EXERCISE FOR WEEK 5 (20.05.2020)

We will use the moodle forum at

`https://moodle.hu-berlin.de/course/view.php?id=95257`

for discussion of the exercises. If you write up a solution that you are happy with, feel to post it to the forum (you can upload a PDF file if you click on *Advanced* next to the *Post to forum* and *Cancel* buttons). You may also use the forum to comment on or ask questions about someone else's solution, or post a partial solution with details you are unsure about, or just to ask questions about the exercises.

1. Fix a Hermitian line bundle (E, J, ω) over S^1, an asymptotic operator $\mathbf{A}$ on E, and a complex-antilinear bundle isomorphism $B : E \to E$.

   (a) Check that $B : E \to E$ is symmetric with respect to the real bundle metric $g = \omega(\cdot, J\cdot)$. *Note: This would not always be true if E had complex rank greater than* 1.

   (b) Part (a) implies that
   $$\mathbf{A}_r := \mathbf{A} - rB$$
   also defines an asymptotic operator for every $r \in \mathbb{R}$. Prove that $\mathbf{A}_r$ is nondegenerate for all $|r| > 0$ sufficiently large.
   Hint: Derive a Weitzenböck formula presenting $\mathbf{A}_r^* \mathbf{A}_r - \mathbf{A}^* \mathbf{A}$ as a zeroth-order operator. It might be easiest to work in a global trivialization.

   (c) Show without computing the Conley-Zehnder index that for any given trivialization τ, $\mu_{\mathrm{CZ}}^{\tau}(\mathbf{A}_r)$ and $\mu_{\mathrm{CZ}}^{\tau}(\mathbf{A}_{-r})$ are the same for $r \gg 0$.

   (d) Show that if $B : E \to E$ is chosen to be complex-linear instead of antilinear, then the result of part (b) is false, i.e. $\mathbf{A}_r$ may be degenerate for arbitrarily large $|r|$.

   (e) In a unitary trivialization τ of (E, J, ω), $B : E \to E$ is identified with a function $B_\tau : S^1 \to \overline{\mathrm{End}}_{\mathbb{C}}(\mathbb{C})$ of the form $B_\tau(t)v = \beta_\tau(t)\bar{v}$ for some loop $\beta_\tau : S^1 \to \mathbb{C}\backslash\{0\}$. Show that for all $|r| > 0$ sufficiently large,
   $$\mu_{\mathrm{CZ}}^{\tau}(\mathbf{A}_r) = \mathrm{wind}(\beta_\tau).$$
   Hint: Certain special cases of this formula are proved in Lemma 5.30 in the notes.