Tangential structures

(numbers of theorems and def. correspond to Freeds notes)

- Main example to keep in mind - orientations on vector bundels.

Obstruction to existence of an orientation (Exerscise 9.2 - Remark 9.3)

Claim For a vector bundle $\begin{matrix} V \\ \downarrow \\ M \end{matrix}$ $\exists$ an orientation iff

$w_1(V) = 0$, where w_1 is a first Stifel-Whitney class.

$$w_1(V) = H^1(M, \mathbb{Z}_2) = Hom(H_1(M,\mathbb{Z}_2), \mathbb{Z}_2) = Hom(\pi_1(M,m), \mathbb{Z}_2)$$

where the last $=$ is true because:

$$\begin{tikzcd} \pi_1 \arrow[r, "ab"] \arrow[dr, "g"'] & H_1 = \tilde{\pi}_1 \arrow[d, "f"] \\ & \mathbb{Z}_2 \end{tikzcd}$$

$\forall f: H_1 = \underset{\text{of } \tilde{\pi}_1}{\text{abelinisation}} \longrightarrow \mathbb{Z}_2$

chose

$g: \pi_1 \longrightarrow \mathbb{Z}_2$ be $g = f \circ ab$

$\forall g: \pi_1 \longrightarrow \mathbb{Z}_2$ $\exists!$ $f: \tilde{\pi}_1 \longrightarrow \mathbb{Z}_2$: $g = f \circ ab$ by universal property of abilinisation.

Constract $w: \pi_1(M,m) \longrightarrow \mathbb{Z}_2$ as follows:

Recall that for V we have $\begin{matrix} \mathcal{O}(V) \\ \downarrow p \\ M \end{matrix}$ - orientation double couver

consider $p^{-1}(m)$ (where $m \in M$ some fixed point).

$p^{-1}(m)$ consists of 2 points $\Rightarrow$ can be indentified with $\mathbb{Z}_2$.

$\forall \gamma \in \pi_1(M,m)$ take a lift $\tilde{\gamma}$ such that $\tilde{\gamma}(0) = 0$

Such lift $\exists!$ by lifting properties of covering.

Define $w(\gamma) = \tilde{\gamma}(1)$.

By the above w can be viewed as an element of $H^1(M, \mathbb{Z}_2)$.

If $w(\gamma) = 0 \Rightarrow \tilde{\gamma}(1) = 0$ $\forall \gamma \in \pi_1 \Rightarrow \mathcal{O}(V) = M \times \mathbb{Z}_2$, as it is not path-connected (otherwise path from 0 to 1 will give after a projection as loop γ: $w(\gamma) = 1$)

If an orientation exist, then $\exists$ section of $\Theta(V) \Rightarrow \Theta(V) = M \times \mathbb{Z}_2$, $\Rightarrow$ it's not pathconnetted $\Rightarrow$ there is no $\tilde{\gamma}$: $\tilde{\gamma}(0)=0$, $\tilde{\gamma}(1)=1$ $\Rightarrow$

$\omega \equiv 0$.

$\Rightarrow$ ω is really an obstruction to existence of orientation.

? Why it's exactly ω_1?

- First, observe, that for f - map of vector bundels:

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V' \arrow[d] \\ M \arrow[r] & M' \end{tikzcd}$$

such that $f^*(V') = V$ we have

$f^*(\omega') = \omega$ - just by construction.

(Could be cheked streightforward)

- Second, as ~~it's~~ for first is true, it's sufficient to show that for $V = E\mathbb{Z}_2$ $M = B\mathbb{Z}_2$ $\omega = \omega_1$.

$B\mathbb{Z}_2 = \mathbb{RP}^\infty$, $H^*(\mathbb{RP}^\infty, \mathbb{Z}_2) = \mathbb{Z}_2[\omega_1]$

$E\mathbb{Z}_2 = S^\infty$, and ~~s~~ it is path-connected $\Rightarrow \omega \neq 0 \Rightarrow$ $\boxed{\omega = \omega_1}$

Reduction of a structure group

Let H, G be a Lie groups and $\rho: H \to G$ homomorphism.

Remark 1 For an orientation it is a homomorphism $GL_n^+ \to GL_n$

Remark 2 Everything above much more intuitive, if ρ is an inclusion.

Motivation: We whatnt to go from principal G bundels to principal H bundels and other way around.

Def (9.8 i) Associated principal bundle

Let $\begin{matrix} Q \\ \downarrow \\ M \end{matrix}$ be principal H bundle. We will construct $\begin{matrix} Q_\rho \\ \downarrow \\ M \end{matrix}$ - principal G-bundle.

Consider $\begin{matrix} Q \times G \\ \downarrow \\ M \end{matrix}$. H acts freely from the right on $Q \times G$:

$(q,g)h := (q\cdot h, \varrho(h)^{-1}g)$ where $q\in Q$, $g\in G$, $h\in H$.

Let $\underline{Q_\varrho := (Q\times G)/H}$

Notice, that this is a ~~G~~ principal G-bundle:

- <u>Fiber</u>: fix fiber of Q and fix it's isomorphism with ~~G~~ H.

Fibers of $(Q\times G)/H$ comes from fibers of Q.

Under the isomorphism of one fixed fiber with H we look at pairs $[(h,g)]$ of $(Q\times G)/H$.

$(h,g)h^{-1} = (e, \varrho(h)g) \Rightarrow [(h,g)] = [(e,\varrho(h)g)]$ in Q_ϱ.

So we see that each fiber is realy G, as by taking $h=e$ and chosing different g we will recieve all G.

- <u>Action of G</u>:

Define $[(q,g)]\tilde{g} = [(q, g\tilde{g})]$ for $q\in Q$, $g,\tilde{g}\in G$

As we multiply from the right, and H acts on G from the left in our definition of action on $Q\times G$, this actions commute and this is a correct definition.

- <u>Transition functions</u>

Let F and $\tilde{F}$ will be two fibers of Q over the same point but under differen indifications ~~of~~ with H, coming from different ~~charts~~ element of covering of M.

(Probably better: Chose covering $\mathcal{U}$ of M: $\mathcal{U} = \{U\}$ and for each U it's preimage in Q is isomorpic with $U\times H$.

Chose a point in $U_1\cap U_2$ and trivialisation of fibers F and $\tilde{F}$)

Then a transition map from F to $\tilde{F}$ is some element h_t from H, as H - a Lie group and Q - principal H-bundle.

($\mathcal{U}$) and trivialisations give rise to trivialisations of Q_ϱ.

~~An~~ Fibers F and $\tilde{F}$ become $F\times G/H$ and $\tilde{F}\times G/H$.

$[(e,g)] \in F\times G/H \to [(h_t, g)] = [(e, \varrho(h_t)g)] \in \tilde{F}\times G/H$

So $\rho(h_t)$ is a new transition map. We will use it later.

- $Q \subset Q_\rho$ when ρ is an inclusion.

  Then $\rho(h) = h$, so ~~all points~~ ~~are~~

$$Q \longrightarrow Q \times G$$
$$q \longmapsto (q, e)$$ is also inclusion after projection
$$Q \times G \to Q \times G/_H.$$

Def (9.8ii) Reduction of the structure group

Let $\begin{matrix} P \\ \downarrow \\ M \end{matrix}$ be a principal G-bundle. Then a reduction to H is

a pair (Q, ϑ) where Q is a principal H-bundle $\begin{matrix} Q \\ \downarrow \\ M \end{matrix}$ and

ϑ is an isomorphism of principal G-bundels

$$\begin{tikzcd} Q_\rho \arrow[rr, "\overset{\vartheta}{\cong}"] \arrow[dr] & & P \arrow[dl] \\ & M & \end{tikzcd}$$

! Reduction is something wich is not nessesary exist.

An isomorphism of reductions of (Q, ϑ) and (Q', ϑ') is an isomorphism of H-bundels Q and Q'.

There is an alternative definition of a reduction:

Chose some trivialisatinon atlas $\mathcal{U} = \{\mathcal{U}, \varphi_{\mathcal{U} \cap \mathcal{U}'}\}$ of P and chose maps from $\mathcal{U} \cap \mathcal{U}'$ to H such that composed with ρ they will give $\varphi_{\mathcal{U} \cap \mathcal{U}'}$.

So a reduction is a way to see already existing transition maps as elements of H.

The proof that this two definitions are equivalent comes from the our ~~p~~ thinking about transitions maps for Q_ρ above.

- Assume again $\rho: H \hookrightarrow G$ is an inclusion.
  Then one can show that
  reductions are in 1-to-1 correspondence with sections of the G/H bundle $P/H \to M$.

Let $\mu: G \longrightarrow G/H$ a projection map.
Notice that for a principal G-bundle P, $P_\mu \simeq P/H$ as G/H bundles.

Suppos we have $\sigma: M \to P/H$ - a section.
P can be viewed as H-bundel on P/H.
Than we can take

$$\begin{tikzcd} \sigma^*(P) \arrow[r] \arrow[d] & P \arrow[d] \\ M \arrow[r, "\sigma"] & P/H \end{tikzcd}$$

an H-bundle. $\sigma^*(P)$.

As σ is a section $\sigma^*(P) \longrightarrow P$ is an inclusion. $\Rightarrow$ we find a reduction

If we have reduction $Q \Rightarrow$ we have an inclusion $Q \subset Q_\rho \simeq P$ wich gives us a section of $Q_\rho/H \simeq P/H$ - as it comes to class of $[e]$.

<u>Some examples</u>

1) Let $\rho: O(n) \hookrightarrow Gl_n(\mathbb{R})$, $V \to M$ - real vector bundle of rank n with metric, $\beta_o(V)$ - the fibration, consistion of orthonormal frames.

Then $\beta_o(V)$ is a ~~reduction~~ Q from definition above, $\beta(V) = P$ (recall: $\beta(V)$ is all frames of V) and so $\beta_o(V)_\rho \simeq \beta(V)$ correspond to V.

2) Let $\rho: SO(n) \hookrightarrow O(n)$ or $Gl_n^+ \hookrightarrow Gl_n$.
From the above reductions in this case correspond to Sections of $B(V)/Gl_n^+$ $\left(\text{or } B_o(V)/SO(n)\right)$, wich are $\theta(V)$ by definition. So the choise of an orientation on V is a choise of reduction of structure group from Gl_n to Gl_n^+, or from $O(n)$ to $SO(n)$.

From now let H, G be compact.

Recall that each $\rho: H \to G$ gives rise to $B\rho: BH \to BG$ – map of classifying spaces. Moreover, it is a fibration, so it ~~has~~ a homotopy lifting property.

$$\begin{tikzcd} B\rho^*(EG) \arrow[r] \arrow[d] & EG \arrow[d] \\ BH \arrow[r, "B\rho"] & BG \end{tikzcd}$$

Consider a pullback of EG by $B\rho$.

<u>Claim</u> $B\rho^*(EG) \xrightarrow{\simeq} (EH)_\rho$ – a bundle, associated to EH by $\rho: H \to G$

I convinced myself in it, looking at a construction of BG and BH.

There were a suggestion during the seminar to look at universal proparties.

<u>Proposition</u> (9.38) Let $\begin{matrix} P \\ \downarrow \\ M \end{matrix}$ is a principal G-bundle and

$f: M \to BG$ is it's clyssifying map.

Then isomorphism classes of reduction to group H are in 1-to-1 correspondence with lifts $\tilde{f}: M \to BH$: $f = B\rho \circ \tilde{f}$

$$\begin{tikzcd} & BH \arrow[d, "B\rho"] \\ M \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & BG \end{tikzcd}$$

- from $\tilde{f}$ to reduction Q:

$$\begin{tikzcd} (EH)_\rho \arrow[drr] \arrow[dddr] & & \\ & P \arrow[r] \arrow[d] & EG \arrow[d] \\ & M \arrow[r, "f"] \arrow[d, "\tilde{f}"] & BG \\ & BH \arrow[ur, "B\rho"'] & \end{tikzcd}$$

Let's look at $\tilde{f}^*(EH_\rho)$:

$$\tilde{f}^*(EH_\rho) \overset{\text{claim above}}{\simeq} \tilde{f}^*(B_\rho^*(EG)) = f^*(EG) = P$$

$\Rightarrow$ $\tilde{f}^*(EH_\rho) = P$, so

Chose $Q = \tilde{f}^*(EH)$ then

$$Q_\rho = \tilde{f}^*(EH)_\rho = \tilde{f}^*(EH_\rho)$$

as a construction of assosiated bundle is natural (as it comes from direct product).

Notice, that if you will chose $\tilde{f}'$ homotopic to $\hat{f}$, then $\tilde{f}^*(EH) \simeq \tilde{f}^*(EH)$ – gives isomorphic reductions.

· from reduction to $\tilde{f}$:

Let g be a classifying map for $\begin{matrix} Q \\ \downarrow \\ M \end{matrix}$: $\begin{tikzcd} Q \arrow[r] \arrow[d] & EH \arrow[d] \\ M \arrow[r, "g"] & BH \end{tikzcd}$, so $Q = g^*(EH)$

Then we have following diagram:

$$\begin{tikzcd} P \simeq Q_\varrho \arrow[r] \arrow[d] & EH_\varrho \arrow[r] \arrow[d] & EG \arrow[d] \\ M \arrow[r, "g"] & BH \arrow[r, "B_\varrho"] & BG \end{tikzcd}$$

as $Q = g^*(EH) \Rightarrow$
$Q_\varrho = g^*(EH_\varrho) = g^*(B_\varrho^*(EG))$, as was above

Then $g \circ B_\varrho$ is a classifing map for Q_ϱ.
Let f be a classifying map for P.
As $Q_\varrho \simeq P \Rightarrow f \sim g \circ B_\varrho$ - homotopy eq.

$$\begin{tikzcd} & BH \arrow[d] \\ M \arrow[ur, "g"] \arrow[r, "g \circ B_\varrho"] \arrow[r, bend right, "f"'] & BG \end{tikzcd}$$

homotopy

By homotopy lifting property $\exists \tilde{f} : M \to BH$ such that $\tilde{f}$ homotopic to g and lifts f.

As we see, homotopy class of $\tilde{f}$ changes isomorphic class of Q. So we are done.

General tangential structures

Recall that we define $BO(n) = \varinjlim Gr(n, n+q)$
where $Gr(n, n+q) \to Gr(n, n+q+1)$ -
as n-dim supspace of $\mathbb{R}^{q+n} \rightsquigarrow$ n-dim subspace of $\mathbb{R}^{n+q} \oplus \mathbb{R}^1$

Also one can define a map $Gr(n, n+q) \to Gr(n+1, n+q+1)$ -
it keeps the quotient of $\mathbb{R}^{n+q}$ by chosen n-dim subspace to be quotient of $\mathbb{R}^{n+q+1}$ by chosen $n+1$-dim subspace.

(If you think about $Gr(n, n+q)$ as $n \times (n+q)$ matrices of rnk n, quotient by $GL(n+q)$ action, then matrix A goes to matrix $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$ - wich represents a class in $Gr(n+1, n+q+1)$

This maps Induce map of colimits: $BO(n) \to BO(n+1)$.
Then we define $BO = \varinjlim BO(n)$

<u>Def</u> (8.45) <u>A n-dim. tangential structure</u> is a topological space $\mathcal{X}(n)$ and a fibration $\pi(n): \mathcal{X}(n) \to BO(n)$

<u>A stable tangential structure</u> is a topological space $\mathcal{X}$ and a fibration $\mathcal{X} \to BO$.

Notice, that after we once fix n-dim structure, then there is a natural way to recive a k-dim structures for all $k < n$: take a pull back of $\mathcal{X}(n)$ to $BO(k)$ under the map of $BO(k) \to BO(n)$.

$$\begin{array}{ccc} \mathcal{X}(k) & \longrightarrow & \mathcal{X}(n) \\ \downarrow & & \downarrow \\ BO(k) & \longrightarrow & BO(n) \end{array}$$

Similary, a choise of $\mathcal{X}$-stable tangential structure immideatly gives us $\mathcal{X}(n)$ for every n: take a pullback to $BO(n)$ under the map $BO(n) \to BO$

$$\begin{array}{ccc} \mathcal{X}(n) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ BO(n) & \longrightarrow & BO \end{array}$$

This suggest another way of thinking abut stable tangential structures: chose n and for all $k \geq n$ chose $\mathcal{X}(k)$ such that the diagram commute and take a direck limit.

$$\begin{array}{ccccccc} \mathcal{X}(n) & \to & \mathcal{X}(n+1) & \longrightarrow & \mathcal{X}(n+2) & \to & \\ \downarrow & & \downarrow & & \downarrow & & \\ BO(n) & \to & BO(n+1) & \to & BO(n+2) & \to & \dots \end{array}$$

(Probaly we need not just commute, but $\mathcal{X}(k)$ is a pullback of $\mathcal{X}(k+1)$).

<u>Def</u> <u>an $\mathcal{X}(n)$ structure on a m-dim manifold M</u> $(n \geq m)$ is a lift $\tilde{f}: M \to \mathcal{X}(n)$ of f - clyssifing map: $M \to BO(n)$ map for $\tilde{T}M = TM \oplus \mathbb{R}^{n-m}$.

$$\begin{array}{ccc} & \overset{\tilde{f}}{\nearrow} & \mathcal{X}(n) \\ & & \downarrow \pi(n) \\ M & \xrightarrow{f} & BO(n) \end{array}$$

<u>A stable structure $\mathcal{X}$</u> is a family of coherent structures $\mathcal{X}(n)$ for n sufficiently large.

Examples 1) (9.50) Take $\mathcal{X} = EO$. Then this gives stable framing of TM - a framing for $TM \oplus \mathbb{R}^k$ for some k, $k \geq 0$

2) (9.49). Trivial structure. Put $\mathcal{X} = BO$.

3) (9.52) An orientation is a stable structure $\mathcal{X} = BSO$.
A an orientation on $\widetilde{TM}$ gives rise to orientation on TM, (if we agree on standart orientation on $\mathbb{R}^k$) $\Rightarrow$ $\mathcal{X}(n) = BSO(n)$

4) For every sequense of groups $G(n) \to G(n+1) \to \dots$ and ~~we co~~ maps $G(n) \to O(n)$: everything commute we can chose $\mathcal{X}(n) = BG(n)$.
One more example: $Spin(n)$ wich is double cover of $SO(n)$.

The ~~homotopy~~ lifting property from (9.38) suggests definition:

Def An <u>isomorpism of $\mathcal{X}(n)$ structures</u> on manifold M is a homotopy between lifting $M \to \mathcal{X}(n)$.

Def <u>The universal $\mathcal{X}(n)$ bundle</u>

Let's fix $\mathcal{X} \xrightarrow{\pi} BO$ and $\mathcal{X}(n) \xrightarrow{\pi(n)} BO(n)$ for $n \in \mathbb{Z}^{\geq 0}$.

$$\begin{tikzcd} \pi(n)^*(S(n)) \arrow[r] \arrow[d] & S(n) \arrow[d] \\ \mathcal{X}(n) \arrow[r, "\pi(n)"] & BO(n) \end{tikzcd}$$

Let $S(n) = EO(n)$ - universal tautological bundle.

Take $\pi(n)^*(S(n))$. It has an $\mathcal{X}(n)$-structure: id map from $\mathcal{X}(n)$ to $\mathcal{X}(n)$.
From now we will denote $\pi(n)^*(S(n))$ as „$S(n)$", and also all it's pullbacks, when it is clear wich maps we consider.

Proposition (9.60). There exist a commutative diagram, so $\mathcal{X}(n)$-structure on M is given by classifing map for $\widetilde{TM}$.

$$\begin{tikzcd} \widetilde{TM} \arrow[r] \arrow[d] & \text{„}S(n)\text{"} \arrow[d] \\ M \arrow[r, "\tilde{f}"] & \mathcal{X}(n) \end{tikzcd}$$

$$\widetilde{TM} = f^*(S(n)) = \tilde{f}^* \pi(n)^* ((S(n)) = \tilde{f}^* \text{„}S(n)\text{"}.$$

$$\begin{tikzcd} & \mathcal{X}(n) \arrow[d, "\pi(n)"] \\ M \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & BO(n) \end{tikzcd}$$

Involution

Notice that $\exists$ involution $Gr(k, n+k) \leftrightarrow Gr(n, n+k)$ wich takes $V \subset \mathbb{R}^{n+k}$, $\dim V = k$ and sends it to $\mathbb{R}^{n+k}/V$.

$\Rightarrow$ it induces an involution of double colimits: $BO \xleftrightarrow{i} BO$.

Denote $\mathcal{X}^\perp = i^*(\mathcal{X})$ for $\mathcal{X}$-stable tangential structure.

$$\begin{tikzcd} \mathcal{X}^\perp \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ BO \arrow[r, "i"] & BO \end{tikzcd}$$

Notice that if $V, V^\perp$ - vector bundles on M, $\dim V = n$, $\dim V^\perp = k$
$V \oplus V^\perp = \underline{\mathbb{R}}^{n+k}$ then the following diagram commute:

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & Gr(n,n+k) \arrow[r] & BO \arrow[d, leftrightarrow, "i"] \\ & Gr(K,n+k) \arrow[r] & BO \end{tikzcd}$$

where first two maps $M \to Gr(n, n+K)$ and $M \to Gr(K, n+K)$ are classifying for V and $V^\perp$ resp.

No we whant to swich between stable tangential $\mathcal{X}$ structures and stable normal $\mathcal{X}^\perp$-structures, like in 5-th lecture.

Def By <u>stable normal $\mathcal{X}^\perp$-structure</u> we mean a map $M \to \mathcal{X}^\perp$ lifting the stable classifying map for ν - normal bundle, arising from embeding of M to $\mathbb{A}^{m+k}$. ($\dim M = m$). Fix ~~this~~ some embeding.

Notice, that stable $\mathcal{X}$-structure is a coherent family of maps from M to $\mathcal{X}(n)$, which are classifying maps for $\widetilde{TM}$ by discussion in (9.60). But maps from $M \to \mathcal{X}(n)$, arising as lifts for $M \to BO(n)$ can be wiewed as a coheren family of maps to $M \to Gr(n,n')$ for n, n' large enough. Then we can repeat a discussion in 9.60, denote by $\mathcal{X}(n,n')$ a pull back of $\mathcal{X}(n)$ under $Gr(n,n') \to BO(n)$ and have following:

$$\begin{tikzcd} \widetilde{TM} \arrow[d] & \mathcal{X}(n,n') \arrow[r] \arrow[d] & \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X}^\perp \arrow[d] \\ M \arrow[r] & Gr(n,n') \arrow[r] & BO \arrow[r, "i"] & BO \end{tikzcd}$$

After composition with i we recieve a map from $M \to \mathcal{X}^\perp$.

$\widetilde{TM} \oplus \nu = \underline{\mathbb{R}}^{\cdot}$ ↖ large.

$\Rightarrow$ By above the composition of all $\to$ in the bottom gives a classifing map for ν $\Rightarrow$ we have ~~a~~ the ~~tangen~~ normal stable structure from tangential one. Notice, that final result does not depend from $M \hookrightarrow \mathbb{A}^{n+k}$.

Back to bordisms

Let now M be a manifold with boundary.
Recall, $T(\partial M) \oplus \nu = TM$ and ν is trivializible.
so $TM = \widetilde{T(\partial M)}$ in our notations.

If M has an $\mathcal{X}(n)$ (or $\mathcal{X}$) structure then ∂M also:
precompose map from $M \to \mathcal{X}(n)$ with $\partial M \hookrightarrow M$ and because $TM = \widetilde{T(\partial M)}$ we automatically have $\mathcal{X}(n)$ (or $\mathcal{X}$) structure on ∂M.

Let's say that Y_0 is bordant to Y_1 as manifolds with $\mathcal{X}$-structure if they are bordant in a usuall sense, X from the definition of bordism has $\mathcal{X}$ structure, and $\mathcal{X}$-structures on Y_i and on Y_i as a part of ∂X are isomorphic.

Freed's claim This is an equivalence relation.
Problem: Remember for an orientation if X a bordism from Y_0 to Y_1, then $-X$ is bordism from Y_1 to Y_0, where $-X$ denotes opposite orientation. How do we know for general $\mathcal{X}$-structure that any other structure exists?

When we change roles of Y_0 and Y_1 $\widetilde{T}Y_{\bullet i} \to \widetilde{T}Y_{\bullet i}$:

$$TY_{\bullet i} \oplus \mathbb{R} \xrightarrow[\mathrm{id} \oplus -\mathrm{id}]{} TY_{\bullet i} \oplus \mathbb{R}.$$

It's an isomorphism as vector bundles.
It could be not an isomorphism of bundles with additional structure (as in case of orientation). In gives new $\mathcal{X}$-structure on Y_i, but we don't know how to extend it ~~to~~ new structure on X.