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Differentialgeometrie II
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Problem Set 8

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Problem 1

On any Lie group G, the *Maurer-Cartan form* is defined as the unique $\mathfrak{g}$-valued left-invariant 1-form $\theta \in \Omega^1(G, \mathfrak{g})$ such that $\theta_e = \mathbb{1}_{\mathfrak{g}}$.

(a) Prove that θ satisfies the so-called *Maurer-Cartan equation*:

$$d\theta + \frac{1}{2}[\theta, \theta] = 0.$$

Hint: The expression on the left is a $\mathfrak{g}$-valued 2-form on G, and it suffices to evaluate it on an arbitrary pair of left-invariant vector fields.

(b) Prove that θ transforms under right translations $R_g : G \to G : h \mapsto hg$ by

$$R_g^* \theta = \mathrm{Ad}_{g^{-1}} \circ \theta \qquad \text{for } g \in G.$$

(c) Can you interpret the Maurer-Cartan equation in terms of connections?

Problem 2

Assume $\pi : E := F^G(TM) \to M$ is the G-frame bundle of the tangent bundle of an n-manifold M, where $TM \to M$ has been equipped with a G-structure for some matrix group $G \subset \mathrm{GL}(n, \mathbb{R})$. Let $\rho : G \to \mathrm{GL}(n, \mathbb{R})$ denote the inclusion, which defines a linear left G-action on $\mathbb{R}^n$ for which TM is isomorphic to the associated vector bundle $E^\rho := (E \times \mathbb{R}^n)/G$. There is a *tautological 1-form*

$$\theta \in \Omega^1(E, \mathbb{R}^n)$$

defined by $\theta_\phi(\xi) := \phi^{-1}(\pi_* \xi)$ for $\xi \in T_\phi E$, where we regard frames $\phi \in E_p$ at points $p \in M$ as vector space isomorphisms $\phi : \mathbb{R}^n \to T_p M$. Given a connection ∇ on TM induced by a choice of principal connection $A \in \Omega^1(E, \mathfrak{g})$ on E, the torsion tensor $T \in \Gamma(T^1_2 M)$ can be interpreted as a bundle-valued 2-form $T \in \Omega^2(M, TM) = \Omega^2(M, E^\rho)$, thus it is naturally equivalent to some ρ-equivariant horizontal 2-form $\tau \in \Omega^2_\rho(E, \mathbb{R}^n)$. The *first structural equation* of Cartan is the relation

$$\tau = d\theta + A \wedge \theta,$$

where the wedge product of $A \in \Omega^1(E, \mathfrak{g})$ with $\theta \in \Omega^1(E, \mathbb{R}^n)$ is defined in terms of the bilinear map $\mathfrak{g} \times \mathbb{R}^n \to \mathbb{R}^n : (X, v) \mapsto \rho_*(X)v$. Prove the equation.
Hint: You can use the same approach that we used to prove the second structural equation in lecture, but there is also a much quicker way. Notice that θ is horizontal and ρ-equivariant. What bundle-valued 1-form on M is it equivalent to?

Problem 3

In many older or more elementary treatments, connections and curvature on vector bundles are described mainly in terms of locally-defined objects that depend on choices of trivializations, without ever mentioning a principal bundle. This exercise is meant to help you *translate between the local picture and the more global perspective that we've adopted in our lectures.*

Assume $\pi : E \to M$ is a principal G-bundle, with a connection 1-form $A \in \Omega^1(E, \mathfrak{g})$ and curvature 2-form $F \in \Omega^2(E, \mathfrak{g})$, and $\{s_\alpha \in \Gamma(E|_{\mathcal{U}_\alpha})\}_{\alpha \in I}$ is a collection of local sections on open sets $\mathcal{U}_\alpha$ that cover M. For any vector space V and $\omega \in \Omega^k(E, V)$ with $k \geqslant 0$, we can pull back ω via the maps $s_\alpha : \mathcal{U}_\alpha \to E$ to define local V-valued k-forms on M,

$$\omega_\alpha := s_\alpha^* \omega \in \Omega^k(\mathcal{U}_\alpha, V), \qquad \alpha \in I$$

The 1-forms $\{A_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathfrak{g})\}_{\alpha \in I}$ and 2-forms $\{F_\alpha \in \Omega^2(\mathcal{U}_\alpha, \mathfrak{g})\}_{\alpha \in I}$ are called the *local* connection and curvature forms respectively. Prove:

(a) The connection on $\pi : E \to M$ is uniquely determined by the collection of local connection forms $\{A_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathfrak{g})\}_{\alpha \in I}$, and its curvature 2-form is similarly determined by the local curvature forms $\{F_\alpha \in \Omega^2(\mathcal{U}_\alpha, \mathfrak{g})\}_{\alpha \in I}$. (Are analogous statements true for *all* forms in $\Omega^*(E, \mathfrak{g})$?)

(b) $F_\alpha = dA_\alpha + \frac{1}{2}[A_\alpha, A_\alpha]$ and $dF_\alpha = [F_\alpha, A_\alpha]$ for each $\alpha \in I$.

Now suppose $\rho : G \to \mathrm{GL}(V)$ is a representation of G on some finite-dimensional vector space V, with induced Lie algebra representation $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$, and let $E^\rho = (E \times V)/G \to M$ denote the associated vector bundle, which carries a connection ∇ determined by $A \in \Omega^1(E, \mathfrak{g})$. As shown in lecture, the local sections $\{s_\alpha \in \Gamma(E|_{\mathcal{U}_\alpha})\}_{\alpha \in I}$ determine a G-bundle atlas $\{\Phi_\alpha : E^\rho|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times V\}_{\alpha \in I}$ for E^ρ, where $\Phi_\alpha^{-1}(p, v) = [s_\alpha(p), v] \in E^\rho_p$ for $p \in \mathcal{U}_\alpha$ and $v \in V$, and the corresponding system of transition functions $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G$ is determined by

$$s_\alpha = s_\beta g_{\beta\alpha} \qquad \text{on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta.$$

For each $\omega \in \Omega^k(M, E^\rho)$, $k \geqslant 0$, let $\hat{\omega} \in \Omega^k_\rho(E, V)$ denote the ρ-equivariant horizontal form that corresponds to it under the natural isomorphism $\Omega^k(M, E^\rho) \cong \Omega^k_\rho(E, V)$, and denote $\omega_\alpha := \hat{\omega}_\alpha = s_\alpha^* \hat{\omega} \in \Omega^k(\mathcal{U}_\alpha, V)$ for each $\alpha \in I$. Given $\omega \in \Omega^k(M, E^\rho)$ and $\alpha, \beta \in I$, prove:

(c) $\omega_\alpha \in \Omega^k(\mathcal{U}_\alpha, V)$ is the local representation of ω with respect to the trivialization Φ_α, meaning

$$\Phi_\alpha(\omega(X_1, \dots, X_k)) = (p, \omega_\alpha(X_1, \dots, X_k)) \qquad \text{for } X_1, \dots, X_k \in T_pM,\ p \in \mathcal{U}_\alpha.$$

(d) $\omega_\beta = \rho(g_{\beta\alpha}) \circ \omega_\alpha$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$.

(e) $(d_\nabla \omega)_\alpha = d\omega_\alpha + A_\alpha \wedge \omega_\alpha$, where the wedge product of $A_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathfrak{g})$ with $\omega_\alpha \in \Omega^k(\mathcal{U}_\alpha, V)$ is defined in terms of the bilinear map $\mathfrak{g} \times V \to V : (X, v) \mapsto \rho_*(X)v$. In particular, for a section $\eta \in \Gamma(E^\rho) = \Omega^0(M, E^\rho)$ and $X \in \mathfrak{X}(\mathcal{U}_\alpha)$, one obtains

$$(\nabla_X \eta)_\alpha = d\eta_\alpha(X) + \rho_*(A_\alpha(X))\eta_\alpha.$$

Finally, prove the following transformation formulas for the local connection and curvature forms: given $p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ and $X, Y \in T_pM$,

(f) $F_\beta(X, Y) = \mathrm{Ad}_{g_{\beta\alpha}(p)} \circ F_\alpha(X, Y)$

(g) $A_\beta(X) = \mathrm{Ad}_{g_{\beta\alpha}(p)} \circ A_\alpha(X) + TL_{g_{\beta\alpha}(p)} \circ Tg_{\alpha\beta}(X)$, where $L_g : G \to G$ denotes left translation $h \mapsto gh$.

In the special case where $G \subset \mathrm{GL}(m, \mathbb{F})$ is a matrix group acting in the obvious way on $V = \mathbb{F}^m$, the transformation formulas of parts (f) and (g) can be written in the simplified form

$$F_\beta = gF_\alpha g^{-1}, \qquad A_\beta = gA_\alpha g^{-1} + g\, dg^{-1},$$

where we abbreviate $g := g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G$. The second formula is known to physicists as a *gauge transformation.*