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Differentialgeometrie II

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Problem Set 2

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Problem 1

For paths $\gamma : [a,b] \to M$ between distinct points $\gamma(a) = p$ and $\gamma(b) = q$ in a Riemannian manifold (M, g), we consider the *length* and *energy* functionals

$$\ell(\gamma) := \int_a^b |\dot\gamma(t)|\, dt, \qquad E(\gamma) := \frac{1}{2}\int_a^b |\dot\gamma(t)|^2\, dt,$$

where by definition $|X| := \sqrt{g(X,X)}$. Suppose $\{\gamma_s : [a,b] \to M\}_{s\in(-\epsilon,\epsilon)}$ is a smooth family of paths from p to q, and write $\gamma := \gamma_0$.

(a) Show that γ satisfies the geodesic equation if and only if $\frac{d}{ds}E(\gamma_s)\big|_{s=0} = 0$ for all possible smooth families of paths $\{\gamma_s\}$ with $\gamma_0 = \gamma$ as described above.

(b) Prove that if γ has constant speed $|\dot\gamma(t)|$ and $\frac{d}{ds}\ell(\gamma_s)\big|_{s=0} = 0$, then $\frac{d}{ds}E(\gamma_s)\big|_{s=0}$ also vanishes, and conclude that length-minimizing paths with constant speed are always geodesics.

(c) Show that if $\gamma : [a,b] \to M$ is not assumed to have constant speed but has minimal length among all smooth paths from p to q, then γ has the same image as an *embedded* path. *Hint: If $\gamma : [a,b] \to M$ is a path with $\dot\gamma = 0$ on some compact subinterval $[t_0,t_1] \subset [a,b]$, what can you say about the shortest path from $\gamma(t_0 - \epsilon)$ to $\gamma(t_1 + \epsilon)$ for $\epsilon > 0$ small?*

Problem 2

Let's review some first-year analysis. Suppose $K \subset \mathbb{R}^k$ is a compact set, $f_j : \mathcal{U}_j \to \mathbb{R}^n$ is a sequence of C^1-smooth maps defined on open neighborhoods $\mathcal{U}_j \subset \mathbb{R}^k$ of K, and $f : \mathcal{U} \to \mathbb{R}^n$ is another such map such that f_j and their first partial derivatives all converge uniformly on K to f and its respective first derivatives.

(a) Prove that if $f|_K : K \to \mathbb{R}^n$ is an injective immersion, then the same is true of $f_j|_K : K \to \mathbb{R}^n$ for all j sufficiently large.
Hint: Show that if $f_j(p_j) = f_j(q_j)$ for some sequences of distinct points $p_j, q_j \in K$, then the derivative $Df(p) : \mathbb{R}^k \to \mathbb{R}^n$ must have nontrivial kernel for some $p \in K$. Use the definition of the derivative.

For smooth manifolds M and N, let us say that a sequence of continuously differentiable maps $f_j \in C^1(M,N)$ is C^1_{loc}-*convergent* to a map $f \in C^1(M,N)$ if for all charts $(\mathcal{U}, x)$ on M and $(\mathcal{V}, y)$ on N, the maps $y \circ f_j \circ x^{-1}$ and their first derivatives converge uniformly to $y \circ f_j \circ x^{-1}$ and its respective first derivatives on all compact subsets of their domains.

(b) Deduce from part (a) that under this condition, if f is an embedding and M is compact, then f_j is also an embedding for all j sufficiently large.

In other words: for a compact manifold M, embeddedness of maps $M \to N$ is a "C^1-open condition".

Problem 3
Find an example of a surface $\Sigma \subset \mathbb{R}^3$ such that, if Σ is endowed with the Riemannian metric g determined by the Euclidean inner product, then (Σ, g) is geodesically complete but its injectivity radius is 0. Prove it with a picture.

Problem 4
Let us say that a subset K of a Riemannian n-manifold (M, g) is *small* if for every pair of points $p, q \in K$, p is contained in a geodesic ball about q of some radius $r < \mathrm{inj}(q)$. Prove:

(a) The intersection of any two small subsets is also small.

(b) The intersection of any two geodesically convex subsets is also geodesically convex.

(c) Any small geodesically convex open subset is diffeomorphic to an n-dimensional star-shaped domain, i.e. a set of the form $\{r\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \in S^{n-1},\ -f(\mathbf{x}) < r < f(\mathbf{x})\}$ for some (not necessarily continuous) function $f : S^{n-1} \to (0, \infty]$.

(d) Every n-dimensional star shaped domain is smoothly contractible.

(e) Every smooth n-manifold admits a so-called *good cover*, i.e. a covering by open subsets whose nonempty finite intersections are all smoothly contractible.

Problem 5
A Riemannian manifold (M, g) is called *homogeneous* if for every pair of points $p, q \in M$, there exists a diffeomorphism $\varphi : M \to M$ satisfying $\varphi^* g = g$ and $\varphi(p) = q$.[1] Show that every homogeneous Riemannian manifold is complete.
Hint: Can the injectivity radius $\mathrm{inj}(p)$ *for* $p \in M$ *get arbitrarily small?*

Problem 6
Show that any flat connection on a vector bundle $E \to M$ determines a natural flat structure (see Definition 32.16 in the notes) on $E \to M$, for which the flat sections on open subsets of M are precisely the parallel sections. Conversely, show that any flat vector bundle over a smooth manifold carries a natural flat connection with the same property.

Problem 7
Recall that on any pseudo-Riemannian manifold (M, g), the fully covariant version $\mathrm{Riem} \in \Gamma(T_4^0 M)$ of the Riemann curvature tensor satisfies the four symmetry relations

(i) $\mathrm{Riem}(V, X, Y, Z) + \mathrm{Riem}(V, Y, X, Z) = 0$

(ii) $\mathrm{Riem}(V, X, Y, Z) + \mathrm{Riem}(Z, X, Y, V) = 0$

(iii) $\mathrm{Riem}(V, X, Y, Z) + \mathrm{Riem}(V, Y, Z, X) + \mathrm{Riem}(V, Z, X, Y) = 0$ *(first Bianchi identity)*

(iv) $\mathrm{Riem}(V, X, Y, Z) = \mathrm{Riem}(Y, Z, V, X)$ *(interchange symmetry)*

Show that in the case $\dim M = 2$, the relations (iii) and (iv) are implied by (i) and (ii).
Hint: If (i) and (ii) are satisfied and $\dim M = 2$*, how many of the components* $R_{ijk\ell}$ *in local coordinates can be nonzero, and how many are actually independent of each other?*

[1] The condition $\varphi^* g = g$ means that φ is an *isometry*, thus it preserves all structures determined by the metric, e.g. geodesics.