Plan of talks

The general goal is to work through the first twelve lectures of Dan Freed's notes on bordism theory, plus a few other topics that seem important to cover along the way. If adjustments to the following schedule prove necessary in the middle of the semester, we will simply do the best we can.

1. April 18: *discussion of possible topics*

2. April 26: *Introduction to bordism theory and background on fiber bundles*
   I will give this talk. I will start by giving precise definitions of the oriented bordism groups Ω_n^{SO} and their unoriented counterparts Ω_n, plus their ring structure, and state some of the known results on computations that we hope to prove by the end of the semester. I will then cover some necessary background material on vector bundles and fiber bundles, hopefully including a useful lemma on how to trivialize bundles over cell complexes.

3. May 3: *Framed bordism and the Pontrjagin-Thom construction* (pages 6–10 of Lecture 2 and Lecture 3)
   This is a classic topic in differential topology and is e.g. also one of the focal points of Milnor's famous book *Topology from the Differentiable Viewpoint.* I first learned about Pontrjagin-Thom as a nice way to compute the set of homotopy classes of maps $M \to S^n$ for a smooth manifold M, and it is that, but one can also view it as a way of transforming a bordism problem into a problem of homotopy theory. We will later use a vast generalization of this idea to understand the groups Ω_n^{SO}. Essential topics to cover:

   - framed submanifolds and the definition of framed bordism
   - statement and proof of the Pontrjagin-Thom isomorphism
   - if time permits: application to the Hopf degree theorem (Theorem 2.37)

   Suggestion: It would be easiest if the student giving this talk is already comfortable with basic facts from differential topology such as Sard's theorem, the tubular neighborhood theorem, approximation of continuous maps by smooth maps, and what the implicit function theorem says for $f^{-1}(p) \subset M$ when p is a regular value of a smooth map $f : M \to N$. You don't need to know how to prove these things, and there wouldn't be time for that anyway, but you need to be able to make them believable to anyone in the room who hasn't seen them before.

4. May 10: *Stabilization and the Freudenthal suspension theorem* (Lecture 4)
   Essential topics to cover:

   - wedge sum, smash product and reduced suspension of pointed spaces
   - statement of the Freudenthal suspension theorem
   - quick review of direct limits (i.e. inductive limits or "colimits")
   - definition of the stable homotopy groups of the sphere
   - based loop space and duality with the suspension
   - stabilization of framed submanifolds and proof of Freudenthal via Pontrjagin-Thom

   A couple of things to note:

   - Freed has a *very* unfortunate typo in the definition of the stable homotopy groups of spheres, Equation (4.25): it should say $\pi_n^s = \operatorname{colim}_{q\to\infty} \pi_{n+q} S^q$.
   - What Freed is calling the *suspension* ΣX is in many standard references (e.g. Hatcher) called the *reduced suspension*, and is different from the ordinary suspension but is homotopy equivalent to it. (Unfortunately I used the same notation ΣX for the ordinary suspension in my Topologie II class last semester, but Hatcher calls that SX and reserves ΣX for the reduced suspension, which is consistent with Freed.) The reduced suspension is the more natural operation to consider when X comes with a base point.

Suggestion: Since this is the first (though certainly not the last) talk that will make use of direct limits, you might want to volunteer for this one if you think direct limits are cool.

5. May 17: *The stable framed bordism ring* (Lecture 5)
Essential topics to cover:
   - The ring structure of π_*^s
   - Stable framings/trivializations of vector bundles
   - Correspondence between stable normal framings and stable tangential framings for submanifolds
   - The stable framed bordism ring Ω_n^{fr} and stable Pontrjagin-Thom theorem
   - Quick sketch of the homotopy exact sequence for fiber bundles
   - The stable orthogonal group and stabilitzation of $\pi_n O(q)$ (Exercise 5.40, via homotopy exact sequence)

   The homotopy exact sequence for fibrations is an important background topic that is needed for this lecture and many times later on, but is unfortunately not explained in Freed's notes. Since many students will not have seen this before, I suggest stating the general theorem (e.g. Theorem 4.41 in Hatcher's *Algebraic Topology* book) and explaining briefly how the connecting homomorphism is defined before using the exact sequence to carry out Freed's Exercise 5.40 on the stabilization of the homotopy groups of orthogonal groups. (It suffices to restrict your attention to fiber bundles, even though the exact sequence is valid for much more general *Serre fibrations*). Freed's notes also cover the following material that should be considered inessential but is interesting, so the more you have time for the better:
   - Twists of framing and the unstable J-homomorphism $\pi_n O(q) \to \pi_{n+q} S^q$
   - The stable J-homomorphism
   - Parallelization of Lie groups and how they represent elements of π_*^s
   - Low-dimensional computations of π_*^s (statements are possible, but proofs would not be realistic)

6. May 24: *Classifying spaces* (Lecture 6 and beginning of Lecture 7)
The main result you need to prove in this talk is actually not stated cleanly in Lecture 6 but is Theorem 7.1 in Lecture 7: it is a statement about classification of principal G-bundles up to isomorphism, but it can be translated into a statement about vector bundles (via the relationship between these and their frame bundles), and this is part of what you should explain. The other thing to mention is that Freed's choice of construction for a classifying space is a bit unorthodox: where most authors take the direct limit of a sequence of finite-dimensional Grassmannians to produce something that is an infinite-dimensional CW-complex but definitively not a manifold, Freed uses an infinite-dimensional Hilbert manifold. The advantage of this is unclear to me, since in later lectures he needs to revert to the CW-decomposition in order to understand the cohomology of these things. But if you enjoy the idea of infinite-dimensional differential geometry, maybe you'll like this. Essential topics to cover:
   - Grassmannians and Stiefel manifolds
   - Contractibility of $St_k(\mathcal{H})$ (or if you prefer, some other principal $GL_k(\mathbb{R})$-bundle with the same homotopy type)
   - existence and uniqueness of classifying maps for principal bundles
   - why classifying principal bundles also classifies vector bundles

   Suggestion: If you find Freed's treatment of this topic too peculiar, you might want to try Chapter 19 of Steenrod's book *The Topology of Fibre Bundles*.

7. June 7: *Characteristic classes 1* (Lecture 7)
A lot of important details in this chapter are left as exercises, but they are very worthwhile exercises! You can skip the initial section on classifying spaces and homotopy invariance (all of which should by now have been covered in previous talks), as well as the final section on characteristic classes for principal G-bundles (which won't be needed until the final talk, and by then we'll be willing to accept a statement without proof). Essential topics to cover:

- universal characteristic classes (on classifying spaces)
- first Chern class on line bundles, formulas for tensor product and dual
- statement (without proof) of the Leray-Hirsch theorem for cohomology of fiber bundles—here it is worth mentioning the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ as an example of a fiber bundle for which the hypotheses of the Leray-Hirsch theorem are not satisfied (and the conclusion is clearly false)
- cohomology of flag bundles and the higher Chern classes
- splitting principle
- Whitney sum formula (Exercise 7.46)
- Chern classes of a conjugate bundle (Exercise 7.48)
- computation of $c(\mathbb{C}P^n)$—you might find it helpful to look at Prop. 8.2 in the next lecture
- stability of Chern classes (Exercise 7.71)

The material on Pontrjagin classes and the L-polynomial should be postponed to the next talk, so you can skip it. Things to note:

- There is no time to prove the Leray-Hirsch theorem on the cohomology of fiber bundles, so you will have to present it as a black box, but some explanation of what it means would be nice. The treatment of this in Hatcher's *Algebraic Topology* book is helpful.
- There is a horrible horrible horrible typo in the Whitney sum formula (Exercise 7.46): it should say
$$c(E_1 \oplus E_2) = c(E_1) \cup c(E_2).$$
Often the right hand side is written simply as "$c(E_1)c(E_2)$"; the cup product is implied.
- Minor typo (or oversight) in (7.45): the notation $c(M)$ doesn't make sense if M is an arbitrary smooth manifold, as TM needs to be a complex vector bundle, i.e. we can talk about $c(M)$ only if M has an *almost complex structure.*

Suggestion: Quite a lot of what you hopefully learned about cohomology in your second semester algebraic topology course will come in useful for this talk, so it would help if you remember that stuff (e.g. the Künneth formula) well.

8. June 14: *Characteristic classes 2* (Lecture 8 and a bit of Lecture 7)
This talk finishes up with some of the material on Chern and Pontrjagin classes in Lecture 7 and then covers the essentials of Lecture 8, including a nice simple proof of the Thom isomorphism theorem via cellular cohomology and the definition of the Euler class. I would recommend skipping the material on smooth plane curves and the K3 surface. The proof that $\chi(M) = \langle e(M), [M] \rangle$ on the last page is unfortunately not comprehensible without more background in intersection theory than is safe to assume, so we will postpone that until the following week. Essential topics to cover:
    - definition of the Pontrjagin classes (and optionally the computation of $p(S^n)$, i.e. Exercise 7.70)
    - Hirzebruch's L-class and the L-genus of $\mathbb{C}P^n$ (this begins on p. 9 of Lecture 7 and continues on p. 1–2 of Lecture 8). The proof of the signature theorem in the second to last talk will depend on this.
    - Thom classes, proof of the Thom isomorphism theorem, the Thom complex
    - definition of the Euler class, sum formula
    - equivalence of the Euler class and top Chern class

9. June 21: *A digression on intersection theory* (notes by Michael Hutchings)
The nicest theorem about the Euler class is one that Freed's notes never quite manage to state: for a smooth oriented vector bundle E over a closed oriented manifold M, $e(M)$ is Poincaré dual to the homology class represented by $s^{-1}(0) \subset M$ for any smooth section $s : M \to E$ that is transverse to the zero-section. Equivalently, this homology class can be interpreted as the intersection product of the zero-section in E with itself. The goal of this talk will be to outline this correspondence, including a rough explanation of the intersection product on homology (which is Poincaré dual to the cup product

on cohomology), and a proof that the self-intersection number of the diagonal in $M \times M$ is $\chi(M)$. (This is essentially a special case of the Lefschetz fixed point theorem.) An important corollary is the formula $\langle e(M), [M] \rangle = \chi(M)$. There may or may not be time to explain (in terms of the Thom isomorphism theorem) why the cup product obtains an intersection-theoretic interpretation under Poincaré duality. If there is enough time, then another nice thing to throw in would be the bordism invariant $\Omega_2 \to \mathbb{Z}_2$ described on p. 3 of Lecture 2, which uses the mod 2 self-intersection number for 1-dimensional submanifolds on a (not necessarily orientable) surface. Hutchings's notes on intersection theory are available at `https://math.berkeley.edu/~hutching/teach/215b-2011/cup.pdf`

10. June 28: *Tangential structures* (Lecture 9)
This lecture contains a lot of material on spin structures that we don't strictly need, but you should focus rather on the definitions in the second half, many of which are somewhat of a challenge to absorb (so try to work out some interesting examples!). Essential topics to cover:

    - the first Stiefel-Whitney class as an obstruction to orientability of real vector bundles (Hint for Exercise 9.2: remember that $H^1(X; G)$ is naturally isomorphic to $\mathrm{Hom}(\pi_1(X), G)$.)
    - reduction of structure groups and some examples, e.g. orientations, bundle metrics, spin structures
    - reduction of structure groups and lifts of classifying maps (Prop. 9.38)
    - n-dimensional tangential structures and stable tangential structures (definitions and examples)
    - relation to stable normal structures
    - definition of the groups $\Omega_*^{\mathcal{X}}$ and Ω_*^G

    A couple of notes:

    - I find Freed's definition of reduction of structure groups on page 2 overly abstract. For me, the natural way to phrase this definition is in terms of transition maps, e.g. see Definition 2.78 in my differential geometry notes,[1] which defines what Freed would call a reduction of the structure group of a fiber bundle $F \hookrightarrow E \to B$ from $\mathrm{Diff}(F)$ to G. (With small changes you could replace $\mathrm{Diff}(F)$ here with some Lie group G' that acts on F and for which you have a homomorphism $\rho : G \to G'$.) It is worth understanding why Freed's definition and mine are equivalent, then you can decide for yourself which version serves your intuition best.
    - For examples of stable tangential structures, you might want to look at (10.28) in the next lecture.

11. July 5: *Thom spectra and $\mathcal{X}$-bordism* (Lecture 10)
This talk introduces fundamental notions from stable homotopy theory in order to write down the general version of the Pontrjagin-Thom isomorphism for translating bordism problems into computations of homotopy groups. The ideas required for proving the isomorphism have by this point already been explained; the hard part is to write down the statement and understand what it says. Essential topics to cover:

    - prespectra and spectra
    - the Thom spectrum
    - the general Pontrjagin-Thom theorem and sketch of proof

12. July 12: *Signature and the Hirzebruch theorem* (Lecture 11)
This talk isn't quite self-contained because it takes a computational result about $\Omega^{SO} \otimes \mathbb{Q}$ as a black box (to be proved next week) in order to prove the signature theorem. It also uses a rational version of the standard Hurewicz theorem relating homotopy groups and homology, which will need to be stated without proof but is not hard to believe. Essential topics to cover:

    - quick review of the signature of a $4k$-manifold, its behavior under disjoint union, connected sum, and products
    - proof that signature is a bordism invariant
    - Pontrjagin numbers as bordism invariants

[1] `https://www.mathematik.hu-berlin.de/~wendl/Winter2016/DiffGeo1/connections_chapter2.pdf`

- proof of the signature theorem modulo the computation of $\Omega^{SO} \otimes \mathbb{Q}$
- the rational Hurewicz theorem (statement without proof)
- computation of $\Omega_4^{SO} \otimes \mathbb{Q}$ and the 4-dimensional case of the Hirzebruch theorem

13. July 19: *The rational oriented bordism ring* (Lecture 12)
Essential topics to cover:

    - the rational cohomology ring of BSO is a polynomial ring generated by Pontrjagin classes—either state this without proof or refer back to the end of Lecture 7
    - computation of the dimension of $\Omega_{4k}^{SO} \otimes \mathbb{Q}$
    - why products of $\mathbb{C}P^{2\ell}$ are generators (sketch)

    Freed doesn't say that much about the last point except that one can use Pontrjagin numbers to show that different products of complex projective spaces corresponding to different partitions of the total dimension are linearly independent in the rational bordism group. Working out the details requires some knowledge of the algebra of symmetric functions which Freed only mentions and refers to Chapter 16 of the book of Milnor and Stasheff on characteristic classes for details. I'll leave it up to you to decide how much of this you want to pursue for a 90 minute talk.