

EXERCISE FOR WEEK 3 (6.05.2020)

We will use the moodle forum at

`https://moodle.hu-berlin.de/course/view.php?id=95257`

for discussion of the exercises. If you write up a solution that you are happy with, feel to post it to the forum (you can upload a PDF file if you click on *Advanced* next to the *Post to forum* and *Cancel* buttons). You may also use the forum to comment on or ask questions about someone else's solution, or post a partial solution with details you are unsure about, or just to ask questions about the exercises.

1. Let $\mathrm{End}(\mathbb{R}^n)$ denote the space of linear transformations $\mathbb{R}^n \to \mathbb{R}^n$, and $\mathrm{End}^{\mathrm{sym}}(\mathbb{R}^n) \subset \mathrm{End}(\mathbb{R}^n)$ the subspace of symmetric linear transformations. By the finite-dimensional versions of results we proved in lecture about Fredholm operators, there are smooth submanifolds

$$\mathrm{End}_k(\mathbb{R}^n) \subset \mathrm{End}(\mathbb{R}^n), \quad \text{and} \quad \mathrm{End}_k^{\mathrm{sym}}(\mathbb{R}^n) \subset \mathrm{End}^{\mathrm{sym}}(\mathbb{R}^n)$$

defined as the sets of transformations with rank k; in particular, $\mathrm{End}_{n-1}(\mathbb{R}^n) \subset \mathrm{End}(\mathbb{R}^n)$ and $\mathrm{End}_{n-1}^{\mathrm{sym}}(\mathbb{R}^n) \subset \mathrm{End}^{\mathrm{sym}}(\mathbb{R}^n)$ are smooth hypersurfaces. Prove:

   (a) If $A : (-\epsilon, \epsilon) \to \mathrm{End}(\mathbb{R}^n)$ is a smooth path with $A(t) \in \mathrm{GL}(n, \mathbb{R})$ for all $t \neq 0$ and $\operatorname{rank} A(0) = n - 1$, then the intersection of this path with $\mathrm{End}_{n-1}(\mathbb{R}^n)$ at $t = 0$ is transverse if and only if $\frac{d}{dt} \det A(t)|_{t=0} \neq 0$.

   (b) $\mathrm{End}_{n-1}(\mathbb{R}^n)$ is co-orientable.

   (c) $\mathrm{End}_{n-1}^{\mathrm{sym}}(\mathbb{R}^n)$ is disconnected.

   (d) $\mathrm{End}_{n-1}(\mathbb{R}^n)$ is connected. (If you prefer, just consider the case $n = 2$.)

   (e) There exists a smooth path $A : [-1, 1] \to \mathrm{GL}(2, \mathbb{R})$ such that $A_\pm := A_{\pm 1}$ are both symmetric and $\mu^{\mathrm{spec}}(A_-, A_+) \neq 0$.

   (f) If $\mathbf{n}$ is a normal vector field along $\mathrm{End}_1(\mathbb{R}^2) \subset \mathrm{End}(\mathbb{R}^2)$ that is tangent to $\mathrm{End}^{\mathrm{sym}}(\mathbb{R}^2)$ at points in $\mathrm{End}_1^{\mathrm{sym}}(\mathbb{R}^2)$, then $\mathbf{n}$ *cannot* match the canonical co-orientation[1] of $\mathrm{End}_1^{\mathrm{sym}}(\mathbb{R}^2)$ on every connected component.

[1] that is, the co-orientation that is used in defining spectral flow for symmetric operators