Symplectic homology and lin. contact homology

(based on „An exact sequence for cti- and sympl. homology" Invent. Math. 2009 by Bourgeois and Oancea)

§1 Motivation

§2 $SH^+_*(W)$, $CH_*(M)$

§3 Filtrations & spectral seq.

§4 Pf of Thm

§5 Examples

<u>Thm (Bourgeois & Oancea)</u> There is a long ex. sequence W^{2n}

$$\cdots \to SH^+_{*-(n-3)}(W) \to CH_*(M) \to$$
$$\to CH_{*-2}(M) \to SH^+_{*-(n-3)-1}(W) \to \cdots$$

<u>Rmk</u>: (i) splits into direct sum indexed by free homotopy classes $S^1 \to W$

(ii) there is a version where replace $CH_*(M)$ with $SH^{S^1}_{*-(n-3)}(W)$

§1 Motivation

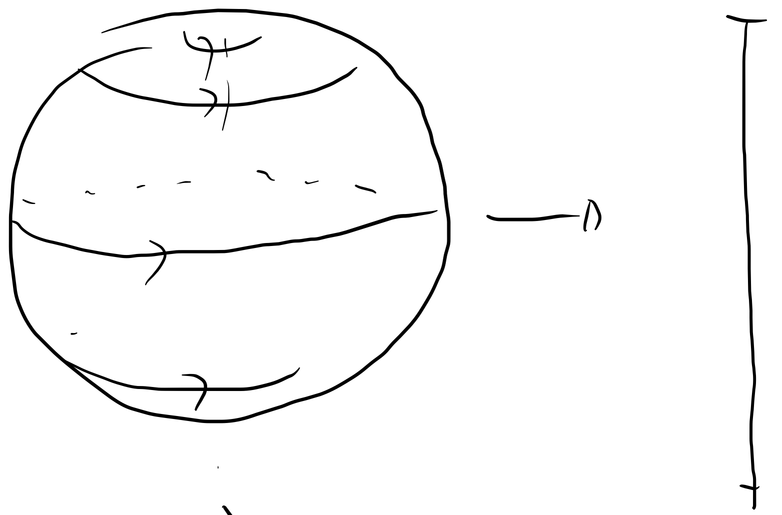
$S^1 = \mathbb{R}/\mathbb{Z}$ Lie group

Q S^1-space (i.e. mfd with S^1 action)

orbit space Q/S^1

if action is not free, then gen. Q/S^1 is not "nice"

eg: $Q = S^2$



Q $\quad$ Q/S^1 contractible

solution (Borel) replace Q with space of same homotopy type but with free action:

Let ES^1 contractible with free S^1-action

$$Q_{S^1} := ES^1 \times Q/S^1$$

where $S^1 \curvearrowright ES^1 \times Q$ diagonally (hence freely) Borel quotient

$$H^*_{S^1}(Q) := H^*(Q_{S^1})$$

equivariant cohomology

Properties

(i) if $S^1 \curvearrowright Q$ freely then $Q_{S^1} \sim Q/S^1$ and $H^*_{S^1}(Q) \cong H^*(Q/S^1)$

(ii) $H^*_{S^1}(Q)$ module over ring $H^*_{S^1}(pt) \cong \mathbb{Z}[u]$ $\deg u = 2$

(iii) Gysin sequence

integration along fibres

$$\cdots \to H^*_{S^1}(Q) \xrightarrow{\cdot \cap u} H^{*+2}_{S^1}(Q) \xrightarrow{\pi^*} H^{*+2}(Q) \xrightarrow{\pi_!} H^{*-1}_{S^1}(Q) \to \cdots$$

cup with class u

Bourgeois & Oancea: Do the same with $H^*(Q)$ with $SH^+_*(W)$ and $H^*_{S^1}(Q)$ with $CH_*(M)$.

§2 $SH_*(W)$

(W^{2n}, ω) cpt. sympl. mfd with ct. type bdary $\partial W = M$ (ie. $\exists Y \in \mathfrak{X}(W)$ st $\mathcal{L}_Y \omega = \omega$) near M

$\lambda := \omega(Y, \cdot)|$

$R \in \mathfrak{X}(M)$ $\begin{cases} \lambda(R) = 1 \\ d\lambda(R, \cdot) = 0 \end{cases}$

$\mathcal{P}(\lambda) := \{$ periodic unparam. R-orbits $\} \cup$

$\mathcal{P}^{\leq \alpha}(\lambda) = \{ \gamma$ with period $\leq \alpha \}$

$\hat{W} := W \cup_M [0,\infty) \times M$ completion

with $d(e^r \lambda)$ on $[0,\infty) \times M$

class of Hamiltonians $\mathcal{H}$

$H: \hat{W} \to \mathbb{R}$ s.t.

(i) $H|_W$ is C^2-small

(ii) $H|_{[0,\infty)\times M} = h \circ pr_{[0,\infty)}$ with

$h(r) = \alpha e^r + \beta \quad \alpha, \beta \in \mathbb{R}$

for $r \gg 1 \quad \alpha \notin spec(\lambda)$

$X_H \in \mathfrak{X}(\hat{W})$ via $dH = \omega(X_H, \cdot)$

on $[0,\infty) \times M$ have $X_H = -e^{r} h'(r) R$

(iii) $h'' - h' > 0$

assume that λ is st'ly non-degenerate (i.e. $\forall \gamma \in \mathcal{P}(\lambda)$

$d\phi^T|_{\xi = \ker\lambda}$ has no

eigenval $=1$ T period of γ

and $\phi^{\cdot}$ flow of R)

$\mathcal{P}(H) := \{ x : S^1 \to \hat{W} \mid \dot{x} = X_H(x) \}$

parametrized Ham. orbits consists of two types

(i) crit pts of $H|W$

(ii) $x(t) = \gamma(-Tt)$ for some $T \leq \alpha$.

hence for each $\gamma \in \mathcal{P}^{\leq \alpha}(\lambda)$ get S^1-family $S_\gamma \subset \mathcal{P}(H)$

Morse-Bott version of Floer hom

aux. Morse func' $f: \mathcal{P}(H) \to \mathbb{R}$

st. $\operatorname{crit} f|_{S_\gamma} = \{\check{\gamma}, \hat{\gamma}\}$

min. max.

assume $\int_{\mathbb{T}^2} u^* \omega = 0 \quad \forall u: \mathbb{T}^2 \to W$

(makes Ham. action func'l well defined)

$\Lambda_\omega :=$ Novikov ring

$:=$ completion of $\mathbb{Q}[H_2(W)]$

$$\ni \sum_{A \in H_2(W)} a_A e^A \qquad a_A \in \mathbb{Q}$$

st. $\{A \mid a_A \neq 0, \omega(A) \leq c\}$ finite for all $c > 0$.

$$BC_*(H) := \bigoplus_{p \in \operatorname{Crit} H} \langle p \rangle \oplus \bigoplus_{\gamma \in \mathcal{P}^{\leq \alpha}(\lambda)} \langle \check{\gamma} \rangle \oplus \langle \hat{\gamma} \rangle$$

free Λ_ω-mod, graded via

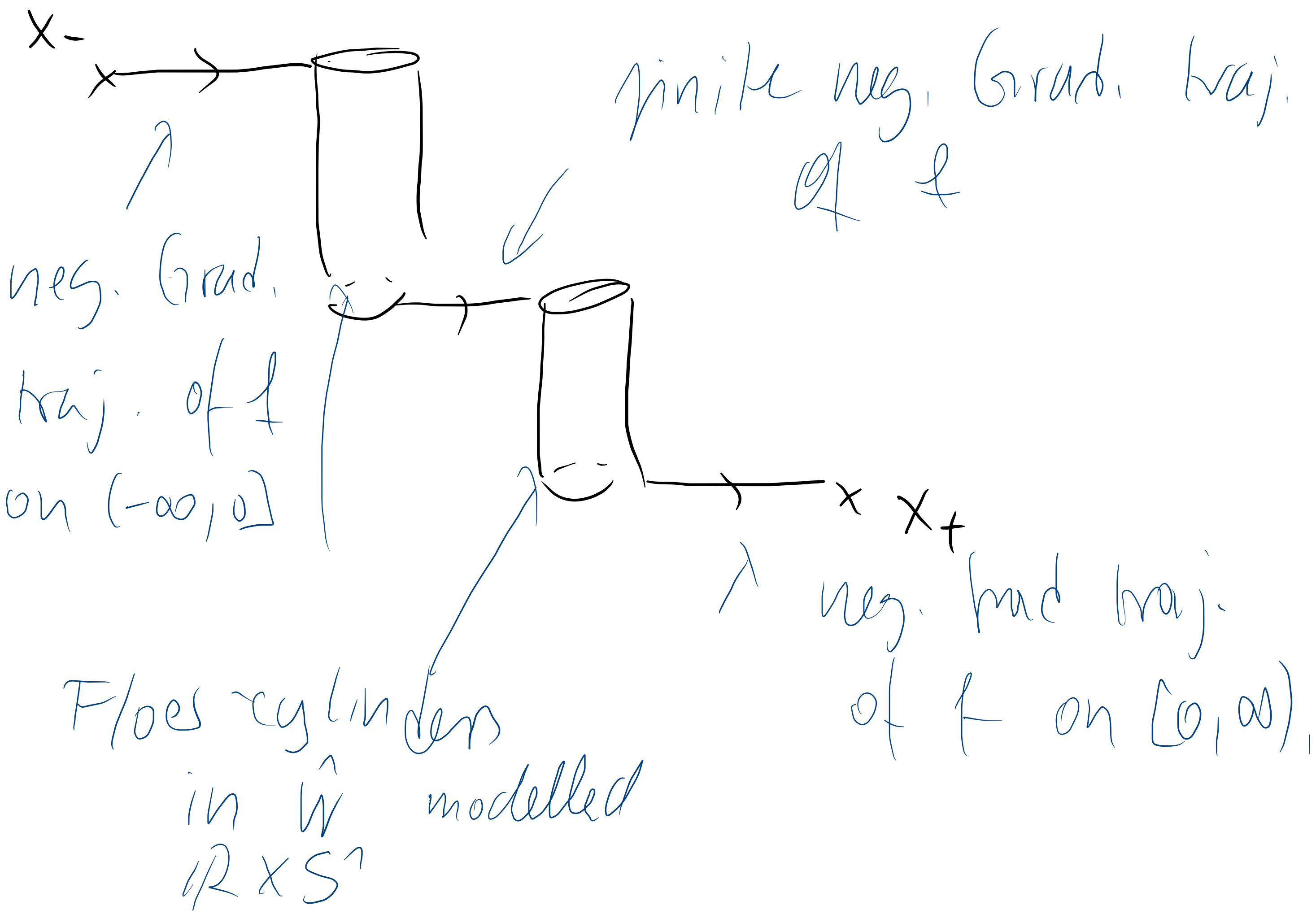
$|p| = \mu_{Morse}(p) - n \quad (= \mu_{CZ}(p))$

$|\check{\gamma}| = \mu_{CZ}(\gamma) \quad |\hat{\gamma}| = \mu_{CZ}(\gamma) + 1$

$|e^A| = -2\, c_1^{TW}(A)$

$\partial_H : BC_*(H) \to BC_{*-1}(H)$

defined via counting Floer-traj. with cascades



Floer traj. with 2 cascades.

$x_-, x_+ \in \mathrm{Crit}\, f \subset \mathcal{P}(H)$

$\partial_H \circ \partial_H = 0$

$\bigoplus_{p \in \mathrm{Crit}\, H} \langle p \rangle$ sub cx. of $BC_*(H)$

have quotient cx.

$$BC_*^+(H) = \frac{BC_*(H)}{\bigoplus_{p \in \mathrm{Crit}\, H} \langle p \rangle}$$

$$SH_*(W) := \varinjlim_{H \in \mathcal{H}} H_*(BC_*(H))$$

$$SH_*^+(W) := \varinjlim_{H \in \mathcal{H}} H_*(BC_*^+(H))$$

get long ex. seq

$$\begin{tikzcd}
SH_*(W) \arrow[rr] & & SH^+_*(W) \arrow[dl, "{[-1]}"] \\
& H_*(W) \arrow[ul] &
\end{tikzcd}$$

$CH_*(\lambda)$

equivariant version of $SH_*(W)$

fix $\mathbb{R} \times S^1 \simeq \mathbb{C}P^1 \setminus \{0, \infty\}$

asymptotic markers

$(\ell_-, \ell_+, \ell_1, \dots, \ell_k)$ are unit vectors

$\ell_- \in T_0 \mathbb{C}P^1$, $\ell_+ \in T_\infty \mathbb{C}P^1$

and $\ell_j \in T_{z_j} \mathbb{C}P^1$

fix base point $*$ in each free homotopy class $S^1 \to M$ and for each $\gamma \in \mathcal{P}(\lambda)$ a map $u_\gamma : [0,1] \times S^1 \to M$

s.t. $u_\gamma(0, \cdot) = *$ $u_\gamma(1, \cdot) = \gamma$.

given $\gamma_-, \gamma_+, \gamma_1, \dots, \gamma_k \in \mathcal{P}(\lambda)$ and $B \in H_2(M)$ consider

$$\hat{\mathcal{M}}^B(\gamma_-, \gamma_+, \gamma_1, \dots, \gamma_k)$$

space of all tuples $(u, \ell_-, \ell_+, \ell_1, \dots, \ell_k)$
where $u : \mathbb{R} \times S^1 \setminus \{z_1, \dots, z_k\} \to M \times \mathbb{R}$
J_∞-holomorphic and sat. asymptotics

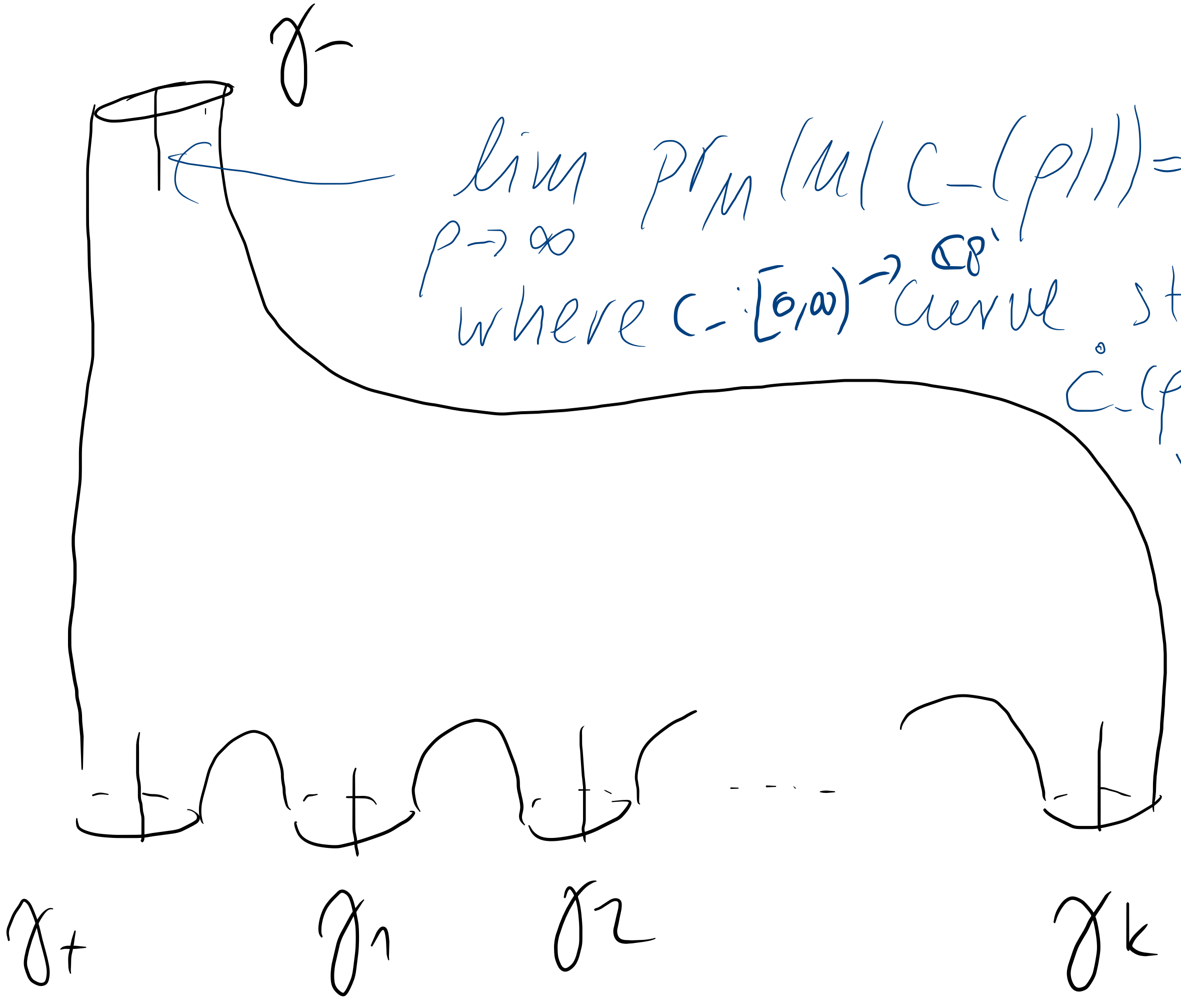
$$\lim_{s \to \pm\infty} pr_{\mathbb{R}}(u(s, \cdot)) = \mp\infty$$

$$\lim_{z \to z_i} pr_{\mathbb{R}}(u(z)) = -\infty$$

$$\lim_{s \to \pm\infty} pr_M(u(s,t)) = \gamma_\pm(-T_\pm \cdot t)$$

$$\lim_{\rho_i \to 0} pr_M(u(\rho_i, \theta_i)) = \gamma_i(T_i t)$$

where (ρ_i, θ_i) polar coord. near z_i



$\lim_{\rho \to \infty} pr_M(u(c_-(\rho))) = P_{\gamma_-}$
where $c_- : [0,\infty) \to \mathbb{C}P^1$ curve s.t. $\dot{c}_-(\rho) \to \ell_-$

$c_- : [0,\infty) \to \mathbb{C}P^1 \setminus \{0, \infty, z_1, \dots, z_k\}$
$c_-(\rho) \to 0$ and $\dot{c}_-(\rho) \to \ell_-$
as $\rho \to +\infty$

where p_γ is a fixed point on the image of γ.

require that

$$[u_{\gamma_1} \# u_{\gamma_2} \# \dots \# u_{\gamma_k} \# u_{\gamma_-}] = [u_{\gamma_+} \# u_B]$$

where u_B curve representing B.

$\mathrm{Aut}(\mathbb{R} \times S^1)$ acts on $\underset{\psi}{\hat{\mathcal{M}}}^B(\gamma_-, \gamma_+, \gamma_1 \dots \gamma_k)$

$$\psi \cdot u = u \circ \psi$$
$$\psi\, l_j = d\psi\, l_j$$

$$\mathcal{M}^B(\gamma_-, \gamma_+, \gamma_1 \dots \gamma_k) = \hat{\mathcal{M}}^B(\gamma_-, \gamma_+, \gamma_1 \dots \gamma_k) / \mathrm{Aut}(\mathbb{R} \times S^1)$$

„space of punct'd. cylinders with asympt. markers"

<u>Thm</u> $\exists \infty$ reg. then $\mathcal{M}^B(\gamma_-, \gamma_+, \gamma_1, \dots, \gamma_k)$ is a mfd. of dimension

$$\mu(\gamma_-) - \mu(\gamma_+) + 2c_1^{\xi}(B) - \sum_{i=1}^{k} \mu(\gamma_i)$$

Def: $\gamma \in \mathcal{P}(\lambda)$ $\quad \gamma^{2k}$ <u>good</u> for $k \in \mathbb{N}$ „anchors"

if $\mu(\gamma^{2k}) = \mu(\gamma^{2k+1}) \mod 2$ $\quad \mathcal{M}^A(\gamma, \emptyset) := \hat{\mathcal{M}}^A(\gamma, \emptyset)/\mathrm{Aut}(\mathbb{C})$

$\forall k$

otherwise γ^{2k} is <u>bad</u>.

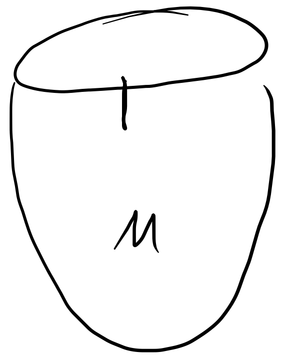
$$C_*(\lambda) := \bigoplus_{\gamma \text{ good}} \langle \gamma \rangle$$

define further $\hat{\mathcal{M}}^A(\gamma, \emptyset)$

as the space of tuples (u, ℓ_∞)

$u : \mathbb{CP}^1 \setminus \{\infty\} \to \hat{W}$

J-hol. plane γ



free Λ_ω module with grading

$|\gamma| = \mu(\gamma) + n - 3 =: \bar{\mu}(\gamma)$

$|e^A| = -2\, c_1^{TW}(A)$

$e : C_*(\lambda) \longrightarrow \Lambda_\omega$ degree 0

$$\gamma \longmapsto \sum_A \#_{alg} \mathcal{M}^A(\gamma, \emptyset) \cdot e^A$$

„augmentation"

bdury operator

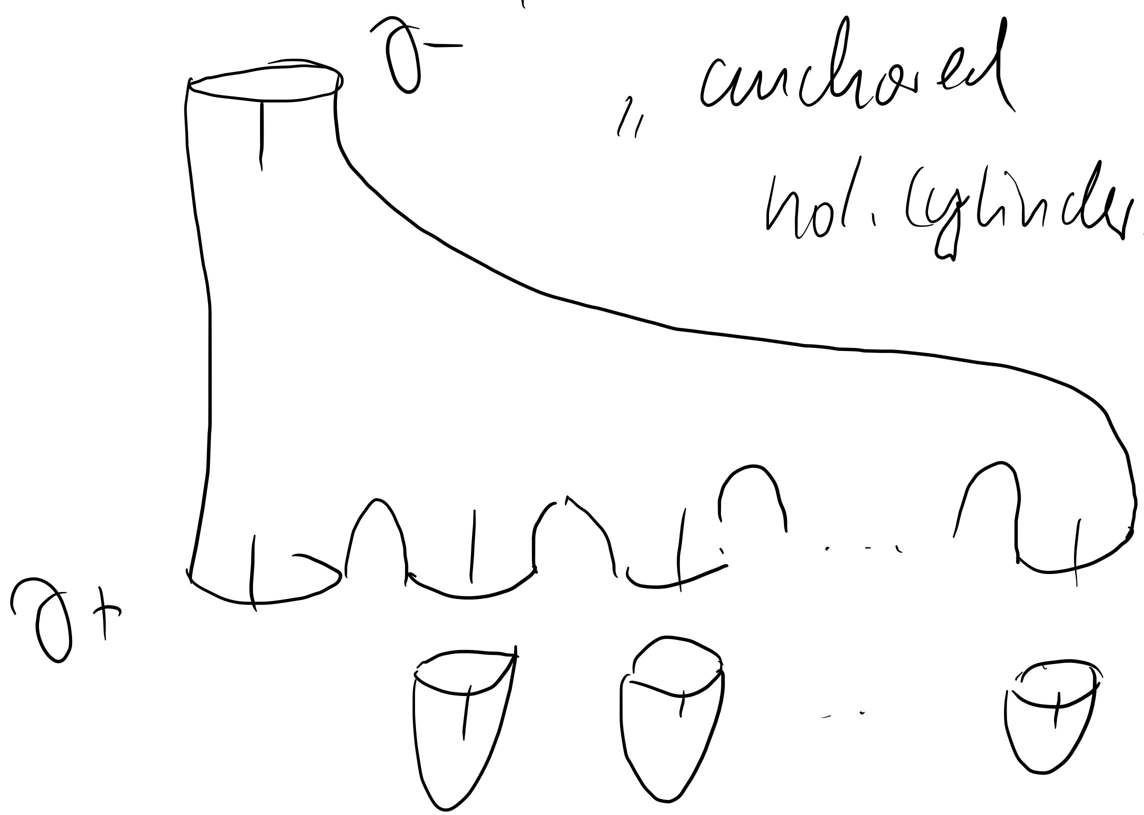
$\partial_\lambda : C_*(\lambda) \longrightarrow C_{*-1}(\lambda)$ via

$$\gamma_- \longmapsto \sum_{\substack{k\geq 0 \\ B\in H_2(M) \\ \gamma_+,\gamma_1,\dots,\gamma_k \in P(\lambda)}} \frac{\#_{alg}\mathcal{M}^B(\gamma_-,\gamma_+,\gamma_1,\dots,\gamma_k)}{K_{\gamma_-}K_{\gamma_1}\cdots K_{\gamma_k}} \cdot e(\gamma_1)e(\gamma_2)\cdots e(\gamma_k)e^B\cdot\gamma_+$$

$K_\gamma :=$ multiplicity of $\gamma \in P(\lambda)$

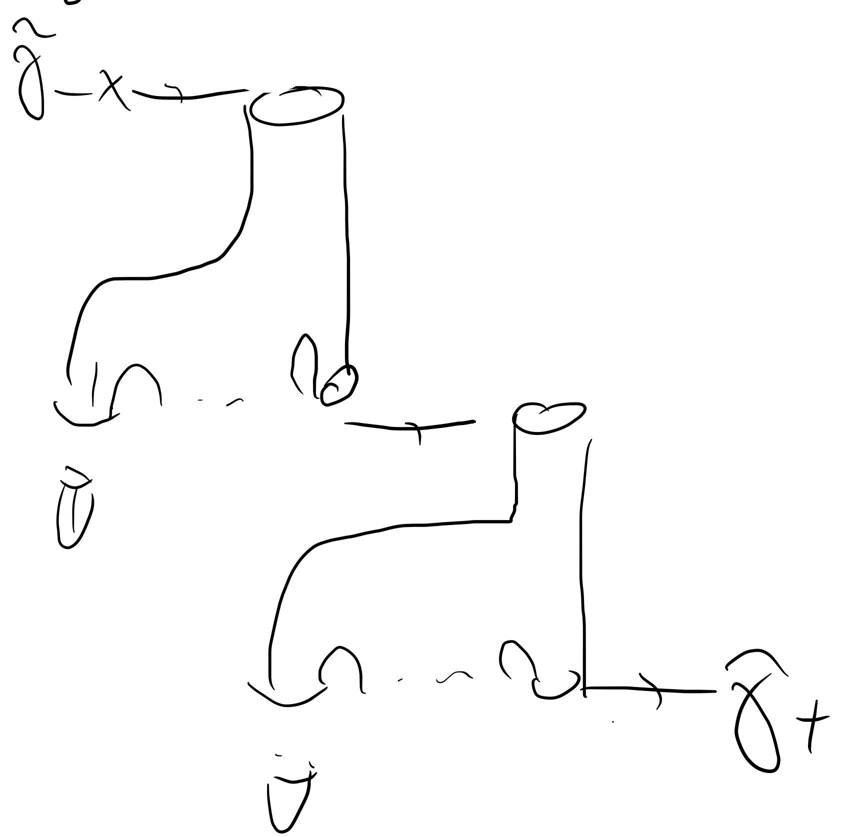
$$\boxed{\begin{array}{l} CH_*(\lambda) := H_*(C_*(\lambda),\partial_\lambda) \\ \text{linearized CH homology} \end{array}}$$

alt description for ∂_λ via count of



" anchored hol. cylinders"

there is also a non-equivariant version of lin. ct. hom. by counting cascades of anchored cylinders



$$B C_*(\alpha) := \bigoplus_{\gamma \in \mathcal{P}(\alpha)} \langle \check{\gamma} \rangle \oplus \langle \hat{\gamma} \rangle$$

„S^1 param. contact homology complex"

§3 Filtrations

(C_*, ∂) cx.

$\mathcal{F}^\ell C_* \subset C_*$ sub cx. $\forall \ell \in \mathbb{Z}$

s.th. $\mathcal{F}^\ell C_* \subset \mathcal{F}^{\ell+1} C_*$

and $\bigcap \mathcal{F}^\ell C_* = 0$ $\quad \varinjlim \mathcal{F}^\ell C_* = C_*$

Filtration

$\Rightarrow$ gives rise to a spectral seq.

$$E^r_{pq} \Rightarrow C_{p+q}$$

in our example

$$F^{\ell} BC^+_*(H) = \bigoplus_{\mu(\gamma) - 2c_1(A)} \mathbb{Q}\, e^A \check{\gamma} \oplus \mathbb{Q}\, e^A \hat{\gamma}$$

gives a filtration of $BC^+_*(H) = \bigoplus \langle \check{\gamma} \rangle \oplus \langle \hat{\gamma} \rangle$

similarly

$$F^{\ell} BC_*(H) = \bigoplus_{\mu(\gamma) - 2c_1(A) \le \ell} \mathbb{Q}\, e^A \check{\gamma} \oplus \mathbb{Q}\, e^A \hat{\gamma}$$