that

$$p(S^n)\, p(N) = p(T\mathbb{R}^{n+1}|_{S^n}).$$

Therefore,

$$p(S^n) = 1$$

## <u>L-class and L-genus</u>

We have seen that any symmetric polynomials in $x_1, \dots, x_k$ defines a polynomial in the chern classes of $E \to M$.
So if we not fix the rank or the dimension of the base, we can encode these classes by formal power series in variable $x$

<u>DEF</u> Consider $L := \dfrac{x}{\tanh x}$ called Hirzebruch's L-polynomial. $x$
if a vector bundle $E \to M$ has formal Chern roots $x_1, x_2, \dots, x_k$ then, $x_i = c_1(L_i)$

$$L(E) := \prod_{j=1}^{k} \frac{x_j}{\tanh x_j}$$

is its L-class.

<u>Remark</u> The term of order $2i$ is a symmetric polynomial of degree $i$ in the variables $x_j$. It is then a polynomial in the elementary symmetric polynomials $c_1, \dots, c_k$ which are

Chern classes of $E$, ie. $L(E) = L(c_1(E), \dots, c_k(E))$
It satisfies $L(E \oplus F) = L(E)L(F)$

<u>Example</u> We compute now $L(E)$ for a rank two complex vector bundle $E \to M$ where $M$ has dimension 4.
Let Chern roots of $E$ be $x_1, x_2$.

The L-polynomial is

$$\frac{x}{\tanh x} = \frac{x \cosh x}{\sinh x} = \frac{x\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)}{x\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \dots\right)} = 1 + \frac{x^2}{3} - \frac{x^4}{45} + \dots$$

and so

$$L(E) = \left(1 + \frac{x_1^2}{3}\right)\left(1 + \frac{x_2^2}{3}\right) = 1 + \frac{x_1^2 + x_2^2}{3} = 1 + \frac{(x_1+x_2)^2 - 2x_2x_1}{3} =$$

$$= 1 + \frac{c_1^2 - 2c_2^2}{3}$$

With $M = \mathbb{CP}^2$ we have
$c(\mathbb{CP}^2) = (1+y)^3 = 1 + \underbrace{3y}_{c_1} + \underbrace{3y^2}_{c_2}$ with $y = c_1(S^*)$
$\Rightarrow L(\mathbb{CP}^2) = 1 + \frac{9y^2 - 6y^2}{3} = 1 + y^2$

<u>DEF.</u> The L-genus is the pairing of the L-class of the tangent bundle of a complex manifold $\underset{\dim M = 2m}{M}$ with its fundamental class $[M] \in H_{2m}(M)$

$$\langle L(M), [M] \rangle$$

<u>LEMMA</u>(B.2) The tangent bundle of $\mathbb{CP}^m$ is stably equivalent to $(S^*)^{\oplus m+1}$

<u>proof</u> Consider the exact sequence

$$0 \to S \to \mathbb{CP}^m \times \mathbb{C}^{m+1} \longrightarrow Q \to 0$$

where $Q$ is the orthogonal complement to $S$ universal line subbundle.

So $Q \oplus S \cong \mathbb{CP}^m \times \mathbb{C}^{m+1}$

Now we tensor with $S^*$

$$0 \to \underset{\mathbb{CP}^m \times \mathbb{C}}{\underset{\parallel}{S \otimes S^*}} \to (\mathbb{CP}^m \times \mathbb{C}^{m+1}) \otimes S^* \to \underset{T\mathbb{CP}^m}{\underset{\parallel}{Q \otimes S^*}} \to 0$$

And then

$$T\mathbb{CP}^m \oplus \underline{\mathbb{C}} \cong (\mathbb{CP}^m \times \mathbb{C}^{m+1}) \otimes S^* = (S^*)^{\oplus m+1}$$

□

PROP (89) The L-genus of $\mathbb{C}P^m$ is such that

$$\langle L(\mathbb{C}P^m), [\mathbb{C}P^m] \rangle = \begin{cases} 1 & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

where $[\mathbb{C}P^m] \in H_{2m}(\mathbb{C}P^m)$ is the fundamental class, defined using the canonical orientation and $L(\mathbb{C}P^m)$ is the L-Polynomial of the tangent bundle.

proof Using lemma 82 and stability of Chern classes we can consider $L((S^*)^{\oplus m+1})$ instead of $L(T\mathbb{C}P^m)$

$(S^*)^{\oplus m+1}$ is a sum of line bundles, and each Chern root is $c_1(S^*) = y \in H^2(\mathbb{C}P^m, \mathbb{Z})$ generator.

we have

$$L(\mathbb{C}P^m) = L((S^*)^{\oplus m+1}) = \left(\frac{y}{\tanh y}\right)^{m+1}$$

Now we have to find the coefficient of $y^m$ in this product and we use the method of residues

$$\frac{d^m f(z)}{dz^m} = \frac{m!}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^{m+1}}\, d\xi$$

where $\gamma$ is a simple closed curve

$$\Rightarrow \frac{1}{m!}\frac{d^m f(y)}{dy^m}\Big|_{y=0} = \frac{1}{2\pi i}\int_\gamma \frac{d\xi}{\xi^{m+1}}\left(\frac{\xi}{\tanh \xi}\right)^{m+1} = \begin{bmatrix} z = \tanh \xi \\ d\xi = \dfrac{dz}{(1-z^2)} \end{bmatrix} =$$

$$= \frac{1}{2\pi i}\int_{\gamma'} \frac{dz}{(1-z^2)\, z^{m+1}} = \frac{1}{2\pi i}\int dz\, \frac{1+z^2+z^4+\dots}{z^{m+1}} = \begin{cases} 1 & , m \text{ even;} \\ 0 & , m \text{ odd,} \end{cases}$$

where $\gamma$ is a small circle around the origin of the complex y-line and the last equal derives because the functions are all holomorphic except $\frac{1}{z}$.

$\square$

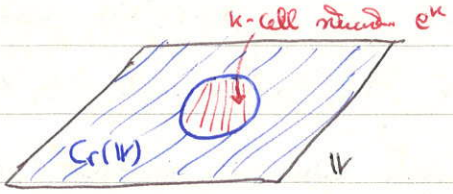
# Thom classes and Thom complex

DEF (Relative cell complex) Let $X$ be topological space and $A \subset X$ a closed subspace, we indicate them writing $(X, A)$.

A *cell structure* on $(X, A)$ is a cell decomposition of $X \setminus A$, i.e. $X$ is obtained from $A$ by successively attaching 0-cell, 1-cell, ..., using $A$ as an extra buildin block

REMARK If the pair $(X, A)$ admits a cell structure, then the homology (r. cohomology) of $(X, A)$ is isomorphic to the reduced homology (r. cohomology) of the quotient $X/_A$ with basepoint $A/A$.

Example Let $\mathbb{V}$ be a real vector space of dim $k$. Suppose $\mathbb{V}$ has an inner product. Define $C_r(\mathbb{V}) := \{ v \in \mathbb{V} \mid \|v\| \geq r,\ r \in \mathbb{R} \}$ and consider the pair $(\mathbb{V}, C_r(\mathbb{V}))$




In this case the $(\mathbb{V}, C_r(\mathbb{V}))$'s cell structure is a $k$-cell (single) called $e^k$, hence his cohomology (relative) is

chain complex $0 \leftarrow \mathbb{Z}\{e^k\} \leftarrow 0 \leftarrow$

$$H^q(\mathbb{V}, C_r(\mathbb{V}); \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & q = k \\ 0 & \text{otherwise} \end{cases}$$

$- 0 \to \mathrm{Hom}(\mathbb{Z}\{e^k\}, \mathbb{Z}) \to 0 \to -$

and $\mathbb{V}/_{C_r(\mathbb{V})}$ is homeomorphic to a $S^k$ ($k$-sphere) with a basepoint.

(*) LEMMA 8.31

NOTE: the cohomology group of $\mathbb{V}$ has two distinguished generators which depend on the choice of $k$-cell $e^k$.

## Thom class

From now we consider $\pi: V \to M$ as a real vector bundle of rank $k$, with an inner product. Then let $(V, C_r(V))$ be the pair with

$C_r(V) := \{ v \in V \mid \|v\| \geq r \ , \ r \in \mathbb{R} \}$.

DEF. A Thom class for $\pi : V \to M$ is a cohomology class $U_V \in H^k(V, C_r(V); \mathbb{Z})$ such that the pullback $\forall p \in M$ $i^*_p U_V$ is a generator of $H^k(V_p, C_r(V_p); \mathbb{Z})$.

Recall that an orientation for a real vector bundle $V \to M$ is a section of the double cover of $O(V) \to X$.

PROP. Let $\pi : V \to M$ be an oriented vector bundle, then exist a Thom class $U_V \in H^k(V, C_r(V); \mathbb{Z})$

proof. We use that a smooth manifold $M$ admits a CW structure, i.e. we can construct it attaching $n$-cell, from the 0-cell.

So denote with $\{e_\alpha\}_{\alpha \in A}$ the set of cells.

We want to prove that even $(V, C_r(V))$ has a cell decomposition $\{f_\alpha\}_{\alpha \in A}$ indexed by the same set $A$, and $\dim f_\alpha = \dim e_\alpha + k$

In that case the cellular chain complex of $(V, C_r(V))$ is just the $k$-shift of the cellular chain complex of $M$.

Hence,

$$\begin{tikzcd}
0 \arrow[r] & 1 \in H^0(M, \mathbb{Z}) \arrow[r] \arrow[d] & H^1(M; \mathbb{Z}) \arrow[r] \arrow[d, "\wr"] & H^2(M; \mathbb{Z}) \arrow[r] & \dots \\
\dots \arrow[r] & U_V \in H^k(V, C_r(V); \mathbb{Z}) \arrow[r] & H^{k+1}(V, C_r(V); \mathbb{Z}) \arrow[r] & H^{k+2}(V, C_r(V); \mathbb{Z}) \arrow[r] & \dots
\end{tikzcd}$$

$\hookrightarrow$ is the Thom class

Now we find the $(V, C_r(V))$ decomposition with cells.

By definition for each cell $e_\alpha$ we have

$$\phi_\alpha : \bar{B}_\alpha \to M$$

a continuous map where $\bar{B}_\alpha$ is a closed ball.

Furthermore

$$\phi_\alpha|_{B_\alpha} \text{ is a homeomorphism onto the cell } e_\alpha \subset M$$

and $M = \bigsqcup \phi_\alpha|_{B_\alpha}$

Then the pullback $\phi_\alpha^* V \to \bar{B}_\alpha$ is trivializable (cfr. Prop 7.3) and choosing a trivialization we have an induced homeomorphism

$$\phi_\alpha^*(V, C_r(V)) \approx (\bar{B}_\alpha \times (\mathbb{V}, C_r(\mathbb{V}))) \approx (\bar{B}_\alpha \times \mathbb{V}, \bar{B}_\alpha \times C_r(\mathbb{V}))$$

where we used that $(V, C_r(V))$ is a fiber bundle over $M$ with typical fiber $(\mathbb{V}, C_r(\mathbb{V}))$.

The cell structure on the pair $(\bar{B}_\alpha \times \mathbb{V}, \bar{B}_\alpha \times C_r(\mathbb{V}))$ has a single cell, wich is the the cartesian product of $\bar{B}_\alpha$ and the $(\mathbb{V}, C_r(\mathbb{V}))$ $k$-cell.

Since the orientation of $V \to M$ induces an orientation of $\phi_\alpha^* V \to \bar{B}_\alpha$, we can consider the cell $e^k$.

Finally,

$$f_\alpha := e_\alpha \times e^k$$

is the cell decomposition of $(V, C_r(V))$

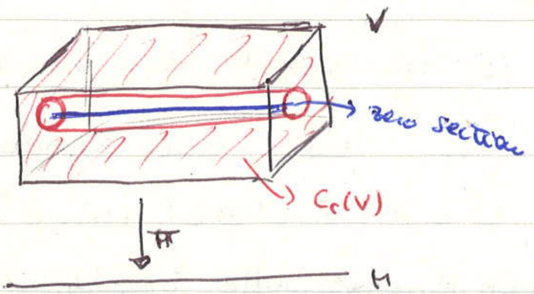
□

(*) Thom isomorphism

**NOTE** we used the theorem the a vector bundle over a contractible space (in our case the closed ball $\bar{B}_\alpha$) is trivializable.

**DEF.** The quotient $V/_{C_r(V)}$ is called <u>Thom complex</u> (or Thom space).
We denote $M^V := V/_{C_r(V)}$

$\pi: V \to M$




The idea behind the definition is that $C_r(V)$ collapses to a point (basepoint).
Now we cannot project $M^V$ on $M$, because there is not basepoint in $M$.

Anyway we have an inclusion from $M$ to $M^V$ induced by the zero section $i: M \to M^V$. (8)

* construction of Thom space

## The Euler class

DEF Let $\pi: V \to M$ be an oriented real vector bundle of rank $k$ with Thom class $U_V$. The Euler class $e(V) \in H^k(M, \mathbb{Z})$ is

$$e(V) := i^*(U_V)$$

So Euler class is the restriction of the Thom class to the zero section of $V$

* PROP 8.47

PROP Let $L \to M$ be a complex line bundle and $L_{\mathbb{R}} \to M$ the underlying oriented rank 2 real vector bundle, then

$$e(L_{\mathbb{R}}) = c_1(L) \in H^2(M, \mathbb{Z})$$

proof. ("sketch") Consider the fiber bundle

$$\mathbb{P}(L^* \oplus \underline{\mathbb{C}}) \to M.$$

Its typical fiber is $\mathbb{CP}^1$, the projective line.

The dual tautological bundle has first chern class $\hat{U} = c_1(S^*)$

We have two section

$j: M \to \mathbb{P}(L^* \oplus \underline{\mathbb{C}})$ $\qquad$ $i: M \to \mathbb{P}(L^* \oplus \underline{\mathbb{C}})$

$p \mapsto$ trivial line $\mathbb{C}$; $\qquad$ $p \mapsto$ line $L^*$

Note that each line $L_p^* \oplus \mathbb{C}$ (except $\mathbb{C}$) is the graph of the linear function $L_p^* \to \mathbb{C}$, so thus an element of $L_p$

Now $j^*(S^*) \to M$ is the trivial line bundle

and $i^*(S^*) \to M$ is the line bundle $L \to M$

Hence, denoted by $U \in H^2(\mathbb{P}(L^* \oplus \mathbb{C}), j(M), \mathbb{Z})$ the Thom class of the underlying rank 2 real vector bundle $L_{\mathbb{R}} \to M$

$$e(L_{\mathbb{R}}) = i^*(U) = i^*(c_1(S^*)) \overset{\text{naturality}}{=} c_1(i^*S^*) = c_1(L)$$

□

**Prop.** Let $V_1, V_2 \to M$ be oriented real vector bundles, then

$$e(V_1 \oplus V_2) = e(V_1)\, e(V_2)$$

proof Let $\pi_1 : V_1 \oplus V_2 \to V_1$ and $\pi_2 : V_1 \oplus V_2 \to V_2$ be projections.

Let $i$ be the zero section of $V_1 \oplus V_2$. Then $\pi_1 \circ i$ and $\pi_2 \circ i$ are the zero section of $V_1$ and $V_2$.

Since $U_{V_1}$ is the Thom class for $V_1 \to M$ and $U_{V_2}$ for $V_2 \to M$, the Thom class of $V_1 \oplus V_2$

$$U_{V_1 \oplus V_2} \in H^2(V_1 \oplus V_2 ; \mathbb{Z}) \quad \text{is} \quad \pi_1^* U_{V_1} \smile \pi_2^* U_{V_2}$$

(cf. Bott, Tu, Differential forms in Algebraic Topology) pag. 65 Prop. 6.19

Hence,

$$e(V_1 \oplus V_2) = i^*(U_{V_1 \oplus V_2}) = i^*\pi_1^*(U_{V_1}) \smile i^*\pi_2^*(U_{V_2}) = e(V_1)\, e(V_2)$$

□

**COROLLARY** Let $E \to M$ be a rank $k$ complex vector bundle, then

$$c_k(E) = e(E_{\mathbb{R}})$$

proof (cf. "Bott, Tu, Differential forms in Algebraic Topology" pag 278)

Let $\pi : F(E) \to E$ be its splitting manifold, we observe that $\pi^* E = L_1 \oplus \dots \oplus L_k$ where the $L_i$'s are line bundles on the split manifold $F(E)$

$$\pi^* c_k(E) \underset{\text{naturality}}{=} c_k(\pi^* E) \underset{\text{Whitney for Chern}}{=} c_1(L_1) \cdots c_1(L_k) =$$

$$\overset{\text{Prop. 8.4B}}{=} e((L_1)_{\mathbb{R}}) \cdots e((L_k)_{\mathbb{R}}) \underset{\text{Whitney for Euler}}{=} e((L_1)_{\mathbb{R}} \oplus \dots \oplus (L_k)_{\mathbb{R}}) =$$

$$= e((\pi^* E)_{\mathbb{R}}) = \pi^* e(E_{\mathbb{R}})$$

We conclude for injectivity of $\pi^*$ on cohomology

$$c_k(E) = e(E_{\mathbb{R}})$$

□